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A LOGARITHMICALLY COMPLETE MONOTONICITY PROPERTY OF THE GAMMA FUNCTION

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Abstract: In this article, we give a logarithmically complete monotonicity property and a supper-additive property of the gamma function.

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1. Introduction

The Euler gamma function is defined and denoted for $\Re(z) > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) := \Gamma'(z)/\Gamma(z)$, is called the psi or digamma function, and $\psi^{(k)}$, for $k \in \mathbb{N}$, are called the polygamma functions.

Recall (see [17]) that a function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o (the interior of I) and $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I^o$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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S. Bernstein proved (see [17, Chapter IV, Section 12]) that a function f on the interval $(0,\infty)$ is completely monotonic if and only if there exists an increasing function $\alpha(t)$ on $[0,\infty)$ such that

$$f(x)=\int_0^\infty e^{-xt}d\alpha(t).$$

Recall also (see [1], [14]) that a positive function f is said to be logarithmically completely monotonic on an interval I if $f\in C(I)$, has derivatives of all orders on I^o and $(-1)^n[\ln f(x)]^{(n)}\geq 0$ for all $x\in I^o$, and $n\in\mathbb{N}$.

It is known (see [10], [14]) that a logarithmically completely monotonic function is also completely monotonic.

There is a rich literature on (logarithmically) completely monotonic functions. For more recent work, see, for example, [4]-[9] and [11], [14], [15], [16].

Let:

$$i)\quad F(x):=\frac{x^x}{e^x\Gamma(x+1)},\quad x\in(0,\infty); \tag{1}$$

$$ii)\quad f(x):=\frac{e^x\Gamma(x+1)}{x^{x+1/2}},\quad x\in(0,\infty); \tag{2}$$

$$iii)\quad g(x):=\frac{(x+1/2)^{x+1/2}}{e^x\Gamma(x+1)},\quad x\in(-1/2,\infty); \tag{3}$$

$$iv)\quad h(x):=\frac{x^{x+1}}{e^x\Gamma(x+1)\sqrt{x-1}},\quad x\in(1,\infty). \tag{4}$$

In [3], it was proved that the above functions are decreasing and logarithmically convex, from which some equivalence sequences to $n!$ with exact equivalence constants were also deduced in [3].

In this article, we shall further prove that these functions are logarithmically complete monotonic. A supper-additive property of the gamma function is also established.

Theorem 1. *The functions $F(x), f(x), g(x)$ and $h(x)$, defined by* (1), (2), (3) *and* (4) *respecttively, are logarithmically completely monotonic.*

Recall that a function f is said to be supper-additive on $(0,\infty)$ if for all $x,y\in(0,\infty)$, $f(x+y)\geq f(x)+f(y)$.

Theorem 2. *The functions $\ln F(x)$ and $\ln g(x)$ are supper-additive on* $(0,\infty)$.

Theorem 3. *The following functions are completely monotonic.*

$$i)\quad g_1(x):=x+\ln\Gamma(x+1)-(x+1/2)\ln x,\quad x\in(0,\infty);$$

$$\begin{aligned} ii)\quad g_2(x) :=& (x+1/2)\ln(x+1/2) - x - \ln\Gamma(x+1) \\ &+ \ln\sqrt{2\pi} - 1/2, \quad x \in (-1/2,\infty); \\ iii)\quad g_3(x) :=& (x+1)\ln x - x - \ln\sqrt{x-1} - \ln\Gamma(x+1) \\ &+ \ln\sqrt{2\pi}, \quad x \in (1,\infty). \end{aligned}$$

2. Some Lemmas

We need the following lemmas to prove our results.

Lemma 1. (see [2, p. 884]) *For $n \in \mathbb{N}, x > 0$:*

$$i)\quad \psi(x) = \ln x - \frac{1}{2x} - \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt}dt.$$

$$ii)\quad \psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}dt.$$

$$iii)\quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

$$iv)\quad \frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-xt}dt.$$

Lemma 2. (see [13, Chapter 1]) *As $x \to \infty$:*

$$i)\quad \ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + O\left(\frac{1}{x}\right). \tag{5}$$

$$ii)\quad \psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{6}$$

Lemma 3. (see [12]) *Suppose that $f > 0$ is completely monotonic on $(0,\infty)$. If $\lim_{x\to 0+} f(x) \le 1$, then $\ln f$ is supperadditive on $(0,\infty)$.*

3. Proofs of Main Results

Proof of Theorem 1. i) It is clear that

$$\ln F(x) = x\ln x - x - \ln\Gamma(x+1), \text{ and} \tag{7}$$

$$[\ln F(x)]' = \ln x - \psi(x+1). \tag{8}$$

By Lemma 1,

$$[\ln F(x)]'' = \frac{1}{x} - \psi'(x+1) = \int_0^\infty \frac{e^t - t - 1}{e^t - 1}e^{-xt}dt.$$

It is easy to show that $e^t - t - 1 > 0, t \in (0, \infty)$. Hence the function $[\ln F(x)]''$ is completely monotonic and the function $[\ln F(x)]'$ is strictly increasing on $(0, \infty)$. From (8) and (6) of Lemma 2, we get $\lim_{x\to\infty}[\ln F(x)]' = 0$. Therefore $[\ln F(x)]' < 0, x \in (0, \infty)$. We have proved that $F(x)$ is logarithmically completely monotonic on $(0, \infty)$.

$$ii) \quad \ln f(x) = \ln \Gamma(x+1) - (x + 1/2)\ln x + x.$$

$$[\ln f(x)]' = \psi(x+1) - \ln x - \frac{1}{2x} = \psi(x) - \ln x + \frac{1}{2x}.$$

By using Lemma 1, we have

$$[\ln f(x)]' = -\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)e^{-xt}dt = -\int_0^\infty \frac{\delta(t)}{2t(e^t - 1)}e^{-xt}dt,$$

where $\delta(t) := te^t - 2e^t + t + 2$.

