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PARTIAL DIFFERENTIAL EQUATIONS WHICH ADMIT INTEGRABLE SYSTEMS

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Abstract: The problem of identifying whether a given nonlinear partial differential equation admits a linear integrable system is studied here. It is shown that the fundamental equations of surface theory can be used to reproduce the compatibility conditions obtained from a linear system in matrix form corresponding to a number of different Lie algebras. It is also shown that the system of equations which has been obtained from the linear matrix problem can also be derived from a set of differential forms.

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1. Introduction

A great number of partial differential equations which are of interest to study on account of their presence in various areas of mathematical physics result as the integrability condition of a particular linear system of equations. There is also a relationship between the solutions of certain types of partial differential equations and the calculation of the form of various kinds of surfaces of constant curvature. This includes surfaces which have either constant mean curvature or Gaussian curvature.

It is known for example that there is a connection between surfaces of negative constant Gaussian curvature in Euclidean three space, the sine-Gordon equation and Bäcklund transformations relevant to the equation. The original

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Bäcklund transformation for the sine-Gordon equation is also a simple geometric construction for pseudospherical surfaces [9], [1]. It also acts on the sine-Gordon equation and leaves it invariant. The nonlinear superposition principle associated with the Bäcklund transformation for the sine-Gordon equation has applications in such physical domains as the theory of ultrashort optical pulse propagation, and Josephson junctions. The generalized Weierstrass representation on the other hand allows inducing surfaces in R^3 which have constant mean curvature [2], [3]. The procedure has been generalized to the case of higher dimensional Euclidean and non-Euclidean spaces as well [8], [4].

Here we will study the problem of identifying whether a given nonlinear partial differential equation (PDE) admits a linear integrable system. It will be of interest here to study the extent to which the fundamental equations of surface theory can be used to reproduce the compatibility conditions obtained from a linear system in matrix form. The approach will be more geometrical in nature and will be formulated in terms of differential forms, which can be compared with the matrix approach [6]. The coefficient matrices for the linear systems of interest here will be based on the Lie algebras $so(3)$ and $so(2,1)$ isomorphic to the Lie algebras $su(2)$ and $sl(2,R)$, which are the ones usually referred to. The integrable systems which correspond to these algebras are typically referred to as the $SO(3)$ and $SO(2,1)$ Lax pair, respectively. Frequently, we will have to refer to the unit sphere in a variety of three-dimensional spaces [7]. To this end, let S^2 be the unit sphere in the Euclidean space R^3. The unit spheres in the Minkowski space $R^{2,1}$, which have curvatures -1 and $+1$, respectively, are referred to as H^2 and $S^{1,1}$. The main conclusion is that a nonlinear PDE which admits an $SO(3)$ or $SO(2,1)$ Lax pair must be the Gauss equation of the spheres S^2, H^2 or $S^{1,1}$, or related to it in a special way. In fact, a general approach which will identify whether or not the nonlinear PDE admits an $SO(3)$ or $SO(2,1)$ integrable system will be given.

2. Nonlinear PDEs Admitting $SO(3)$ Lax Pairs

Let us write the general form of a PDE in terms of the unknown function φ in the form

$$F(\varphi, \varphi_x, \varphi_t, \varphi_{xx}, \varphi_{tx}, \varphi_{tt}, \cdots) = 0. \tag{1}$$

Here φ_x, φ_t, φ_{xx}, φ_{xt}, $\varphi_{tt}, \cdots$ are partial derivatives of φ with respect to x and t. Consider first the case in which an $SO(3)$ Lax pair is admitted. If there exist two three by three antisymmetric matrices

$$U = \begin{pmatrix} 0 & C & B \\ -C & 0 & A \\ -B & -A & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & F & E \\ -F & 0 & D \\ -E & -D & 0 \end{pmatrix}, \tag{2}$$

such that the two linear systems

$$\Phi_t = U\Phi, \qquad \Phi_x = V\Phi, \tag{3}$$

are completely integrable when φ satisfies the PDE (1), then the PDE in (1) is said to admit an $SO(3)$ Lax pair (2). Here A, B, C, D, E and F are suitable functions of φ and its derivatives up to a certain order, and the notation for the matrices in (2) is adopted from [6]. The function Φ in (3) can be thought of as a function in either R^3 or in $SO(3)$ made up of vectors consisting of the elements which generate the Darboux frame for the surface. These can consist of two tangent vectors to the surface and the normal as will be shown. In fact, all possible PDEs of the form (1) which admit Lax pairs will be determined.

