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FERMAT'S, MERSENNE'S, AND WARING'S PROBLEMS IN OBSERVER'S MATHEMATICS

Dmitriy Khots[1], Boris Khots[2] §
[1]3710 S. 202-nd Avenue Omaha, NE 68130, USA
e-mail: dkhots@cox.net
[2]4725 121-st Street Des Moines, IA 50323, USA
e-mail: bkhots@cccglobal.com

Abstract: This work considers analogies of certain problems in number theory, such as Fermat's Last Problem, Mersenne's and Fermat's numbers problems, and Waring's problem, in a setting of arithmetic provided by Observer's Mathematics. Certain results and commutations pertaining to solutions of these problems are provided.

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1. Introduction

The main goal of this paper is to prove the following four theorems:

Theorem 1. (Analogy of Fermat's Last Problem) *For any integer n, $n \geq 2$, and for any integer m, $m \geq 3$, $m \in W_n$ (see below for the definition of W_n) there exist positive a, b, $c \in W_n$, such that $a^m +_n b^m = c^m$ (operation $+_n$ is defined below).*

Theorem 2. (Analogy of Mersenne's Numbers Problem) *There exist integers n, $k \geq 2$, Mersenne's numbers M_k, with $\{k, M_k\} \in W_n$, and positive $a \in W_n$, such that $M_k = a^2$.*

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§Correspondence author

Theorem 3. (Analogy of Fermat's Numbers Problem) *There exist integers n, $k \geq 2$, Fermat's numbers F_k, $\{k, F_k\} \in W_n$, and positive $a \in W_n$, such that $F_k = a^2$.*

Theorem 4. (Analogy of Waring's Problem) *For any integer k, $k \geq 2$, there exist integer n, $n \geq 2$, $(k \in W_n)$ and some $x \in W_n$ such that any equality of the form $x = a_1^k + a_2^k + \ldots + a_l^k$ is not possible for any integer $l \in W_n$ and any positive numbers $a_1, a_2, \ldots, a_l \in W_n$.*

In addition, we state certain problems that arise as consequences of the introduced theory.

2. Observer's Mathematics

Mathematics of Relativity – Observer's Mathematics was introduced by authors in Khots et al [1], and then was published in Khots et al [2] and Khots et al [3]. This work was created as an attempt to do away with the concept of infinity, in particular, from arithmetic. The arithmetic that we introduced coincides with the standard arithmetic until it meets the "allowed" boundaries of space. It turns out that the "boundary conditions" could cardinally affect the computational process in general. We consider various applications of this arithmetic to algebra, geometry, topology, analysis, and logic as well as to physics, data mining and control systems. In particular, the application of this arithmetic to data mining/modeling allows for more accurate and robust models than the classical methods.

The main arithmetical definitions and statements of Observer's Mathematics are as follows.

Let W_n be a set of all finite decimal fractions of length $2n$, hence, visually W_n can be described as $W_n = \{\underbrace{\star \cdots \star}_{n}.\underbrace{\star \cdots \star}_{n}\}$. For $c = c_0.c_1...c_n$, $d = d_0.d_1...d_n \in W_n$ we introduce addition and subtraction $(\pm_n)$ as follows.

$$c \pm_n d = \begin{cases} c \pm d, \text{ if } c \pm d \in W_n\,, \\ \text{not defined, if } c \pm d \notin W_n \end{cases}$$

(the $\pm$ sign is a standard arithmetic sign). We will write

$$((\ldots(c_1 +_n c_2)\ldots) +_n c_N) = \sum_{i=1}^{N}{}^{n} c_i$$

for $c_1, ..., c_N$ iff the contents of any parenthesis are in W_n.

Next, multiplication is defined for $c = c_0.c_1...c_n$, $d = d_0.d_1...d_n \in W_n$.

$$c \times_n d = \sum_{k=0}^{n} {}^n \sum_{m=0}^{n-k} {}^n 0.\underbrace{0...0}_{k-1} c_k \cdot 0.\underbrace{0...0}_{m-1} d_m \,,$$

where $c, d \geq 0$, $c_0 \cdot d_0 \in W_n$, $0.\underbrace{0...0}_{k-1} c_k \cdot 0.\underbrace{0...0}_{m-1} d_m$ is the standard product, and $k = m = 0$ means that $0.\underbrace{0...0}_{k-1} c_k = c_0$ and $0.\underbrace{0...0}_{m-1} d_m = d_0$. If either $c < 0$ or $d < 0$, then we compute $|c| \times_n |d|$ and define $c \times_n d = \pm |c| \times_n |d|$, where the sign $\pm$ is defined as usual. Note, if the content of at least one parentheses (in previous formula) is not in W_n, then $c \times_n d$ is not defined.