Direct computation shows that $\delta(t) > 0, t \in (0, \infty)$. Therefore $-[\ln f(x)]'$ is completely monotonic on $(0, \infty)$. That is, for $k \in \mathbb{N}_0$ and $x > 0$,

$$(-1)^k\big\{-[\ln f(x)]'\big\}^{(k)} = (-1)^{k+1}[\ln f(x)]^{(k+1)} \geq 0\,.$$

So we have proved that $f(x)$ is logarithmically completely monotonic on $(0, \infty)$.

$$iii) \quad \ln g(x) = (x + 1/2)\ln(x + 1/2) - x - \ln \Gamma(x+1), \tag{9}$$

$$[\ln g(x)]' = \ln(x + 1/2) - \psi(x+1), \tag{10}$$

$$[\ln g(x)]'' = \frac{1}{x + 1/2} - \psi'(x+1)\,. \tag{11}$$

By Lemma 1, for $x > -1/2$,

$$[\ln g(x)]'' = \int_0^\infty \frac{e^t - te^{t/2} - 1}{e^t - 1}e^{-(x+1/2)t}dt.$$

Simple computation shows that $e^t - te^{t/2} - 1 > 0, t \in (0, \infty)$. Hence $[\ln g(x)]''$ is completely monotonic and $[\ln g(x)]'$ is strictly increasing on $(-1/2, \infty)$. From (10) and (6) of Lemma 2, we have $\lim_{x\to\infty}[\ln g(x)]' = 0$. Therefore $[\ln g(x)]' < 0, x \in (-1/2, \infty)$. We have proved that $g(x), x \in (-1/2, \infty)$ is logarithmically completely monotonic.

$$iv) \quad \ln h(x) = (x+1)\ln x - \frac{1}{2}\ln(x-1) - x - \ln \Gamma(x+1).$$

By using Lemma 1

$$[\ln h(x)]' = \ln x - \psi(x+1) + \frac{1}{x} - \frac{1}{2(x-1)} \tag{12}$$

$$= \ln x - \psi(x) - \frac{1}{2(x-1)}, \tag{13}$$

and

$$[\ln h(x)]'' = \frac{1}{x} - \psi'(x) + \frac{1}{2(x-1)^2}$$
$$= \int_0^\infty \left(1 - \frac{t}{1-e^{-t}} + \frac{te^t}{2}\right) e^{-xt} dt = \int_0^\infty \frac{p(t)}{2(e^t-1)} e^{-xt} dt,$$

where $p(t) := te^{2t} - 3te^t + 2e^t - 2$. It is easy to show that $p(t) > 0, t \in (0, \infty)$. Hence $[\ln h(x)]''$ is completely monotonic and $[\ln h(x)]'$ is strictly increasing on $(1, \infty)$. Combining (13) and (6) of Lemma 2 yields $\lim_{x\to\infty}[\ln h(x)]' = 0$. Therefore $[\ln h(x)]' < 0, x \in (1, \infty)$. The proof is complete. □

Proof of Theorem 2. It is clear that $\lim_{x\to 0+} F(x) = 1$ and $\lim_{x\to 0+} g(x) = 1/\sqrt{2}$. Since logarithmically complete monotonicity implies complete monotonicity, from Theorem 1 and Lemma 3, we see that Theorem 2 is true. □

Proof of Theorem 3. Since $g_1(x) = \ln f(x), g_2(x) = \ln g(x) + \ln\sqrt{2\pi} - 1/2$ and $g_3(x) = \ln h(x) + \ln\sqrt{2\pi}$, from Theorem 1, to prove our results, we only need to show that $\lim_{x\to\infty} g_1(x) \geq 0, \lim_{x\to\infty} g_2(x) \geq 0$ and $\lim_{x\to\infty} g_3(x) \geq 0$.

By (5) of Lemma 2, as $x \to \infty$,

$$g_1(x) = (x + \frac{1}{2})\ln(1 + \frac{1}{x}) - 1 + \ln\sqrt{2\pi} + O(\frac{1}{x+1}),$$
$$g_2(x) = (x + \frac{1}{2})\ln\frac{x+1/2}{x+1} + \frac{1}{2} + O(\frac{1}{x+1}), \text{ and}$$
$$g_3(x) = (x+1)\ln\frac{x}{x+1} + \frac{1}{2}\ln\frac{x+1}{x-1} + 1 + O(\frac{1}{x+1}).$$

Hence $\lim_{x\to\infty} g_1(x) = \ln\sqrt{2\pi}, \lim_{x\to\infty} g_2(x) = 0$ and $\lim_{x\to\infty} g_3(x) = 0$. □

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