Introduce three mutually orthogonal unit vectors in R^3 referred to as $\mathbf{p}(t,x)$, $\mathbf{m}(t,x)$ and $\mathbf{n}(t,x)$ such that $\mathbf{p}(t,x)$ is the parametric equation of the unit sphere S^2 and the pair $\mathbf{m}(t,x)$, $\mathbf{n}(t,x)$ denote tangent vectors to this sphere. Then Φ has the vector form

$$\Phi = \begin{pmatrix} \mathbf{p} \\ \mathbf{m} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \tag{4}$$

and the system of equations which results from using (4) in (3) is given by

$$\begin{aligned} \mathbf{p}_t &= C\mathbf{m} + B\mathbf{n}, & \mathbf{p}_x &= F\mathbf{m} + E\mathbf{n}, \\ \mathbf{m}_t &= -C\mathbf{p} + A\mathbf{n}, & \mathbf{m}_x &= -F\mathbf{p} + D\mathbf{n}, \\ \mathbf{n}_t &= -B\mathbf{p} - A\mathbf{m}, & \mathbf{n}_x &= -E\mathbf{p} - D\mathbf{m}. \end{aligned} \tag{5}$$

The integrability condition for (3) in terms of the matrices U and V in (2) is given as

$$U_x - V_t + [U,V] = 0. \tag{6}$$

In terms of the components of the matrices U and V, (6) takes the form

$$\begin{aligned} C_x - F_t + AE - BD &= 0, \\ B_x - E_t + CD - AF &= 0, \\ A_x - D_t + BF - CE &= 0. \end{aligned} \tag{7}$$

One of the objectives here is to see to what extent the equations in (5) and (7) can be obtained from the fundamental equations for surfaces by developing a system of one forms. Considering a surface S with local coordinates (t,x), we have an orthogonal frame $\{\mathbf{r}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ along the surface S, where $\mathbf{e}_3$ is the normal vector. The fundamental equations of S are given by

$$\begin{aligned} d\mathbf{r} &= \omega^a \mathbf{e}_a, \\ d\mathbf{e}_a &= \omega_a^b \mathbf{e}_b + \omega_a^3 \mathbf{e}_3, \quad a, b = 1, 2, \\ d\mathbf{e}_3 &= \omega_3^a \mathbf{e}_a. \end{aligned} \tag{8}$$

Taking the exterior derivative of system (8), the following equations are obtained

$$\begin{aligned} d\omega^a + \omega_b^a \wedge \omega^b &= 0, \\ d\omega_b^a + \omega_c^a \wedge \omega_b^c - \omega_b^3 \wedge \omega_3^a &= 0, \\ d\omega_a^3 - \omega_a^b \wedge \omega_b^3 &= 0. \end{aligned} \tag{9}$$

The first equation in (9) determines the connection and the last two are the Gauss and Codazzi equation, respectively. These basic equations will be used throughout such that only the one forms ω_μ^ν depending on the group under consideration are varied.

Theorem 1. *In Euclidean three space, the ω_j^i are one-forms which satisfy $\omega_j^i = -\omega_i^j$ and the fundamental equations of S generate (5) provided that the one-forms in (8) are given by*

$$\omega^1 = C\,dt + F\,dx, \qquad \omega^2 = B\,dt + E\,dx, \qquad \omega_1^2 = A\,dt + D\,dx. \tag{10}$$

Proof. Beginning with the vector $\mathbf{p} = \mathbf{e}_3$ and using $\omega^a = \omega_3^a$ for this space,

$$\begin{aligned} d\mathbf{p} = \mathbf{p}_t\,dt + \mathbf{p}_x\,dx = \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2 &= (C\,dt + F\,dx)\mathbf{e}_1 + (B\,dt + E\,dx)\mathbf{e}_2 \\ &= (C\mathbf{m} + B\mathbf{n})\,dt + (F\mathbf{m} + E\mathbf{n})\,dx. \end{aligned}$$

Equating the components of the differentials, we obtain the first two components of (5). Similarly, calculating $d\mathbf{e}_1$, we obtain

$$d\mathbf{e}_1 = \mathbf{m}_t\,dt + \mathbf{m}_x\,dx = (A\mathbf{n} - C\mathbf{p})\,dt + (D\mathbf{n} - F\mathbf{p})\,dx,$$

and

$$d\mathbf{e}_2 = \mathbf{n}_t\,dt + \mathbf{n}_x\,dx = -(A\mathbf{m} + B\mathbf{p})\,dt - (D\mathbf{m} + E\mathbf{p})\,dx.$$

These give the final four equations of (5). Of course, ω^1 and ω^2 are independent.

