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INTEGRATION QUESTION RELATED TO B-VALUED GENERALIZED FUNCTIONALS

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Abstract: In this paper, we discuss the integration of an abstract function valued in B-valued generalized functional space. We obtain the condition of Bochner integrable about this function.

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1. Introduction

Vector-valued functionals of white noise, as was pointed out by Obata [3], play an important role in application of white noise analysis to many research fields. Hence it is of interest to make studies on vector-valued generalized functionals of white noise, which make up a considerable part of vector-valued generalized functionals of white noise.

Wang and Huang [5] characterized Banach space-valued generalized functionals of white noise by using their moments. Wang [4] gave chaotic decompositions of Banach space-valued generalized functionals of white noise. Chen [1] give a necessary and sufficient condition of strong convergent for a sequence of Banach space-valued generalized functionals of white noise, and discussed the continuity of abstract functions valued in Banach space-valued generalized functional space. In the present paper, we discuss the integration of the abstract function valued in B-valued generalized functional space.

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2. The Framework of White Noise Analysis

In this section we outline the framework of white noise analysis where we work. Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields, respectively.

Let H be a real separable Hilbert space with $|\cdot|_0$ and inner product $\langle\cdot,\cdot\rangle$. Let A be a positive self-adjoint operator on H such that there exists an orthonormal basis $\{e_j\}_{j\geq 1}$ for H satisfying the following conditions:

(1) $Ae_j = \lambda_j e_j$, $j = 1, 2, \cdots$;

(2) $1 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$;

(3) $\sum_{j=1}^{\infty} \lambda_j^{-\alpha} < +\infty$ for some positive constant α.

For each $p \in \mathbb{R}$, define $|\cdot|_p \equiv |A^p \cdot|_0$ and let E_p be the completion of $\mathrm{Dom}A^p$ with respect to $|\cdot|_p$. Then E_p is a real Hilbert space for each $p \in \mathbb{R}$ and moreover E_p and E_{-p} can be viewed as each other's topological dual.

Let E be the projective limit of $\{E_p|p \geq 0\}$ and E^* the inductive limit of $\{E_{-p}|p \geq 0\}$. Then E and E^* can be regarded as each other's topological dual and moreover

$$E \subset H \subset E^* \tag{1}$$

constitues a Gel'fand triple. We denote by $\langle\cdot,\cdot\rangle$ the canonical bilinear form on $E^* \times E$ which is consistent with the inner product of H.

By Minlos Theorem, there exists a Gaussian measure on E^* such that

$$\int_{E^*} e^{i\langle x,\xi\rangle} d\mu(x) = e^{-\frac{1}{2}|\xi|_0^2}, \quad \xi \in E_{\mathbb{C}}. \tag{2}$$

The measure space (E^*, μ) is known as the white noise space, see [2].

Lemma 1. (see [2], Wiener-Itö-Segal) *Let $\Gamma(H_{\mathbb{C}})$ be the symmetric Fock space over $H_{\mathbb{C}}$. Then there exists a isometric isomorphism $\mathcal{I} : \Gamma(H_{\mathbb{C}}) \to (L^2)$ such that*

$$\mathcal{I}(\bigoplus_{n=0}^{\infty} f_n) = \sum_{n=0}^{\infty} \mathcal{I}_n(f_n), \quad \bigoplus_{n=0}^{\infty} f_n \in \Gamma(H_{\mathbb{C}}). \tag{3}$$

Here the series on the righthand side converge in the norm of (L^2).

Now let $\Gamma(A)$ be the second quantization operator of A defined by

$$\Gamma(A)\varphi = \mathcal{I}(\bigoplus_{n=0}^{\infty} A^{\otimes n}\xi_n), \quad \varphi \in \mathrm{D}[\Gamma(A)], \tag{4}$$

where $\varphi = \mathcal{I}(\bigoplus_{n=0}^{\infty} \xi_n) \in \mathrm{D}[\Gamma(A)]$. Then $\Gamma(A)$ is a self-adjoint operator in (L^2) with inverse $\Gamma(A^{-1})$. Similarly, for each $p \in \mathbb{R}$, define $\|\cdot\|_p \equiv \|\Gamma(A)^p \cdot\|_0$, and let (E_p) be the completion of $\mathrm{D}[\Gamma(A)^p]$ with the norm $\|\cdot\|_p$. Then (E_p) is a completion Hilbert space for each $p \in \mathbb{R}$ and (E_p) and (E_{-p}) can be viewed as each other's dual. Let(E) be the projective limit of $\{(E_p)|p \geq 0\}$ and $(E)^*$ be the the inductive limit of $\{(E_{-p})|p \geq 0\}$. Then we have the following inclusion relation

