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COMPREHENSIVE CONVERGENCE

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Abstract: An account of some important convergence structures is presented. Among them we discuss set-convergence spaces in the sense of Wyler and preuniform convergence spaces in the sense of Preuß. Both form topological universes, and they seem to be good candidates for an intrinsic study within the realm of convenient topology. By bringing them together we consider as a basic concept uniform filters converging to bounded subsets. Thus, in special cases, we recover the constructs of set-convergence spaces and preuniform convergence spaces and moreover obtain an interesting generalization of Cauchy-spaces, here considered as b-filter spaces. This now enables us to simultaneously express generalized "topological" and "uniform" aspects by common means.

This paper is dedicated to my TOP-father Gerhard Preuß
on the occasion of his sixtyfifth birthday

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1. Introduction

In the past constructs of various "convergence types" were considered in order to discover more "convenient" categories besides the classical ones of topological or uniform spaces. In one direction, the realm of convenient topology, strong

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topological universes were studied, i.e., concrete categories where initial structures exist, fibers are small, and which satisfy a terminal separator property. Consequently, natural function spaces exist in such categories (i.e., they are Cartesian closed), quotients are stable under products, and in addition such categories are extensional.

Moreover, a certain symmetry was proposed, leading to symmetric convergence structures, together with various generalizations of symmetric topological structures, as well as to uniform convergence structures and various generalizations to uniform structures.

Among them the nearness spaces, merotopic spaces and Cauchy spaces seem to be of great interest.

In a second direction, referred to "non-symmetric convenient topology" by Preuß [34], strong topological universes are available, in which non-symmetric convergence structures, such as topological structures and their various generalizations, e.g., limit spaces, pseudotopological spaces as well as set-convergence spaces and also supernearness spaces play an important role. Moreover, uniform convergence structures such as quasiuniformities and various generalizations can be dealt with.

In both cases, all the universes considered can easily be described by means of suitable axioms. Now having the corresponding constructs, some nice properties arising from the classical ones, like compactness or completeness, are described in order to obtain a general "compactification theory", or a "completion theory", respectively. Moreover, in some cases a comprehensive "extension theory" was created in order to describe both processes of compactification and completion in common terms.

On the other hand, if a topological construct fails to have certain convenience properties, e.g., being Cartesian closed or extensional, respectively, it is often possible to embed the given topological construct in a new one with the desired properties. The minimal such extensions will be called the corresponding "hulls". So, by construction, if the topological universe hull of a construct $\mathcal{C}$ exists, it is the smallest topological universe $\mathcal{B}$ in which $\mathcal{C}$ is finally dense.

For example, the topological universe hull of ***TOP*** turns out to be the construct ***PSTOP*** of pseudotopological spaces introduced by Choquet in 1948. The topological universe hull of the construct ***STOP*** of supertopological spaces was determined in 1989 by Wyler to be the construct of "Choquet set-convergence spaces" (cf., [40]).

A categorical approach to the problems mentioned above is the study of closure operators on categories in the sense of Dikranian and Giuli. By employing (generalized) filters, "raster convergence" is investigated, which turns out to behave analogously to filter convergence in a topological space. This leads us to a treatment of separation and compactness from a more general point of view. My dear friend and colleague Joseph Slapal will handle these problems within an associated paper.

By bringing together set-convergence spaces and preuniform convergence spaces in the sense of Preuß, which form a strong topological universe that contains the categories of topological spaces as well as that of uniform spaces, we fill the gap between them by introducing a new category of so-called "b-convergence spaces". As a basic concept we consider uniform filters converging to bounded subsets. Thus, in special cases, we recover the constructs of set-convergence spaces (Choquet set-convergence spaces) and preuniform convergence spaces (semiuniform convergence spaces), respectively. This now enables us to simultaneously express generalized "topological" and "uniform" aspects by common means, but, as pointed out above, with respect to the branches of convenient topology and non-symmetric convenient topology as well.

2. Categorical Concepts

As usual, PX denotes the powerset of a set X, and we use $\mathcal{B}^X$ to denote a collection of *bounded* subsets of X, also known as B-sets. Explicitly, $\mathcal{B}^X \subseteq PX$ satisfies the following axioms:

(B_1) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;

(B_2) $\emptyset \in \mathcal{B}^X$;

(B_3) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.

If $\mathcal{B}^X$ and $\mathcal{B}^Y$ are B-sets on X and Y, respectively, a function $f : X \longrightarrow Y$ is called *bounded*, if it preserves bounded sets.

The category ***BOUND*** with pairs $(X, \mathcal{B}^X)$ consisting of a set X and a corresponding B-set as objects and bounded maps as morphisms is a topological universe, which means it is a Cartesian closed and extensional and hence has universal one-point extensions.

By the way, a concrete category $\mathcal{C}$ is called *topological* iff it satisfies the following conditions:

(CT1) "Existence of Initial Structures". For any set X, any family $(X_i, T_i)_I$ of $\mathcal{C}$-objects indexed by a class I, and any family $(X f_i X_i)_I$ of maps indexed by I, there exists a unique $\mathcal{C}$-structure T on X that is *initial* with respect to $(X, f_i, (X, T_i), I)$, i.e. for any $\mathcal{C}$-object (Y, S) a function $g : Y \longrightarrow X$ is a $\mathcal{C}$-morphism from (Y, S) to (X, T) iff for every $i \in I$ the composite map $f_i \circ g : Y \longrightarrow X_i$ is a $\mathcal{C}$-morphism from (Y, S) to (X_i, T_i).

(CT2) "Fibre-Smallness". For any set X the $\mathcal{C}$-fiber, i.e., the class of all $\mathcal{C}$-structures on X is a set.

(CT3) "Terminal Separator Property". For any set X with cardinality one there exists precisely one $\mathcal{C}$-structure.

Moreover, a topological category (construct) $\mathcal{C}$ is Cartesian closed (i.e., has natural function space structures), provided that for any pair (A, B) of $\mathcal{C}$-objects the set $\boldsymbol{Mor}(A, B)$ of all $\mathcal{C}$-morphisms from A to B can be endowed with the structure of a $\mathcal{C}$-object, denoted by $\boldsymbol{Pow}(A, B)$ and called *power-object* or a *natural function space*, such that the following are satisfied:

(1) The evaluation map $e : A \times \boldsymbol{Pow}(A, B) \longrightarrow B$ defined by $e(a, f) := f(a)$ for each pair $(a, f) \in A \times \boldsymbol{Pow}(A, B)$ is a $\mathcal{C}$-morphism.

(2) For each $\mathcal{C}$-object C and each $\mathcal{C}$-morphism $f : A \times C \longrightarrow B$ the map $f^* : C \longrightarrow \boldsymbol{Pow}(A, B)$ defined by $f^*(c)(a) := f(a, c)$ is a $\mathcal{C}$-morphism.

For a topological category $\mathcal{C}$ with *universal one-point extensions* the last expression means that every $\mathcal{C}$-object A can be embedded via the addition of a single point ∞ into an object $A^* := A \cup \{\infty\}$ such that the following hold:

For every $\mathcal{C}$-morphism $f : U \longrightarrow A$ from a subspace U of a $\mathcal{C}$-object B into A the unique function $f^* : B \longrightarrow A^*$ defined by

$$f^*(b) := \begin{cases} f(b), & \text{if } b \in U; \\ \infty, & \text{if } b \notin U \end{cases}$$

is a $\mathcal{C}$-morphism.

