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COHOMOLOGICAL DIMENSION OF QUASI-AFFINE VARIETIES

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Abstract: Let Y be an affine and smooth n-dimensional algebraic scheme of finite type over a field K and X a non-empty open subset of Y. Here we prove that $cd(X)$ is the first integer t such that $h^i(X, \mathcal{O}_X) = 0$ for all $i > t$.

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Let Y be an affine and smooth n-dimensional algebraic scheme of finite type over a field K and X a non-empty open subset of Y. Recall that the cohomological dimension $cd(X)$ of X is the maximal integer t such that $H^i(X, F) = 0$ for all $i > t$ and all quasi-coherent sheaves F on X and that to test $cd(X)$ it is sufficient to use all coherent sheaves F on X (see [1], p. 405). As in [4] our aim is to use just one sheaf to test the cohomological dimension of X. Here we will give a quick proof of the following result.

Theorem 1. *Let Y be an affine and smooth n-dimensional algebraic scheme of finite type over a field K and X a non-empty open subset of Y. Assume $n > 0$. Then $cd(X) \geq n - 1$ and $cd(X)$ is the first integer t such that $h^i(X, \mathcal{O}_X) = 0$ for all $i > t$.*

Proof. Since X is not complete, $cd(X) \leq n - 1$ by a theorem of Lichtenbaum ([2], Theorem at p. 98). Let $t \geq 0$ be the maximal integer i such that $h^i(X, \mathcal{O}_X) \neq 0$. Fix a coherent sheaf F on X. It is sufficient to check that

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$h^{t+1}(X,F) = 0$ and we may assume that $h^i(X,G) = 0$ for all $i \ge t+2$ and all coherent sheaves G on X.

Claim. *F is spanned by its global sections.*

Proof of Claim. Let $i : X \to Y$ be the inclusion. Since Y is Noetherian, $i_*(F)$ is quasi-coherent (see [3], Proposition II.5.8 (c)). Since Y is affine, the characterization of quasi-coherent sheaves on Y gives that $i_*(F)$ is spanned. Since X is open in Y, $i_*(F)|X = F$. Hence F is spanned. Alternatively, there is an extension G of F to a cohent sheaf D on Y ([3], Example II.5.15) and then apply Theorem A of Serre to the coherent sheaf D to get the claim.

Since Y is noetherian, X is quasi-compact ([3], Remark II.5.14.1). Hence the claim implies that F is generated by finitely many global sections. Hence there is an integer $m > 0$ and an exact sequence on X:

$$0 \to G \to \mathcal{O}_X^{\oplus m} \to F \to 0 \tag{1}$$

with G a coherent sheaf. Since $h^i(X,G) = 0$ for all $i \ge t+2$ and $h^i(X,\mathcal{O}_X) = 0$ for all $i \ge t+1$, (1) gives the thesis. □

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