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DOMINANT IRREDUCIBLE COMPONENTS OF THE HILBERT SCHEMES OF SCROLLS

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Abstract: Fix integers $g \geq 4$ and $r \geq 2$. Let $H'[d, g, r]$ denote the open subset of $\mathrm{Hilb}(\mathbf{P}^{d+r-rg-1})_{red}$ parametrizing the smooth r-dimensional and degree d non-degenerate scrolls over a general smooth genus g curve. Here we prove that if $d \geq r^2g+rg$ the algebraic set $H'[d, g, r]$ has at least r irreducible components.

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For all integers $g \geq 2$, $r \geq 2$ and $d \geq gr + r + 2$ let $H[d, g, r]$ denote the open subset of $\mathrm{Hilb}(\mathbf{P}^{d+r-rg-1})_{red}$ parametrizing the smooth r-dimensional non-degenerate scrolls over a smooth genus g curve and with degree d. Let $\mu_{d,g,r} : H[d, g, r] \to \mathcal{M}_g$ denote the moduli map. For any $C \in \mathcal{M}_g$ set $H[d, C, r] := \mu_{d,g,r}^{-1}(C)_{red}$. Let $H'[d, g, r]$ denote the union of the irreducible components Γ of $H'[d, g, r]$ such that $\mu_{d,g,r}(\Gamma)$ is dense in $\mathcal{M}_g$.

Here we will prove the following result.

Theorem 1. *Fix integers $g \geq 4$, $r \geq 2$, $d \geq r^2g + rg$ and a general $C \in \mathcal{M}_g$. Then:*

(a) $H[d, C, r]$ has at least r irreducible components;

(b) $H'[d, g, r]$ has at least r irreducible components.

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We work over an algebraically closed field $\mathbb{K}$. For any smooth genus g curve X let $M(X;r,d)$ denote the set of all stable vector bundles on X with rank r and degree d. $M(X;r,d)$ is an integral variety of dimension $r^2(g-1)+1$. For all integers $r > k \geq 1$ and any rank r vector bundle E on a smooth curve set $s_k(E) := k \cdot \deg(E) - r \cdot \deg(F)$, where F is any rank k subsheaf of E with maximal degree (see [6]). $s_k(E)$ is called the k-th order of stability of E.

The case $r = 2$ (and hence $f = 1$) of the following example is just [1], Example 5.12.

Example 1. Fix integers $g \geq 4$, $r > f \geq 1$, $d \geq gr + r$, and a general genus g curve C. Set $a := \lfloor g/4 \rfloor$, $e := g - 4a$ and $m := g + 3 - a = 3a + 3 + e$. Hence $0 \leq e \leq 3$ and $\rho(g, 3, m) = e$. Since C is general, $W^3_m(C)$ is non-empty, it has pure dimension e, a dense open subset A of it is formed by very ample line bundles M with $h^0(C, M) = 4$, and it is irreducible if $e > 0$. Notice that $h^1(C, M) = a$ for any $M \in A$. Since $d \geq gr+r$, $d-(r-f)m \geq f(g-1)+2f+2$. Hence $h^1(C, F) = 0$ for a general $F \in M(C; f, d - (r - f)m)$ and there is a non-empty open subset B_f of $M(C; f, d - (r - f)m)$ such that $h^1(C, F) = 0$, $h^0(C, F) = d - (r - f)m + f(1 - g)$ and $\mathcal{O}_{\mathbf{P}(F)}(1)$ is very ample for every $F \in B_f$. Let $D_f[C]$ denote the set of all $(r - f + 2)$-ples $(F, M_1, \ldots, M_{r-f}, V)$, where $F \in B_f$, $M_i \in A$ for all $1 \leq i \leq f$ and V is a $(d - gr + r)$-dimensional linear subspace of $H^0(C, F \oplus \oplus_{i=1}^{r-f} M_i)$. Since $h^0(C, F \otimes M) = d + r - rg + (r - f)a$ and the Grassmannian of all $(d+r-rg)$-dimensional linear subspaces of $\mathbb{K}^{\oplus(d+r-rg+(r-f)a)}$ has dimension $a(d + r - rg)$, D is non-empty and it has pure dimension $z_{d,f} := (r - f)a(d + r - rg) + f^2(g - 1) + 1 + e$. We have $z_f \geq r^2(g-1)+1 \geq r^2(g-1)+1$. Notice that for $z_{d,f} > z_{d,h}$ if $1 \leq f < h < r$ and $d \geq 10r + 10$. if and only if $a(d + r - rg) + e \geq (2r - 1)(g - 1)$. Write D_f for unique closed algebraic subset of $H'[d, g, r]$ containing all $D_f[C]$ for C general in $\mathcal{M}_g$. In characteristic zero $D_f[C]$ is irreducible even if $e = 0$ by an irreducibility theorem due to Eisenbud and Harris (see [2]), but we will not use it.