For basic literature concerning the above mentioned definitions, the reader is referred to the book of Preuß [32].

3. Convergence Concepts

Now, more precisely, we recall the definition of a superneighborhood-system (supertopology) on a set X.

Definition 1. A *superneighborhood-system* on a set X is determined by a pair $(\mathcal{M}, \Theta)$, where $\mathcal{M}$ is a set of so-called *bounded* subsets of X, and $\Theta : \mathcal{M} \longrightarrow \boldsymbol{Fil}(X)$ is a function into the set of all filters on X (including the zerofilter PX) such that the following properties are satisfied

(ST1) $\Theta(\emptyset) = PX$;

(ST2) If $A \in \mathcal{M}$ and $U \in \Theta(A)$ then $A \subseteq U$.

A superneighborhood-system $(\mathcal{M}, \Theta)$ is called a *supertopology* on X iff in addition

(ST3) If $A \in \mathcal{M}$ and $U \in \Theta(A)$, then there exists $V \in \Theta(A)$ such that $U \in \Theta(B)$ for each $B \in \mathcal{M}$ for which $B \subseteq V$.

The triple $(X, \mathcal{M}, \Theta)$ is called a *superneighborhood space* (*supertopological space*) and a function $f : X \longrightarrow Y$ between such spaces $(X, \mathcal{M}^X, \Theta_X)$ and $(Y, \mathcal{M}^Y, \Theta_Y)$ is called *continuous*, if it maps bounded sets in X to bounded sets in Y, and if for any $A \in \mathcal{M}^X$

$$V \in \Theta_Y(f[A]) \qquad \text{implies} \qquad f^{-1}[V] \in \Theta_X(A) \; .$$

The category of superneighborhood spaces (supertopological spaces) and continuous maps is denoted by ***SNBD*** and ***STOP***, respectively.

This Definition is not the original one given by Doĭcinov. The condition (ST1) had to be added to insure that constant maps are continuous, or that singletons carry a unique structure. Further it should be noted that Θ need not be order-reversing or antitonic, i.e., for $A_1 \subseteq A_2$ need not imply $\Theta(A_2) \subseteq \Theta(A_1)$ for $A_1, A_2 \in \mathcal{M}$.

Then Doĭcinov embeds ***TOP*** and ***PROX*** into ***STOP*** as full and isomorphism-closed subcategories.

The topological universe hull of the construct ***STOP*** of supertopological spaces was determined by Wyler in 1989 [40]. It is what he called the construct ***ΨSTOP*** of "Choquet set-convergence spaces".

Definition 2. The objects of ***ΨSTOP*** are triples $(X, \mathcal{M}^X, q)$ where, as for supertopological spaces, X is a set and $\mathcal{M}^X$ is a ***B***-set. Instead of having a neighborhood-system for bounded sets, however, we now have a relating q from filters on X to bounded sets satisfying the following conditions:

(ΨST1) $\dot{A}\, q\, A$ for any $A \in \mathcal{M}^X$, where $\dot{A} := \{ B \subseteq X : B \supseteq A \}$;

(ΨST2) If $\mathcal{F}$ is a filter on X, then $\mathcal{F}\, q\, \emptyset$ if and only if $\mathcal{F} = PX$;

(ΨST3) If $A \in \mathcal{M}^X$ and $\mathcal{F}$ is a filter on X such that every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$ satisfies $\mathcal{U}\, q\, A$, then $\mathcal{F}\, q\, A$.

A function $f : X \longrightarrow Y$ between Choquet set-convergence spaces $(X, \mathcal{M}^X, q_X)$ and $(Y, \mathcal{M}^Y, q_Y)$ is said to be *continuous*, if it maps bounded sets to bounded sets, and if for any filter $\mathcal{F}$ on X and any set $A \in \mathcal{M}^X$: $\mathcal{F}\, q_X\, A$

implies $f(\mathcal{F})\, q_Y\, f[A]$.

By introducing "merotopic spaces" and uniformly continuous maps Katětov provided an elegant solution for describing both topological and uniform structures. The basic idea was to present an axiomatization of collections of subsets that contain arbitrarily small members, which were called "micromeric".

Definition 3. A *merotopic* structure on a set X is determined by giving a non-empty set Γ of collections of subsets of X satisfying the following requirements:

(M1) $\emptyset \notin \Gamma$;

(M2) for each $x \in X$ we have $\{\{x\}\} \in \Gamma$;

(M3) if $\mathcal{A} \in \Gamma$ corefines $\mathcal{B} \subseteq PX$ (i.e., if for each $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq A$), then $\mathcal{B} \in \Gamma$;

(M4) if $\mathcal{A}$ and $\mathcal{B}$ are collections such that $\mathcal{A} \cup \mathcal{B} \in \Gamma$, then $\mathcal{A} \in \Gamma$ or $\mathcal{B} \in \Gamma$.

The pair (X, Γ) is called a *merotopic space* and the members of Γ are usually referred to as *micromeric* collections.

A function $f : X \longrightarrow Y$ between merotopic spaces (X, Γ_X) and (Y, Γ_Y) is called *uniformly continuous*, if f preserves micromeric collections, i.e., $\mathcal{A} \in \Gamma_X$ implies $f\mathcal{A} \in \Gamma_Y$.

The category consisting of merotopic spaces and uniformly continuous maps is denoted by ***MER***.

As it turns out, the setting of merotopic spaces is of such generality that every symmetric convergence can be described as a merotopic space. Moreover, the strength of Katětov's theory lies in the fact that different but equivalent formulations of the nearness spaces in the sense of Herrlich are available. Thus uniform structures can be described in such a manner.

The nearness approach is most useful when considering extensions of a space, e.g., completions. The micromeric approach is the one which is most directly applicable to filters. Thus, a concept of symmetrical convergence can be defined in any merotopic space.

It turns out that the corresponding category ***FIL*** of filtermerotopic spaces (filter spaces) is Cartesian closed and that the corresponding function space structure is one of continuous convergence. On the other hand, certain topological extensions are on one-to-one correspondence with so-called grill-determined nearness spaces. The corresponding category ***GRILL*** is (in fact) isomorphic to ***FIL***.

As mentioned, supertopological spaces in the sense of Doĭchinov generalize topological spaces and proximity spaces as well. He shows that there is a one-to-one correspondence between the family of all locally compact extensions of X (defined up to equivalence) and the family of all LC-supertopologies on X which agree with its topology.

Recently, supernear spaces were introduced by the author in order to define a common generalization of nearness spaces and supertopological spaces as well. Now, in this context, it is possible to express the already known results about topological extensions in terms of supernear spaces. Moreover, we obtain an isomorphism between grill-defined presupernear spaces and so-called b-filter spaces. Suitable specialization then results in the above mentioned correspondence between ***GRILL*** and ***FIL***.

Recall that $\mathcal{G} \subseteq PX$ is called a *grill* on the set X (G. Choquet), provided that:

(G1) $\emptyset \notin \mathcal{G}$;

(G2)$G_1 \cup G_2 \in \mathcal{G}$ iff $G_1 \in \mathcal{G}$ or $\mathcal{G}_2 \in \mathcal{G}$.