$$(E) \subset (E_q) \subset (E_p) \subset (L^2) \subset (E_{-p}) \subset (E_{-q}) \subset (E)^*,$$

where $0 \leq p \leq q$. Moreover (E) is a nuclear countably Hilbert space and $(E)^*$ can be regarded as the topological dual of (E). Hence we come to a second Gel'fand triple

$$(E) \subset (L^2) \subset (E)^*. \tag{5}$$

Which is known as the framework of white noise analysis over $E \subset H \subset E^*$. Usually, elements of (E) are called testing functionals while elements of $(E)^*$ are referred to as generalized functionals.

The exponential map $\mathcal{E} : E_{\mathbb{C}} \to (E)$ is defined as follows

$$\mathcal{E}(\xi) = e^{\langle \cdot,\xi\rangle - \frac{1}{2}\langle \xi,\xi\rangle}, \quad \xi \in E_{\mathbb{C}}. \tag{6}$$

For $\xi \in E_{\mathbb{C}}$, $\mathcal{E}(\xi)$ is known as the exponential functional associated with ξ. It is known that $\{\mathcal{E}(\xi)|\xi \in E_{\mathbb{C}}\}$ is a total subset of (E).

Henceforth we always assume that X is a complex Banach space with $\|\cdot\|_X$. By an X-valued generalized functional we mean a continuous linear mapping form (E) to X. As usual, we denote by $\mathcal{L}[(E), X]$ the space of X-valued generalized functionals.

For a X-valued generalized functional $T \in \mathcal{L}[(E), X]$, its S-transform $\widehat{T}$ is defined by

$$\widehat{T}(\xi) = T \circ \mathcal{E}(\xi), \quad \xi \in E_{\mathbb{C}}. \tag{7}$$

Here $T \circ \mathcal{E}(\xi)$ is the composition of T and $\mathcal{E}(\xi)$ is the exponential map.

Lemma 2. (see [1]) *Let $T \in \mathcal{L}[(E), X]$ and $p \geq 0$, $a, M > 0$ be such that*

$$\|\widehat{T}(\xi)\|_X \leq M e^{a|\xi|_p^2}, \quad \xi \in E_{\mathbb{C}}. \tag{8}$$

Then, for $q \geq p$ with $2e^2 a\|A^{-(q-p)}\|_{HS}^2 < 1$, the following estimation holds

$$\|T\varphi\|_X \leq M(1 - 2e^2 a\|A^{-(q-p)}\|_{HS}^2)^{-\frac{1}{2}} \|\varphi\|_q, \quad \varphi \in (E). \tag{9}$$

3. Main Results

In the present section we mainly discuss the Bochner integral about the abstract function valued in B-valued generalized functional space.

Definition 1. Let $(\Omega, \mathcal{F}, \lambda)$ be a compact measure space, $T(\cdot) : \Omega \to \mathcal{L}[(E)_{\mathbb{C}}, X]$ is a function in X-valued generalized functional space. Then $T(\cdot)$ is said to be strong Bochner integrable if for each $\varphi \in (E)_{\mathbb{C}}$, the function $T(\cdot)\varphi : \Omega \to X$ is Bochner integrable with respect to λ. In this case, the relation

$$\varphi \to \int_{\Omega} T(\omega)\varphi d\lambda(\omega)$$

is a linear operator from $(E)_{\mathbb{C}}$ to X. This is called strong Bochner integral of $T(\cdot)$ with respect to λ, and is denoted by $(\mathtt{B}) \int_{\Omega} T(\omega) d\lambda(\omega)$.