The first equation of (9) reproduces two of the compatibility conditions since

$$\begin{aligned} d\omega^1 &= (C_x - F_t)\,dx \wedge dt, \\ \omega_2^1 \wedge \omega^2 &= (-A\,dt - D\,dx) \wedge (B\,dt + E\,dx) = (AE - BD)\,dx \wedge dt. \end{aligned}$$

Substituting these into the first equation in (9) gives the first equation of (7).

Similarly,

$$d\omega^2 = (B_x - E_t)\, dx \wedge dt,$$
$$\omega_1^2 \wedge \omega^1 = (A\, dt + D\, dx) \wedge (C\, dt + F\, dx) = (CD - AF)\, dx \wedge dt,$$

and these clearly give the second equation in (7). The second equation in (9) determines the third equation of (7) since $d\omega_1^2 = (A_x - D_t)\, dx \wedge dt$ and $\omega_1^3 \wedge \omega_3^2 = -(BF - EC)\, dx \wedge dt$. This finishes the proof of (7) formulated in terms of differential forms. □

To make a link with the curvature, the Gauss equation for a Euclidean space-like surface can be written as

$$d\omega_2^1 = R_{1212}\, \omega^1 \wedge \omega^2 = K\, \omega^1 \wedge \omega^2.$$

Substituting the forms obtained here and taking $K = 1$, we obtain $d\omega_2^1 = (A_x - D_t)\, dt \wedge dx$ and $\omega^1 \wedge \omega^2 = (-BF + CE)\, dt \wedge dx$ which yields exactly the third equation in (7).

Solving the first two equations in (7) for the quantities A and D, respectively, we obtain

$$A = \frac{1}{CE - BF}[(E_t - B_x)B + (F_t - C_x)C],$$
$$D = \frac{1}{CE - BF}[(E_t - B_x)E + (F_t - C_x)F]. \qquad (11)$$

Substituting A and D from (11) into the last equation of (7) gives the following second order result

$$\left(\frac{E_t - B_x}{EC - BF}B + \frac{F_t - C_x}{EC - BF}C\right)_x - \left(\frac{E_t - B_x}{EC - BF}E + \frac{F_t - C_x}{EC - BF}F\right)_t + BF - CE = 0. \qquad (12)$$

This is the Gauss equation of the sphere S^2. This shows that the nonlinear PDE (12), which is the Gauss equation of the sphere S^2, admits an $SO(3)$ Lax pair.

It is convenient to refer to a PDE $H(\varphi, \varphi_t, \varphi_x, \cdots) = 0$ as a subequation of another equation $G(\varphi, \varphi_t, \varphi_x, \cdots) = 0$ if every solution of $H = 0$ also satisfies $G = 0$. Clearly, if $H = 0$ admits a Lax pair, then $H = 0$ must be a subequation of each equation of (6). Conversely, if for given B, C, E, F with $CE - BF \neq 0$, $H = 0$ is a subequation of (12), then $H = 0$ admits a Lax pair in which A, D are defined by (11). In this sense, all possible PDEs admitting $SO(3)$ Lax pairs with $CE - BF \neq 0$ have been determined.

Defining the matrix

$$M = \begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix}, \tag{13}$$

then if rank $(M) = 2$, we can assume that $EC - BF \neq 0$. When rank $(M) = 1$, the second row must be a multiple of the first. In this case, we have

$$D = \sigma A, \qquad E = \sigma B, \qquad F = \sigma C. \tag{14}$$

Substituting (14) into the compatibility conditions (7), the following conservation laws result

$$A_x - (\sigma A)_t = 0, \quad B_x - (\sigma B)_t = 0, \quad C_x - (\sigma C)_t = 0.$$

Since the integrability condition (6) consists of only one equation, we can suppose that (1) is the first equation of (7), namely $A_x - (\sigma A)_t = 0$. This is the integrability condition of the system

$$\psi_t = A\psi, \qquad \psi_x = \sigma A\psi. \tag{15}$$

In (15), ψ is a real function and (15) is a $U(1)$ Lax pair. Let us summarize these results in the form of a theorem.

Theorem 2. *All nonlinear PDEs admitting $SO(3)$ integrable systems can be obtained in the following ways:*

(i) When rank (M) =2, the nonlinear equation is the Gauss equation of $S^2 \subset R^3$ or its subequation and B, C, E, F in (12) are any given functions of φ and the derivatives of φ and the derivatives of φ up to a certain order.