3. Analogy of Fermat's Last Problem

This result was presented by authors at the International Congress of Mathematicians in Madrid in 2006 (see Khots et al [4]).

To begin, we present a few notes. It is obvious that the classical Fermat's Last Problem (for any integer m, $m \geq 3$, there do not exist positive integers a, b, c, such that $a^m + b^m = c^m$) may be reformulated not just for integers a, b, c, but for any real rational numbers a, b, c.

Note, in Observer's Mathematics the power operation is not always associative. For illustrative purposes, we give a W_2 example. Consider $1.49 \in W_2$. Then $1.49 \times_2 1.49 = 2.14$ and $1.49 \times_2 2.14 = 3.16$. On the other hand, $1.49 \times_2 3.16 = 4.67$ and $2.14 \times_2 2.14 = 4.57$, i.e. $((1.49 \times_2 1.49) \times_2 1.49) \times_2 1.49 \neq (1.49 \times_2 1.49) \times_2 (1.49 \times_2 1.49)$.

We now re-state Theorem 1:

For any integer n, $n \geq 2$, and for any integer m, $m \geq 3$, $m \in W_n$ there exist positive a, b, $c \in W_n$, such that $a^m +_n b^m = c^m$. Here x^m means $\underbrace{((\ldots(x \times_n x) \times_n \ldots) \times_n x))}_{m}$.

Proof. Put $a = b = c = 0.\underbrace{0\ldots0}_{k}1$, where $1 \leq k \leq m$, $k \times m \in W_n$, $k \times m > n$ (note, "$\times$" is the multiplication sign in standard arithmetic). Then $a^m = b^m = c^m = 0$, hence $a^m +_n b^m = c^m$. $\square$

Note that the full set of solutions of the equation $a^m +_n b^m = c^m$ is not simple.

For example, if $n = 2$, we can calculate that

$$1^3 +_2 1^3 = 1.28^3\,, \qquad 1^3 +_2 1.21^3 = 1.41^3\,,$$
$$1.2^3 +_2 1.03^3 = 1.41^3\,, \quad 1^{20} +_2 1^{20} = 1.05^{20}\,,$$
$$1^{25} +_2 1^{25} = 1.04^{25}\,, \qquad 1^{50} +_2 1^{50} = 1.02^{50}\,.$$

For $n = 3$, we can calculate that

$$1^{17} +_3 1^{17} = 1.044^{17}\,, \qquad 1^{22} +_3 1^{22} = 1.034^{22}\,,$$
$$1^{50} +_3 1^{50} = 1.016^{50}\,, \qquad 1^{200} +_3 1^{200} = 1.005^{200}\,,$$
$$1^{250} +_3 1^{250} = 1.004^{250}\,, \quad 1^{500} +_3 1^{500} = 1.002^{500}\,.$$

For $n = 4$, we can calculate that

$$1^{2000} +_4 1^{2000} = 1.0005^{2000}\,, \quad 1^{2500} +_4 1^{2500} = 1.0004^{2500}\,,$$
$$1^{5000} +_4 1^{5000} = 1.0002^{5000}\,.$$

For $n = 8$, we can calculate that

$$1.8601023^3 +_8 1.35432561^3 = 2.07390372^3\,,$$
$$1.02345678^3 +_8 1.25160402^3 = 1.44746886^3\,,$$
$$1.13687002^3 +_8 1.57041392^3 = 1.74814264^3\,,$$
$$1.00056781^4 +_8 1.42300976^4 = 1.50297066^4\,,$$
$$1.85643209^4 +_8 1.67843218^4 = 2.10979538^4\,,$$
$$1.85643209^5 +_8 1.55566643^5 = 1.98939654^5\,.$$

For $n = 16$, we can calculate that

$$1.42309901648308913^3 +_{16} 1.57041392556390733^3 = 1.8903509118894252^3\,.$$

Note that the main reason of cardinal difference between standard Mathematics and Observer's Mathematics results is the following. The negative solution of classical Fermat's problem requires Axiom of Choice to be valid. But in Observer's Mathematics this Axiom is invalid (see Khots et al [2]).

4. Analogy of Mersenne's and Fermat's Numbers Problems

Mersenne's numbers are defined as $M_k = 2^k - 1$, with $k = 1, 2, \ldots$. The following question is still open: is every Mersenne's number square-free?

Fermat's numbers are defined as $F_k = 2^{2^k} + 1$, $k = 0, 1, 2, \ldots$. The following question is still open: is every Fermat's number square-free?

We begin with some comments. It is obvious that if some integer number is square-free in the set of all real integers, than this number is square-free in the set of all real rational numbers. We now re-state Theorem 2:

There exist integers n, $k \geq 2$, Mersenne's numbers M_k, with $\{k, M_k\} \in W_n$, and positive $a \in W_n$, such that $M_k = a^2$.