Remark 1. According to Definition 1, if the function $T(\cdot){:}\Omega{\to}\mathcal{L}[(E)_{\mathbb{C}}, X]$ is Bochner integrable with respect to λ, then we have

$$[(\mathtt{B}) \int_{\Omega} T(\omega) d\lambda(\omega)]\varphi = (\mathtt{B}) \int_{\Omega} T(\omega)\varphi d\lambda(\omega), \quad \varphi \in (E)_{\mathbb{C}}.$$

Theorem 1. *Let $(\Omega, \mathcal{F}, \lambda)$ be a compact measure space, $T(\cdot) : (\Omega, \mathcal{F}, \lambda) \to \mathcal{L}[(E)_{\mathbb{C}}, X]$ be a function in B-valued generalized functional space. Let $T(\cdot)$ satisfy the follwing two conditions:*

(1) $\forall \xi \in E_{\mathbb{C}}$, $\widehat{T}(\cdot)(\xi) : \Omega \to X$ is measurable in a strong sense;

(2) there exist $p \geq 0$, $a > 0$, $M \in L^1(\Omega)$ and $M \geq 0$, such that

$$\|\widehat{T(\omega)}(\xi)\|_X \leq M(\omega) e^{a|\xi|_p^2}$$

for each $\xi \in E_{\mathbb{C}}$ and $a \cdot e$ $\omega \in \Omega$.

Then $T(\cdot)$ is strong Bochner integrable with respect to λ, and the following estimation holds

$$\|[(\mathtt{B}) \int_{\Omega} T(\omega) d\lambda(\omega)]\varphi\|_X \leq \frac{\int M(\omega) d\lambda(\omega)}{(1 - 2e^2 a \|A^{-(q-p)}\|_{HS}^2)^{\frac{1}{2}}} \|\varphi\|_q \,, \qquad \varphi \in (E_q)_{\mathbb{C}}. \quad (10)$$

Here $q \geq p$ and $2e^2 a \|A^{-(q-p)}\|_{HS}^2 < 1$. In particular, we have

$$(\mathtt{B}) \int_{\Omega} T(\omega) d\lambda(\omega) \in \mathcal{L}[(E)_{\mathbb{C}}, X].$$

Proof. We first prove that for each $\varphi \in (E)_{\mathbb{C}}$, the function $T(\cdot)\varphi$ is strong measurable. In fact, for each $\varphi \in \operatorname{span}\{\mathcal{E}(\xi) | \xi \in E_{\mathbb{C}}\}$, $T(\cdot)\varphi$ is strong measurable, and span$\{\mathcal{E}(\xi)|\xi \in E_{\mathbb{C}}\}$ is dense in $(E)_{\mathbb{C}}$, which implies that $T(\cdot)\varphi$ is strong measurable for each $\varphi \in (E)_{\mathbb{C}}$. So the real valued function $\|T(\omega)\varphi\|_X$ is measurable.

Take $q \geq p$ with $2e^2 a\|A^{-(q-p)}\|_{HS}^2 < 1$, then by Lemma 2, we have

$$\|T(\omega)\varphi\|_X \leq M(\omega)(1 - 2e^2 a\|A^{-(q-p)}\|_{HS}^2)^{-\frac{1}{2}}\|\varphi\|_q, \quad \varphi \in (E)_{\mathbb{C}}. \tag{11}$$

On the other hand, by condition, we know $M(\omega)$ is a nonnegative integrable function, so the real valued $\|T(\omega)\varphi\|_X$ is Lebesgue integrable. And $T(\cdot)\varphi$ is strong Bochner with respect to λ.

To prove (10), let $\varphi \in (E_q)_{\mathbb{C}}$, then

$$[(\mathtt{B})\int_\Omega T(\omega)d\lambda(\omega)]\varphi = (\mathtt{B})\int_\Omega T(\omega)\varphi d\lambda(\omega), \quad \varphi \in (E)_{\mathbb{C}}.$$

According to the condition (2), we have

$$\|[(\mathtt{B})\int_\Omega T(\omega)d\lambda(\omega)]\varphi\|_X \leq \frac{\int M(\omega)d\lambda(\omega)}{(1 - 2e^2 a\|A^{-(q-p)}\|_{HS}^2)^{\frac{1}{2}}}\|\varphi\|_q\,, \quad \varphi \in (E_q)_{\mathbb{C}}.$$

This completes the proof. □

Remark 2. Apply the result obtained above, a wick stochastic differential equation

$$\begin{cases} \frac{dT(t)}{dt} = a(t, T(t)) + b(t, T(t)) \diamond N(t)\,, \quad t \in \mathbb{R}_+\,, \\ T(0) = Z \end{cases}$$

in term of B-valued generalized functional can be considered, such as the existence and uniqueness of solutions can be established. The continuity and continuous dependence on initial values of solutions can be proved. Such works will appear in a forthcoming paper.

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