$\boldsymbol{GRL}(X)$ denotes the set of all grills on X.

Definition 4. In this context a *presupernear space* is a pair $(\mathcal{B}^X, N)$, where $\mathcal{B}^X$ is a B-set on a set X and $N : \mathcal{B}^X \longrightarrow P(P(P(X)))$ is a function satisfying the following conditions:

(SN1) $\mathcal{N}_2 \ll \mathcal{N}_1 \in N(B)$ implies $\mathcal{N}_2 \in N(B)$;

(SN2) $N(\emptyset) = \{\emptyset\}$ and $\mathcal{B}^X \notin N(B)$ for all $B \in \mathcal{B}^X$;

(SN3) $B_2 \subseteq B_1 \in \mathcal{B}^X$ implies $N(B_2) \subseteq N(B_1)$;

(SN4) $x \in X$ implies $\{\{x\}\} \in N(\{x\})$;

A presupernear space $(\mathcal{B}^X, N)$ is called *grill-defined*, if in addition

(G) for each $\mathcal{N} \in N(B)$ there exists a grill $\mathcal{G} \in \boldsymbol{GRL}(X)$ with $\mathcal{N} \subseteq \mathcal{G}$ and $\mathcal{G} \in N(B)$.

Given a pair $(\mathcal{B}^X, N)$, $(\mathcal{B}^Y, S)$ of presupernear spaces, a bounded map $f : \mathcal{B}^X \longrightarrow \mathcal{B}^Y$ is called a *supernear map*, or *sn-map* for short, iff

(SN) $B \in \mathcal{B}^X$ and $\mathcal{N} \in N(B)$ implies $\{\, f[F] : F \in \mathcal{N}\,\} \in S(f[B])$.

PSN denotes the corresponding category of presupernear spaces and sn-maps; its full subcategory ***G-PSN*** is spanned by the grill-defined presupernear spaces.

Now, for each grill-defined presupernear space $(\mathcal{B}^X, N)$ let us define a corresponding *b-Cauchy-relation* $q_N \subseteq \boldsymbol{FIL}(X) \times \mathcal{B}^X$ by setting $\mathcal{F}\, q_N\, B$ iff there exists $\mathcal{N} \in N(B)$ with $sec\mathcal{N} \subseteq \mathcal{F}$, where $\boldsymbol{FIL}(X)$ is the set of all filters on X and $sec\mathcal{N} := \{ T \subseteq X : \forall S \in \mathcal{N}. S \cap T \neq \emptyset \}$. Then q_N satisfies the following conditions

(bc1) $\mathcal{F} \, q_N \, \emptyset$ iff $\mathcal{F} = PX$;

(bc2) $B_2 \subseteq B_1 \in \mathcal{B}^X$ and $\mathcal{F} \, q_N \, B_2$ implies $\mathcal{F} \, q_N \, B_1$;

(bc3) $\mathcal{F}_1 \, q_N \, B$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \in \boldsymbol{FIL}(X)$ implies $\mathcal{F}_2 \, q_N \, B$;

(bc4) $x \in X$ implies $\dot{x} \, q_N \, \{x\}$, where $\dot{x} := \{ T \subseteq X : x \in T \}$.

Conversely, let p be such a b-Cauchy relation on $\mathcal{B}^X$, then for each $B \in \mathcal{B}^X$ we set

$$\boldsymbol{Sp}(B) := \{ \mathcal{S} \in P(P(X)) : \exists \mathcal{F} \in \boldsymbol{FIL}(X). \mathcal{F} \, p \, B \wedge \mathcal{F} \subseteq sec \, \mathcal{S} \} .$$

Consequently, we get a bijection between the set of all grill-defined presupernear operators and the set of all isotone b-Cauchy relations on $\mathcal{B}^X$.

We note that q_N is isotone, which means in particular that q_N satisfies axiom (bc2). If we omit this requirement, we call the resulting objects $(\mathcal{B}^X, q)$ *b-filter spaces*.

A map between b-filter spaces will be referred to as a *c-continuous* map, if it preserves the corresponding filters; we denote the resulting category by ***b-FMER***, or ***b-FIL***, respectively. As a corollary we then find that ***G-PSN*** is isomorphic to a full subcategory of ***b-FMER***.

Theorem 5. *By setting $\mathcal{B}^X := \{\emptyset\} \cup \{ \{x\} : x \in X \}$, each b-Cauchy relation on $\mathcal{B}^X$ leads us to a corresponding "generalized" convergence relation and vice versa, so that the category* ***GCONV*** *of generalized convergence spaces and related maps is isomorphic to a full subcategory of* ***b-FMER****.*

Corollary 6. ***GRILL*** *is isomorphic to a full subcategory of* ***G-PSN*** *and consequently it can also be fully embedded into* ***b-FMER****.*

Proof. We set $\mathcal{B}^X := PX$ and define a prenearness ξ_N on X as follows:

$$\mathcal{N} \in \xi_N \quad \text{iff} \quad \mathcal{N} \in \bigcap \{ N(F) : F \in \mathcal{N} \} .$$

Conversely we set

$$M_\eta(B) := \begin{cases} \{\emptyset\}, & \text{if } B = \emptyset; \\ \{ \mathcal{S} \subseteq PX : \{B\} \cup \mathcal{S} \in \eta \}, & \text{otherwise.} \end{cases}$$

Thus ***PNEAR***, the category of prenearness spaces and nearness-preserving maps is isomorphic to a full subcategory of ***PSN***, and in this context it turns out that ***GRILL*** can be considered as its full subcategory of ***PSN***. □

Corollary 7. ***SRG****, the category of surrounding spaces and continuous* *maps is isomorphic to a full subcategory of* ***b-FMER***.

Remark 8. By the way, a *surrounding space*, or equivalently a *neighborhood space* in the sense of Tozzy and Wyler [39] is a pair $(\mathcal{B}^X, \Theta)$ with B-set $\mathcal{B}^X$ and a function $\Theta : \mathcal{B}^X \longrightarrow \boldsymbol{FIL}(X)$ satisfying the following axioms:

(SR1) $\Theta(\emptyset) = PX$;

(SR2) $x \in X$ implies $x \in \bigcap\{ U \subseteq X : U \in \Theta(\{x\}) \}$;

(SR3) $B_2 \subseteq B_1 \in \mathcal{B}^X$ implies $\Theta(B_1) \subseteq \Theta(B_2)$.

For each $B \in \mathcal{B}^X$ the set $\Theta(B)$ denotes the *surrounding-system* of B with respect to Θ. Note that in addition to the axioms of a superneighborhood system the function Θ is antitonic (see especially axiom (SR3)!). In the case $\mathcal{B}^X = \{\emptyset\} \cup \{ \{x\} : x \in X \}$, Θ defines a *pretopology* on X related to the corresponding Hausdorff-axioms.

Continuous functions between surrounding spaces are defined in the obvious way.

By setting $\mathcal{F}\, q_\Theta\, B$ iff $\Theta(B) \subseteq \mathcal{F}$ for each $B \in \mathcal{B}^X$, we obtain a related isotone b-Cauchy relation on $\mathcal{B}^X$ that is *surrounded* in the following sense:

(SR) $B \in \mathcal{B}^X$ implies $\bigcap\{ \mathcal{F} \in \boldsymbol{FIL}(X) : \mathcal{F}\, q_\Theta\, B \}\, q_\Theta\, B$.