Proof. For $k = 3$, $M_3 = 7$. Then for $n = 2$ the set $\{3, 7\} \subset W_2$ and $2.66 \times_2 2.66 = 7$, i.e., $a = 2.66$. For $k = 2$, $M_2 = 3$. Then for $n = 3$ the set $\{2, 3\} \subset W_3$ and $1.734 \times_3 1.734 = 3$, i.e., $a = 1.734$. For $k = 3$, $M_3 = 7$. Then for $n = 3$ the set $\{3, 7\} \subset W_3$ and $2.648 \times_3 2.648 = 7$, i.e., $a = 2.648$. For $k = 5$, $M_5 = 31$. Then for $n = 3$ the set $\{5, 31\} \subset W_3$ and $5.569 \times_3 5.569 = 31$, i.e., $a = 5.569$. For $k = 8$, $M_8 = 255$. Then for $n = 4$ the set $\{8, 255\} \subset W_4$ and $15.9688 \times_4 15.9688 = 255$, i.e., $a = 15.9688$. □

We now re-state Theorem 3:

There exist integers n, $k \geq 2$, Fermat's numbers F_k, $\{k, F_k\} \in W_n$, and positive $a \in W_n$, such that $F_k = a^2$.

Proof. For $k = 1$, $F_1 = 5$. Then for $n = 2$ the set $\{1, 5\} \subset W_2$ and $2.24 \times_2 2.24 = 5$, i.e., $a = 2.24$. For $k = 0$, $F_1 = 3$. Then for $n = 3$ the set $\{0, 3\} \subset W_3$ and $1.734 \times_2 1.734 = 3$, i.e., $a = 1.734$. For $k = 1$, $F_1 = 5$. Then for $n = 3$ the set $\{1, 5\} \subset W_3$ and $2.237 \times_2 2.237 = 5$, i.e., $a = 2.237$. For $k = 0$, $F_1 = 3$. Then for $n = 5$ the set $\{0, 3\} \subset W_5$ and $1.73209 \times_2 1.73209 = 3$, i.e., $a = 1.73209$. For $k = 3$, $F_1 = 257$. Then for $n = 5$ the set $\{3, 257\} \subset W_5$ and $16.03122 \times_2 16.03122 = 257$, i.e., $a = 16.03122$. □

5. Analogy of Waring's Problem

It is known (Lagrange) that the minimum number of squares to express all positive integers is four. What is the minimum number of k-th powers necessary to express all positive integers? This is a classical Waring's problem, in standard arithmetic.

We now re-state Theorem 4:

For any integer k, $k \geq 2$, there exist integer n, $n \geq 2$, ($k \in W_n$) and some $x \in W_n$ such that any equality of the form $x = a_1^k + a_2^k + \ldots + a_l^k$ is not possible for any integer $l \in W_n$ and any positive numbers $a_1, a_2, \ldots, a_l \in W_n$.

Proof. For given integer $k \geq 2$ take $n = k + 1$. So, $k \in W_n$, and take $x = 0.\underbrace{0\ldots0}_{k}1$. There is no positive $a \in W_n$ such that $a^k = x$, because a^k equals to 0 or is greater than x. Hence, if we have any integer $l \geq 1$ and any positive a_1, a_2, ..., $a_l \in W_n$, we see that $a_1^k + a_2^k + \ldots + a_l^k$ equals to 0 or is greater than x. □

Note that for $n = 2$ and for any $x \in W_2$, $x \in [0, 1]$, there do not exist more than four numbers $a, b, c, d \in W_2$, such that $x = ((a^2 +_2 b^2) +_2 c^2) +_2 d^2$. It is easy to check this statement by direct calculation.

We can now formulate three problems that arise in view of the work above.

Problem 1. For any positive integers n, k, $k \geq 3$, $k \in W_n$, develop an algorithm that finds all solutions to the equation $a^k +_n b^k = c^k$, such that $a, b, c > 0$, $a, b, c \in W_n$ and a^k,b^k,$c^k \in W_n$.

Problem 2. For any positive integer n, find all Mersenne (M_k) and Fermat (F_k) numbers such that k is an integer with $k \geq 3$, $k \in W_n$, and such that $M_k \in W_n$, $F_k \in W_n$ and $M_k = a^2$, $F_k = b^2$ for some $a, b \in W_n$.

Problem 3. For any positive integers n, k, $k \geq 2$, $k \in W_n$, find the set A_{nk} of all $x \in W_n$ such that $x = a_1^k + a_2^k + \ldots + a_l^k$, $l \leq 4$, $l \in W_n$ and positive numbers $a_1, a_2, \ldots, a_l \in W_n$, $a_1^k, a_2^k, \ldots, a_l^k \in W_n$.

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