Remark 1. Fix integers $g \geq 2$, d and $r > k \leq 1$. Let ϵ be the unique integer such that $0 \leq \epsilon \leq r-1$ and $\epsilon + k(r-k)(g-1) \equiv kd \pmod{r}$. Let C be any smooth genus g curve. Fix a general $E \in M(C; r, d)$. Then $s_k(E) = k(r - k)(g - 1) + \epsilon$ (see [4], [8], [7], Remark 3.14).

The vector bundles appearing in the families $D_f[C]$ have the following property.

Proposition 1. *Fix integers $g \geq 4$, $r > f \geq x \geq 1$, $d \geq 2gr + 2r$, and a general genus g curve C. Take the set-up of Remark 1 and fix a general element*

$(F, M_1, \dots, M_{r-f}, V)$ of $D_f[C]$. Set $E := F \oplus \oplus_{i=1}^{r-f} M_i$. Let G be a maximal degree rank x subsheaf of E. Then $G \subseteq F$. If $1 \le k < f$ let ϵ_k be the only integer such that $0 \le \epsilon_k \le f-1$ and $z_k := \epsilon_k + k(f-k)(g-1) \equiv k(d-(r-f)m)$ (mod f). Let y_k be the only integer such that $z_k = f(d-(r-m)f) - ry_k$. Then $s_k(E) = kd - ry_k$ if $1 \le k < f$ and $s_f(E) = fd - r(d-(r-f)m)$.

Proof. The generality of $(F, M_1, \dots, M_{r-f}, V)$ implies that F is a general element of $M(C; f, d-(r-f)m)$. Hence $s_k(F) = z_k$ for all $1 \le k \le f-1$ (Remark 1). Hence the second part is true if we prove that G is contained in F. Set $A := G \cap F$. Since F is saturated in E, it is sufficient to prove that $\mathrm{rank}(A) = x$. Assume $y := \mathrm{rank}(A) < x$. Since $\oplus_{i=1}^{r-f} M_i$ is polystable and with slope m, e have $\deg(G) \le \deg(A) + (x-y)m$. Fix $P \in C$. Notice that $\mu(F) = (d-(r-f)m)/f \ge g+m+1$. Hence $F \otimes M_1^*(-P)$ is spanned. Hence the inclusion $A \to F$ may be extended to a rank x morphism $u : A \otimes M_1(P)^{\oplus(x-y)}$. The rank x subsheaf $u(A \otimes M_1(P)^{\oplus(x-y)})$ of F and hence of E has degree $\deg(A) + (x-y)(m+1)$, contradiction. □

Proof of Theorem 1. Let $\mathcal{H}_{d,C,r}$ denote the set of all pairs $(E, H^0(C,E))$, where E is a rank r vector bundle on C such that $\deg(E) = d$ and $h^1(C,E) = 0$. The semicontinuity theorem for cohomology shows that $\mathcal{H}_{d,C,r}$ is an irreducible component of $H[d,C,r]$. Varying C in $\mathcal{M}_g$ we obtain in this way an irreducible component of $H'[d,g,r]$. Hence to prove part (a) it is sufficient to prove that for all integers $1 \le x < y \le y-1$ and all irreducible components T_x of $D_x[C]$ and T_y of $D_y[C]$ there is no irreducible family T of $H[d,C,r]$ containing both T_x and T_y. Assume the existence of such component T and take a general $(E,V) \in T$. The semicontinuity theorem for cohomology gives $h^1(C,E) \le (r-y)a$. Hence for fixed E the family of possible V's has dimension at most $(r-y)a(d+r-rg)$. The algebraic group $\mathrm{Aut}(E)$ acts on the Grassmannian of all $(d+r-rg)$-dimensional linear subspaces of $H^0(C,E)$. We have $\dim(T) \ge \dim(T_x)0x^2(g-1)+1+e+(r-x)a(d+r-rg)$. Since $a \ge 1$, $x < y$ and $d \ge r^2g + rg$, we give that E should depend from more than $r^2(g-1)+1$ parameters [3], p. 25, [5], and the contradiction comes from a suitable stratification of this moduli stack and the invariance for the stabilizers $\mathrm{Aut}(G)$ of the fibers. Alternatively, take any algebraic space Z inducing a local complete deformation of E. Stratify Z by the dimension of the automorphism group of the associated vector bundle. If Y is any such stratum and z is the corresponding dimension, then $\dim(Y) - z \le r^2(g-1)+1$. Since the automorphism goup does not contribute to $\dim(T)$, we got a contradiction. To prove part (b) it is sufficient to use D_x and D_y instead of $D_x[C]$ and $D_y[C]$. □

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