Conversely, given such a b-Cauchy relation p on $\mathcal{B}^X$, we define a surrounding system on $\mathcal{B}$ by setting

$$Q_p(B) := \bigcap\{ \mathcal{F} \in \boldsymbol{FIL}(X) : \mathcal{F}\, p\, B \}$$

for each $B \in \mathcal{B}^X$.

Consequently, this establishes the corresponding isomorphism between the category ***SRG*** and a full subcategory of ***b-FMER***.

Remark 9. As noted earlier, the construct ***ΨSTOP*** of Choquet setconvergence spaces in the sense of Wyler is the topological universe hull of the construct ***STOP***.

In [40] Wyler also introduced the so-called set-convergence spaces, i.e., triples $(X, \mathcal{M}^X, q)$ with a relation q from filters on X to bounded subsets satisfying the axioms (ΨST1), (ΨST2) and in addition

(ΨST3) If $A \in \mathcal{M}^X$ and $\mathcal{F}_1\, q\, A$ with $\mathcal{F}_1 \subseteq \mathcal{F}_2 \in \boldsymbol{FIL}(X)$, then $\mathcal{F}_2\, q\, A$.

Thus, the corresponding category ***SETCONV*** not only contains ***ΨSTOP*** as a full subcategory, but also ***SNBD*** and ***GCONV***.

Remark 10. If we call a b-Cauchy relation q on a B-set $\mathcal{B}^X$ *set-defined* iff

(S) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ implies $\dot{B}\, q\, B$,

the corresponding set-defined b-filter spaces coincide with the set-convergence spaces. ***SETCONV*** therefore can be considered as as full and isomorphism-closed subcategory of ***b-FMER***.

Consequently, we propose ***b-FMER*** as a suitable candidate for a common study of convergence of “topological types”.

As already mentioned in the introduction, the category ***PUCONV*** of pre-uniform convergence spaces in the sense of Preuß in particular contains both the categories ***GCONV*** and ***UNIF***, the category of uniform spaces, as nicely embedded full subcategories. Consequently, topological and uniform aspects can be handled within this category simultaneously. Moreover, ***PUCONV*** is a strong topological universe having the nice properties of being extensional and Cartesian closed; moreover, the product of quotients in ***PUCONV*** is again a quotient. Hence this category seems to be a candidate for a fundamental framework in the realm of “convenient topology”.

Figure 1 below displays the relationships between all categories mentioned so far. Now the question naturally arises, how ***SETCONV*** or ***b-FMER***, respectively, and ***PUCONV*** are connected.

4. b-Convergence Spaces

(i) If one has a set-convergence space $(X, \mathcal{M}^X, q)$, then one can also consider a function τ_q from $\mathcal{M}^X$ into $P(\boldsymbol{FIL}(X \times X))$ by defining for each $A \in \mathcal{M}$

$$\tau_q(A) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) : \exists \mathcal{F} \in \boldsymbol{FIL}(X).\, \mathcal{F}\, q\, A \wedge \dot{A} \times \mathcal{F} \subseteq \mathcal{U}\} .$$

(ii) In the special case of a neighborhood-system $(\mathcal{M}^X, \Theta)$ on X, we analogously set for each $A \in \mathcal{M}^X$

$$\tau_\Theta(A) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X)) : \dot{A} \times \Theta(A) \subseteq \mathcal{U}\} .$$

(iii) More generally, if we consider a b-filter space $(\mathcal{B}^X, p)$, then for each $B \in \mathcal{B}^X$ we put

$$\tau_p(B) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X)) : \exists \mathcal{F} \in \boldsymbol{FIL}(X).\, \mathcal{F}\, p\, B \wedge \mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}\} .$$

(iv) At last consider an equiconvergence space (X, μ) in the sense of [13], i.e., for a set X, $\overset{\mu}{:} X \longrightarrow P(\boldsymbol{FIL}(X \times X))$ is a function satisfying the following two conditions:

(EC1) $x \in X, \mathcal{U} \in \mu(x)$ and $\mathcal{U} \subseteq \mathcal{V} \in \boldsymbol{FIL}(X \times X)$ imply $\mathcal{V} \in \mu(x)$;

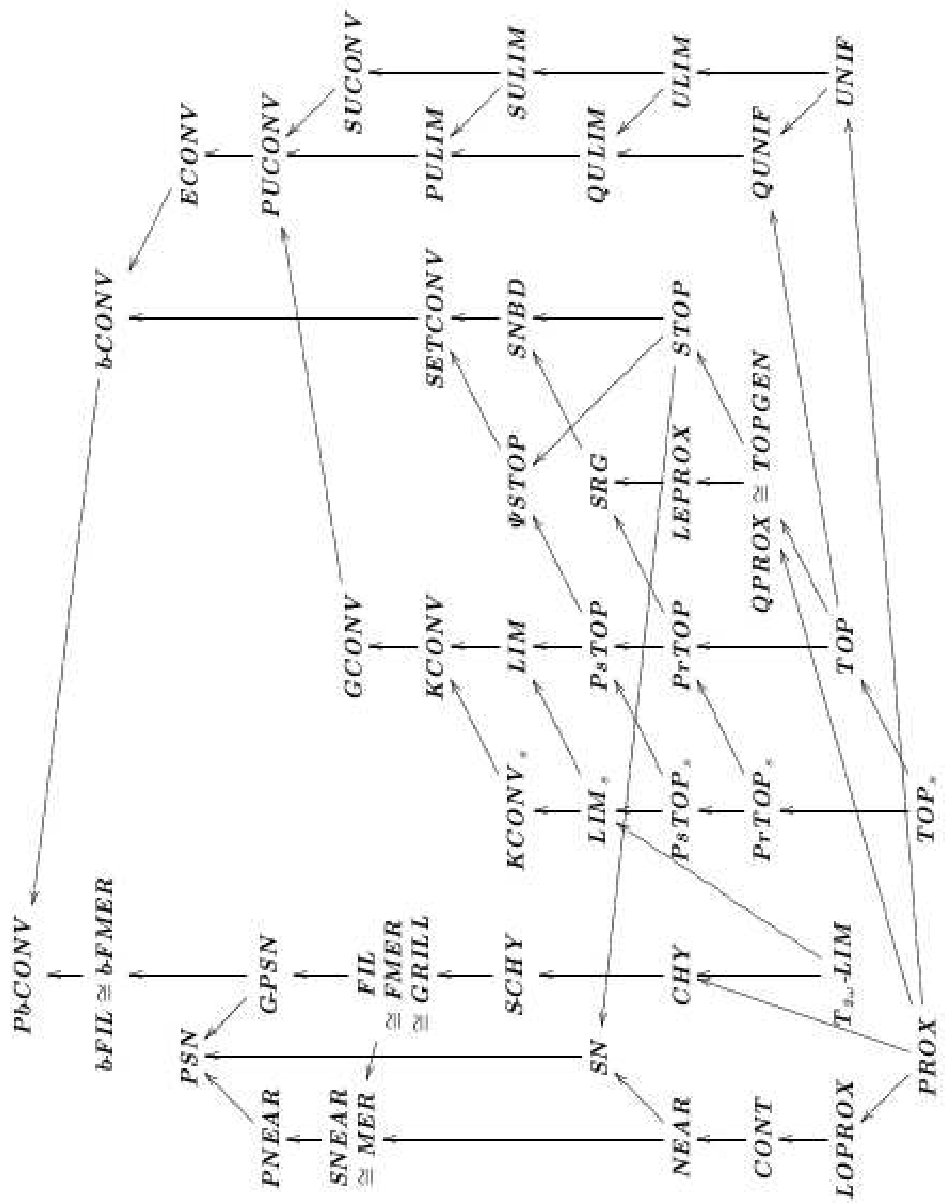

Figure 1: Relationships of the categories mentioned in Section 3

(EC2) $x \in X$ implies $\dot{x} \times \dot{x} \in \mu(x)$.

Equicontinuous maps between equiconvergence spaces (X, μ) and (Y, η) are then defined in the obvious way, i.e., $x \in X$ and $\mathcal{U} \in \mu(x)$ imply $(f \times f)(\mathcal{U}) \in \eta(f(x))$; where

$$(f \times f)(\mathcal{U}) := \{ R \subseteq Y \times Y : (f \times f)^{-1}[R] \in \mathcal{U} \}$$

with

$$(f \times f)^{-1}[R] := \{ (x, x'): \in X \times X : (f(x), f(x')) \in R \}.$$

ECONV denotes the corresponding category. In this context we note that the category ***PUCONV*** of preuniform convergence spaces (and uniformly continuous maps) is isomorphic to a full subcategory that is bireflective in ***ECONV***. Also remember that ***ECONV*** is a topological universe.

Now we put $\mathcal{B}^X := \{\emptyset\} \cup \{ \{x\} : x \in X \}$ and define a function $\tau_\mu : \mathcal{B}^X \longrightarrow P(\boldsymbol{FIL}(X \times X))$ by setting

$$\tau_\mu(\emptyset) := \{P(X \times X)\}$$
$$\tau_\mu(\{x\}) := \mu(x) \quad \text{for each } x \in X$$

In all of the mentioned cases the corresponding function τ satisfies the following axioms:

(b-Con1) $\tau(\emptyset) = \{P(X \times X)\}$

(b-Con2) $B \in \mathcal{B}^X, \mathcal{U} \in \tau(B)$ and $\mathcal{U} \subseteq \mathcal{V} \in \boldsymbol{FIL}(X \times X)$ implies $\mathcal{V} \in \tau(B)$;

(b-Con3) $x \in X \implies \dot{x} \times \dot{x} \in \tau(\{x\})$.

Motivated by these examples, we define a *generalized* concept of convergence.

Definition 11. We call a function τ from a B-set $\mathcal{B}^X$ into the set $P(\boldsymbol{FIL}(X \times X))$ satisfying the axioms (b-Con1)-(b-Con3) a *pre-b-convergence* on $\mathcal{B}^X$, and the pair $(\mathcal{B}^X, \tau)$ a *pre-b-convergence space*.

Given two pre-b-convergence spaces $(\mathcal{B}^X, \tau_X)$ and $(\mathcal{B}^Y, \tau_Y)$, a function $f : X \longrightarrow Y$ is called *b-continuous* iff it is bounded, which means:

(bc1) $\{ f[B] : B \in \mathcal{B}^X \} \subseteq \mathcal{B}^Y$, and additionally we have

(bc2) $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau_X(B)$ implies $(f \times f)(\mathcal{U}) \in \tau_Y(f[B])$; where

$$(f \times f)(\mathcal{U}) := \{ V \subseteq Y \times Y : (f \times f)^{-1}[V] \in \mathcal{U} \}$$

with $(f \times f)^{-1}[V] := \{ (x_1, x_2) \in X \times X : (f(x_1), f(x_2)) \in V \}$.

We denote the corresponding category by ***Pb-CONV***.

Definition 12. Especially, we call a function τ from a B-set $\mathcal{B}^X$ into the set $P(\boldsymbol{FIL}(X \times X))$ a *b-convergence* and the pair $(\mathcal{B}^X, \tau)$ *b-convergence space* iff it satisfies the following axioms:

(bc1) $B \in \mathcal{B}^X$ impies $\dot{B} \times \dot{B} \in \tau(B)$, where $\dot{B} := \{ T \subseteq X : T \supseteq B \}$;

(bc2) $\mathcal{U} \in \tau(\emptyset)$ impies $\mathcal{U} = P(X \times X)$;

(bc3) $\mathcal{U}_1 \in \tau(B)$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2 \in \boldsymbol{FIL}(X \times X)$ implies $\mathcal{U}_2 \in \tau(B)$.

The full subcategory of ***Pb-CONV*** spanned by the b-convergence spaces is denoted by ***b-CONV***.

Theorem 13. *The category* ***SETCONV*** *is isomorphic to a full subcategory of* ***b-CONV***.

Proof. In view of our remarks above, each set-convergence space $(X, \mathcal{M}, q)$ leads us to define a corresponding b-convergence by setting

$$\tau_q(A) := \{ \mathcal{U} \in \boldsymbol{FIL}(X \times X) : \exists \mathcal{F} \in \boldsymbol{FIL}(X).\, \mathcal{F}\, q\, A \wedge \dot{A} \times \mathcal{F} \subseteq \mathcal{U} \} \,.$$

Conversely, given a b-convergence space $(\mathcal{B}^X, \Gamma)$, we define the underlying set-convergence as follows:

$$\mathcal{F}\, p_\Gamma\, B \qquad \text{iff} \qquad \exists \mathcal{U} \in \Gamma(B).\ \dot{B} \times \mathcal{F} \supseteq \mathcal{U} \,.$$

Moreover, we note that the b-convergence above is *set-related* in the sense that it also satisfies

(SR) $A \in \mathcal{M}^X$ and $\mathcal{U} \in \tau_q(A)$ implies $\mathcal{F}\, p_{\tau_q}\, A$ and $\dot{A} \times \mathcal{F} \subseteq \mathcal{U}$ for some $\mathcal{F} \in \boldsymbol{FIL}(X)$.

Thus, ***SETCONV*** is isomorphic to the full subcategory ***SETb-CONV*** of ***b-CONV*** with the set-related b-convergence spaces as objects.

It remains to show that a function $f : X \longrightarrow Y$ is continuous from $(X, \mathcal{M}^X, q_X)$ to $(Y, \mathcal{M}^Y, q_Y)$ iff it is continuous from $(\mathcal{M}^X, \tau_{q_X})$ to $(\mathcal{M}^Y, \tau_{q_Y})$.

$\Longrightarrow$: Consider $\mathcal{U} \in \tau_{q_X}(B)$ and without restriction $B \in \mathcal{M}^X \setminus \{\emptyset\}$. There exists $\mathcal{F} \in \boldsymbol{FIL}(X)$ such that $\mathcal{F}\, q\, B$ and $\dot{B} \times \mathcal{F} \subseteq \mathcal{U}$. By hypothesis we have $f(\mathcal{F})\, q_Y\, f[B]$. In order to show $(f \times f)(\mathcal{U}) \in \tau_{q_Y} f[B]$ we will prove that $f(\dot{B}) \times f(\mathcal{F}) \subseteq (f \times f)(\dot{B} \times \mathcal{F})$. Any element $V \in f(\dot{B}) \times f(\mathcal{F})$ satisfies $V \supseteq f[B] \times f[F]$ for some $F \in \mathcal{F}$. Therefore $(f \times f)^{-1}[V] \supseteq (f \times f)^{-1}[[f[B] \times f[F]] \supseteq B \times F \in \dot{B} \times \mathcal{F}$, and consequently $V \in f(\dot{B} \times \mathcal{F})$, which was to be shown.

$\Longleftarrow$: Conversely, let $\mathcal{F}\, q_X\, B$ for some $B \neq \emptyset$. By hypothesis we have $(f \times f)(\dot{B} \times \mathcal{F}) \in \tau_{q_Y}(f[B])$, hence there exists $\mathcal{F}' \in \boldsymbol{FIL}$ with $\mathcal{F}'\, q_Y\, f[B]$ and $f\dot{[B]} \times \mathcal{F}' \subseteq (f \times f)(\dot{B} \times \mathcal{F})$. It remains to prove $\mathcal{F}' \subseteq f(\mathcal{F})$. But $F' \in \mathcal{F}'$ implies $f[B] \times F' \supseteq (f \times f)([B \times F])$ for some $F \in \mathcal{F}$. Now we claim $f[F] \subseteq F'$. For $y \in f[F]$ select some $x \in F$ with $f(x) = y$. Then $b \in B$ implies $(f \times f)(b, x) = (f(b), f(x)) = (f(b), y) \in f[B] \times F'$, from which we conclude $y \in F'$. $\square$

Definition 14. A b-convergence $P : \mathcal{B}^X\tau \longrightarrow (\boldsymbol{FIL}(X \times X))$ on a B-set $\mathcal{B}^X$ is called *generated*, and the pair $(\mathcal{B}^X, \tau)$ is called a *generated b-convergence space*, iff τ satisfies

(g) $B \in \mathcal{B}^X$ implies $\bigcap\{\,\mathcal{U} \in \boldsymbol{FIL}(X \times X) : \mathcal{U} \in \tau(B)\,\} \in \tau(B)$

Theorem 15. *__SNBD__ is isomorphic to a full subcategory of __b-CONV__.*

Proof. With respect to Example (ii) above, τ_Θ is set-related and also generated. □

Remark 16. In this context superneighborhood spaces can be identified with special generated b-convergence spaces and furthermore "diagonal" filters, or more precisely, "uniform structures", can also be described in such a manner, as we will see again later.

Proposition 17. *As already mentioned (see Example (iv) above), each equiconvergence space (X, μ) gives rise to a corresponding b-convergence space by restricting $\mathcal{B}^X$ to the set $\mathcal{B}^X := \{\emptyset\} \cup \{\,\{x\} : x \in X\,\}$ together with the "naturally" defined b-convergence on it.*

Conversely, each b-convergence space $(\mathcal{B}^X, \tau)$ leads us to such an "underlying" set $\mathcal{D}^X := \{\emptyset\} \cup \{\,\{x\} : x \in X\,\}$, which can be endowed with a b-convergence by restricting τ to $\mathcal{D}^X$.

Definition 18. For a set X we call each b-convergence space of the form $(\mathcal{D}^X, \tau)$ *discrete*.

Lemma 19. *The full subcategory __DISb-CONV__ of __b-CONV__ spanned by the discrete b-convergence spaces is bicoreflective in __b-CONV__.*

Proof. Straightforeward. □

Theorem 20. *The categories __ECONV__ and __DISb-CONV__ are isomorphic.*

Proof. With respect to Proposition 17 we only note that for a given discrete b-convergence space $(\mathcal{D}^X, \tau)$ we can define a corresponding equiconvergence space (X, μ_τ) by setting $\mu_\tau(x) := \tau(\{x\})$. □

Remark 21. As already mentioned in [13], the category ***PUCONV*** is isomorphic to a full subcategory that is bireflective in ***ECONV***.

Here we ony note that each preuniform convergence structure J_X on a set X defines an equiconvergence function μ_{J_X} by setting

$$\mu_{J_X}(x) := J_X \qquad \text{for each } x \in X\,.$$

Conversely, given such a "constant" function η, we put

$$\mathcal{L}_\eta := \bigcup \{\, \eta(x) : x \in X \,\}\,.$$

The above mentioned definition now estabishes the desired isomorphism. In the "uniform" case we note that J_X is "generated" by a uniformity like $\mathcal{U}$. Consequently, μ_{J_K} is "generated" as well, which leads us to consider this notion also for b-convergence spaces.

Theorem 22. *The category* ***b-FIL*** *is isomorphic to a full subcategory of* ***Pb-CONV****.*

Proof. In view of Example (iii) above we note that each b-filter space $(\mathcal{B}^X, p)$ leads us to a corresponding pre-b-convergence space by setting:

$$\tau_p(B) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) : \exists \mathcal{F} \in \boldsymbol{FIL}(X).\ \mathcal{F}\, p\, B \wedge \mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}\,\}$$

for each $B \in \mathcal{B}^X$. Conversely, we set

$$\mathcal{F}\, c_\Gamma\, B \qquad \text{iff} \qquad \mathcal{F} \times \mathcal{F} \in \Gamma(B)$$

for a filter $\mathcal{F} \in \boldsymbol{FIL}(X)$ and a bounded set $B \in \mathcal{B}^X$. The rest is easily verified. Note also that τ_p in particular is *Cauchy-defined*, which means that for each $\mathcal{U} \in \Gamma(B)$ there exists $\mathcal{C} \in \boldsymbol{FIL}(X)$ such that $\mathcal{C}\, c_\Gamma\, B$ and $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}$. □

Remark 23. Now a first goal for obtaining a common concept for studying uniform and topological aspects in a *general* manner seems to be reached.

Remark 24. At last it should be noted that for set-convergences we can define interesting "supplements" within the realm of b-convergence spaces by setting

(1) $\mathcal{F}\, q_\tau\, B$ iff $\mathcal{F} \times \dot{B} \in \tau(B)$;

(2) $\mathcal{F}\, q_\tau\, B$ iff $(\dot{B} \cap \mathcal{F}) \times (\dot{B} \cap \mathcal{F}) \in \tau(B)$.

Only in special cases (by considering in particular b-filter spaces) the convergences defined above coincide.

Turning to pre-b-convergences τ we note that these induce an underlying isotone *Kent* pre-b-convergence naturally defined by setting

$$\tau_{Ke}(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\, \mathcal{V} \in \boldsymbol{FIL}(X \times X) : \exists x \in B\, \exists \mathcal{U} \in \tau(\{x\}) & \\ \qquad\qquad \dot{x} \times \dot{x} \cap \mathcal{U} \subseteq \mathcal{V}\,\}, & \text{if } B \neq \emptyset. \end{cases}$$

The distinguishing property of a *Kent* pre-b-convergence τ is given by:

(K) $x \in X$ and $\mathcal{U} \in \tau(\{x\})$ implies $(\dot{x} \times \dot{x}) \cap \mathcal{U} \in \tau(\{x\})$.

On the other hand, isotone pre-b-convergence spaces also appear – as already seen – in connection with grill-defined presupernear spaces.

Moving from singletons to bounded sets, we note that each b-convergense τ has an underlying *Kent*° b-convergence by defining

$$\tau_{KE}(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\, \mathcal{V} \in \boldsymbol{FIL}(X \times X) : \exists \mathcal{U} \in \tau(B).\, \dot{B} \times \dot{B} \cap \mathcal{U} \subseteq \mathcal{V} \,\}, & \text{if } B \neq \emptyset. \end{cases}$$

In this case a *Kent*° b-convergence τ is characterized by

(KE) $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau(\{B\})$ implies $(\dot{B} \times \dot{B}) \cap \mathcal{U} \in \tau(B)$.

5. Categorical and Other Remarks

Theorem 25. *The (concrete) construct **bCONV** is a topological category.*

Proof. Evidently, ***bCONV*** satisfies axiom (CT2) of being "fiber-small".

(CT3) Note that for any set X with cardinality one there exists precisely one b-convergence on $\mathcal{B}^X = \{\emptyset, X\}$.

(CT1) "Existence of initial structures": For a B-set $\mathcal{B}^X$, any family $(\mathcal{B}^{X_i}, \tau_i)_{i \in I}$ of b-convergence spaces, and any family $(f_i : \mathcal{B}^X \longrightarrow \mathcal{B}^{X_i})_{i \in I}$ of bounded maps there exists a unique b-convergence $\tau_{f_i}^{-1}$ on $\mathcal{B}^X$ that is initial with respect to the given data $(\mathcal{B}^X, f_i, (\mathcal{B}^{X_i}, \tau_i), I)$, i.e., such that for any b-convergence space $(\mathcal{B}^Y, \tau)$ a bounded map $g : Y \longrightarrow X$ is b-continuous from $(\mathcal{B}^Y, \tau)$ to $(\mathcal{B}^X, \tau_{f_i}^{-1})$, if for every $i \in I$ the composite map $f_i \circ g$ is b-continuous from $(\mathcal{B}^Y, \tau)$ to $(\mathcal{B}^{X_i}, \tau_i)$. We define the "initial" b-convergence by setting

$$\tau_{f_i}^{-1}(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) : \forall i \in I. & \\ \qquad (f_i \times f_i)(\mathcal{U}) \in \tau_i(f_i[B]) \,\}, & \text{if } B \neq \emptyset. \end{cases}$$

For simplicity, we only check axiom (b-Con1). For $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $j \in I$ we have to verify $(f_j \times f_j)(\dot{B} \times \dot{B}) \in \tau_j(f_j[B])$. It suffices to show the following inclusion:

$$f_j\dot{[B]} \times f_j\dot{[B]} \subseteq (f_j \times f_j)(\dot{B} \times \dot{B}) \ .$$

But this is clear, when taking into account that B is contained in $f_j^{-1}[f_j[B]]$. □

Theorem 26. *The category* ***SETb-CONV*** *is a bicoreflective subconstruct of* ***b-CONV****.*

Proof. For a b-convergence space $(\mathcal{B}^X, \tau)$ we set

$$\tau_{set}(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\, \mathcal{V} \in \boldsymbol{FIL}(X \times X) : \exists \mathcal{F} \in \boldsymbol{FIL}(X). \\ \qquad \mathcal{F}\, p_\tau\, B \,\wedge\, \dot{B} \times \mathcal{F} \subseteq \mathcal{V}\,\}, & \text{if } B \neq \emptyset. \end{cases}$$

Hence 1_X is b-continuous from $(\mathcal{B}^X, \tau_{set})$ to $(\mathcal{B}^X, \tau)$. The case of $B = \emptyset$ is trivial. For $B \neq \emptyset$ and $\mathcal{U} \in \tau_{set}(B)$ there exists a filter $\mathcal{F}$ on X such that $\mathcal{F}\, p_\tau\, B$ and $\dot{B} \times \mathcal{F} \subseteq \mathcal{V}$. We then have $\dot{B} \times \mathcal{F} \supseteq \mathcal{U}$ for some $\mathcal{U} \in \tau(B)$, which implies $\mathcal{U} \subseteq \mathcal{V}$ and thus $\mathcal{V} \in \tau(B)$.

Let $(\mathcal{B}^Y, \Gamma)$ be a set-related b-convergence space and $f : (\mathcal{B}^Y, \Gamma) \longrightarrow (\mathcal{B}^X, \tau)$ a corresponding b-continuous map. We must verify that $f : Y \longrightarrow X$ is b-continuous from $(\mathcal{B}^Y, \Gamma)$ to $(\mathcal{B}^X, \tau_{set})$ as well. If $\mathcal{V} \in \Gamma(B)$ for $B \neq \emptyset$ we have $\mathcal{F}\, p_\Gamma\, B$ and $\dot{B} \times \mathcal{F} \subseteq \mathcal{V}$ for some $\mathcal{F} \in \boldsymbol{FIL}(Y)$.

By definition of p_Γ there exists $\mathcal{U} \in \Gamma(B)$ such that $\dot{B} \times \mathcal{F} \supseteq \mathcal{U}$, hence by hypothesis$(f \times f)(\mathcal{U}) \in \tau(B)$. We have $f(\dot{B}) \times f(\mathcal{F}) \supseteq (f \times f)(\mathcal{U})$, which implies $f(\mathcal{F})\, p_\tau\, f[B]$. But from $\dot{B} \times \mathcal{F} \subseteq \mathcal{V}$ we get $f(\dot{B}) \times f(\mathcal{F}) \subseteq (f \times f)(\mathcal{U})$, which concludes the proof. □

Corollary 27. ***SETb-CONV*** *is closed under the formation of quotients and coproducts in* ***b-CONV*** *and contains all discrete* ***b-CONV****-objects.* □

Corollary 28. *If a source* $(f_i : (\mathcal{B}^X, \tau_{f_i^{-1}}) \longrightarrow (\mathcal{B}^{X_i}, \tau_i))_{i \in I}$ *is initial in the category* ***b-CONV****, then so is the source* $(f_i : (\mathcal{B}^X, p_{\tau_{f_i^{-1}}}) \longrightarrow (\mathcal{B}^{X_i}, p_{\tau_i}))_{i \in I}$ *in* ***SETb-CONV****.*

Theorem 29. *The category* ***CPb-CONV*** *of Cauchy-defined pre-b-convergence spaces is a bicoreflective subcategory of* ***Pb-CONV****.*

Proof. In view of Theorem 22 ***CPb-CONV*** denotes the category that is isomorphic to ***b-FIL***. Then, for a pre-b-convergence space $(\mathcal{B}^X, \tau)$ we set

$$\tau_{Cau}(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) : \exists \mathcal{C} \in \boldsymbol{FIL}(X). \\ \qquad \mathcal{C}\, c_\tau\, B \,\wedge\, \mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}\,\} & \text{if } B \neq \emptyset. \end{cases}$$

Hence 1_X is b-continuous from $(\mathcal{B}^X, \tau_{Cau})$ to $(\mathcal{B}^X, \tau)$. Without loss of generality consider $B \neq \emptyset$ and $\mathcal{V} \in \tau_{set}(B)$. Then there exists a filter $\mathcal{C}$ on X such that $\mathcal{C}\, c_\tau\, B$ and $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}$. But then $\mathcal{C} \times \mathcal{C} \in \tau(B)$ shows $\mathcal{U} \in \tau(B)$ as desired.

Now let $(\mathcal{B}^Y, \Gamma)$ be a Cauchy-defined pre-b-convergence space and $f : (\mathcal{B}^Y, \Gamma) \longrightarrow (\mathcal{B}^X, \tau)$ be a corresponding b-continuous map. We must verify that $f : Y \longrightarrow X$ is b-continuous from $(\mathcal{B}^Y, \Gamma)$ to $(\mathcal{B}^X, \tau_{Cau})$ as well. The case of $B = \emptyset$ is clear, so consider $\mathcal{U} \in \Gamma(B)$ with $B \neq \emptyset$. Then we have $\mathcal{C}\, c_\Gamma\, B$ and $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}$ for some $\mathcal{C} \in \boldsymbol{FIL}(Y)$, and hence $\mathcal{C} \times \mathcal{C} \in \Gamma(B)$. By hypothesis this implies $(f \times f)(\mathcal{C} \times \mathcal{C}) \in \tau(f[B])$. We set $\mathcal{C}^* := f(\mathcal{C})$, which proves the desired result. □

Corollary 30. *The category* ***CPb-CONV*** *is closed under the formation of quotients and coproducts in* ***Pb-CONV*** *and contains all discrete* ***Pb-CONV****-objects.* □

Corollary 31. *If a source* $(f_i : (\mathcal{B}^X, \tau_{f_i^{-1}}) \longrightarrow (\mathcal{B}^{X_i}, \tau_i))_{i \in I}$ *is initial in the category* ***Pb-CONV****, then so is the source* $(f_i : (\mathcal{B}^X, c_{\tau_{f_i^{-1}}}) \longrightarrow (\mathcal{B}^{X_i}, c_{\tau_i}))_{i \in I}$ *in* ***CPb-CONV****.*

An analogous argument establishes ***CPb-CONV*** as initially complete.

Remark 32. In view of some earlier remarks concerning a general theory of "compactification" or "completion" theory, respectively, we only mention here that the relevant basic notions can be formulated in ***b-CONV***.

Definition 33. For a b-convergence space $(\mathcal{B}^X, \tau)$, a filter $\mathcal{F} \in \boldsymbol{FIL}(X)$ is called τ*-convergent* iff

(c) $\exists B \in \mathcal{B}^X . \dot{B} \times \mathcal{F} \in \tau(B)$;

and a τ*-Cauchy filter* iff

(cf) $\exists B \in \mathcal{B}^X . \mathcal{F} \times \mathcal{F} \in \tau(B)$.

Remark 34. Then we call a b-convergence space $(\mathcal{B}^X, \tau)$

(i) *compact* iff every ultrafilter is τ-convergent, and

(ii) *complete* iff every τ-Cauchy filter is τ-convergent.

Another related property is addressed by the following definition. τ is called *pre-compact* iff every ultrafilter is a τ-Cauchy filter.

Moreover, we call a b-convergence space *symmetric* iff

(s) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U} \in \tau(B)$ implies $\mathcal{U}^{-1} \in \tau(B)$,

where $\mathcal{U}^{-1}$ denotes the uniform filter generated by the set $\{U^{-1} : U \in \mathcal{U}\}$, and *strong* iff

(str) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U}, \mathcal{V} \in \tau(B)$ implies $\mathcal{U} \circ \mathcal{V} \in \tau(B)$

(whenever $\mathcal{U} \circ \mathcal{V}$ exists, i.e., provided $U \circ V := \{ (x,y) : \exists z \in X. (x,z) \in V \wedge (z,y) \in U \} \neq \emptyset$ for every $U \in \mathcal{U}$ and every $V \in \mathcal{V}$), where $\mathcal{U} \circ \mathcal{V}$ is the filter generated by the set $\{ U \circ V : U \in \mathcal{U}, V \in \mathcal{V} \}$.

At last we mention that $(\mathcal{B}^X, \tau)$ is called a *b-limit space* iff

(lim) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U}, \mathcal{V} \in \tau(B)$ imply $\mathcal{U} \cap \mathcal{V} \in \tau(B)$.

Combining corresponding properties in special cases we recover well-known "topological" convergences or "uniform" convergences, respectively.

Discussion 35. To see, whether ***b-CONV*** is a topological universe, we have to check extensionality and Cartesian closedness.

In the second case, for two b-convergence spaces $(\mathcal{B}^X, \tau_X)$ and $(\mathcal{B}^Y, \tau_Y)$ we propose to consider the set

$$[\mathcal{B}^X, \mathcal{B}^Y]_b := \{ f : X \longrightarrow Y : \text{ is b-continuous from } (\mathcal{B}^X, \tau_X) \text{ to } (\mathcal{B}^Y, \tau_Y) \}$$

We define a b-convergence on the corresponding B-set $\mathcal{B}^{X^Y}$ by setting for each $B^* \in \mathcal{B}^{X^Y}$

$$\tau(B^*) := \{ \mathcal{U}^* \in \boldsymbol{FIL}([\mathcal{B}^X, \mathcal{B}^Y]_b \times [\mathcal{B}^X, \mathcal{B}^Y]_b) : \forall B \in \mathcal{B}^X \, \forall \mathcal{U} \in \tau_X(B). \; e(\mathcal{U} \times \mathcal{U}^*) \in \tau_Y(B^*(B)) \},$$

where $e(\mathcal{U} \times \mathcal{U}^*)$ denotes the filter generated by $\{ e[U \times U^*] : U \in \mathcal{U}, U^* \in \mathcal{U}^* \}$ with

$$\begin{aligned} e[U \times U^*] :=& \{ e((x,x'),(f,f')) : (x,x') \in U, (f,f') \in U^* \} \\ =& \{ (f(x), f'(x')) : (x,x') \in U, (f,f') \in U^* \} \end{aligned}$$

and $B^*(B) := \{ f(b) : f \in B^*, b \in B \}$.

In the first case, let $(\mathcal{B}^X, \tau)$ be a b-convergence space. Put

$$X^* := X \cup \{\infty\}$$

with $\infty \notin X$, and, moreover, set

$$\mathcal{B}^* := \mathcal{B}^X \cup \{\{\infty\}\} \ .$$

Now, for each $B^* \in \mathcal{B}^*$

$$\tau^*(B^*) := \begin{cases} \{ P(X^* \times X^*) \}, & \text{if } B^* = \emptyset; \\ \{ \mathcal{R} \in \boldsymbol{FIL}(X^* \times X^*) : \exists B \in \mathcal{B}^X \, \exists \mathcal{U} \in \tau(B). & \\ \qquad \big(\mathcal{U}^* \subseteq \mathcal{R} \vee \{(\infty,\infty)\}^* \in \mathcal{R} \big) \} & \\ \quad \cup \{ \dot{\infty} \times \dot{\infty} \}, & \text{if } B^* \neq \emptyset, \end{cases}$$

where sets of the form $U^* := U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$ constitute the filter $\mathcal{U}^*$.

Further investigations will appear in a separate paper.

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