(ii) When rank (M)=1, the nonlinear equation can be chosen to be the equation of a conservation law $M_t + N_x = 0$, where $N \neq 0$.

If B, C, E, F are given functions of φ and their derivatives up to a certain order such that $BE - CF \neq 0$, then this theorem gives a straightforward way of building all nonlinear PDEs which admit $SO(3)$ Lax pairs. Substituting this set of functions into (12), the corresponding nonlinear equation (1) is obtained.

As an illustration of how (12) is applied, the negative sine-Gordon equation can be easily obtained by selecting $B = F = 0$, $C = \cos(\varphi/2)$ and $E = \sin(\varphi/2)$ and substituting into (12). Moreover, taking $C = \varphi$, $E = \varphi^2$, $B = F = 0$, upon substituting these into (12) and simplifying, we obtain the PDE given as follows

$$(2 + \varphi^2)\varphi_t + \frac{\varphi_{xxt}}{\varphi^2} - 2\frac{\varphi_x \varphi_{xt}}{\varphi^3} = 0.$$

3. Nonlinear PDEs Admitting $SO(2,1)$ Lax Pairs

Consider the nonlinear PDEs of the form (1) which now admit the $SO(2,1)$ Lax pairs with the identical structure as (3) but with U and V taking values in the Lie algebra $so(2,1)$. The case in which the integrability condition for (3) is the Gauss equation for $S^{1,1} \subset R^{2,1}$ will be examined first. The set of matrices analogous to the set (2) is given by

$$U = \begin{pmatrix} 0 & C & B \\ -C & 0 & A \\ B & A & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & F & E \\ -F & 0 & D \\ E & D & 0 \end{pmatrix}. \tag{16}$$

These of course differ from the matrices in (2). It suffices to take rank $(M) = 2$ so that we have $BF - CE \neq 0$. In the case that rank $(M) = 1$, the same results obtained in the previous section will apply.

In this case, let $\{\mathbf{p}, \mathbf{m}, \mathbf{n}\}$ be orthonormal frames in $R^{2,1}$ with $\mathbf{p}^2 = \mathbf{m}^2 = -\mathbf{n}^2 = 1$, then $\mathbf{p}(t,x)$ can be interpreted as a parametric representation of $S^{1,2}$, since its equation is $\mathbf{p}^2 = 1$ and the pair $\mathbf{m}(t,x)$, $\mathbf{n}(t,x)$ are tangent vectors to $S^{1,1}$. From (16), the equations for the orthonormal frame are

$$\begin{aligned} \mathbf{p}_t &= C\mathbf{m} + B\mathbf{n}, & \mathbf{p}_x &= F\mathbf{m} + E\mathbf{n}, \\ \mathbf{m}_t &= -C\mathbf{p} + A\mathbf{n}, & \mathbf{m}_x &= -F\mathbf{p} + D\mathbf{n}, \\ \mathbf{n}_t &= B\mathbf{p} + A\mathbf{m}, & \mathbf{n}_x &= E\mathbf{p} + D\mathbf{m}. \end{aligned} \tag{17}$$

The compatibility condition for system (3) generates the following three equations,

$$\begin{aligned} &C_x - F_t + BD - AE = 0, \\ &B_x - E_t + CD - AF = 0, \\ &A_x - D_t + BF - CE = 0. \end{aligned} \tag{18}$$

Solving for A and D from the first two equations in (18), we find that

$$\begin{aligned} A &= \frac{1}{BF - CE}(B(B_x - E_t) + C(F_t - C_x)), \\ D &= \frac{1}{BF - CE}(F(F_t - C_x) + E(B_x - E_t)). \end{aligned} \tag{19}$$

Substituting (19) into the third equation of (18), it follows that,

$$\left(\frac{B(E_t - B_x) + C(C_x - F_t)}{CE - BF}\right)_x - \left(\frac{F(C_x - F_t) + E(E_t - B_x)}{CE - BF}\right)_t + BF - CE = 0. \tag{20}$$

Equation (20) is the Gauss equation for $S^{1,1}$.

Theorem 3. *In the space $R^{2,1}$, the ω_j^i in the fundamental equations of surface theory generate (17) and (18) provided that these one-forms are given by $\omega^1 = C\,dt + F\,dx$, $\omega^2 = B\,dt + E\,dx$, $\omega_1^2 = A\,dt + D\,dx = \omega_2^1$ with $\omega_1^3 = -C\,dt - F\,dx$, and $\omega_2^3 = B\,dt + E\,dx$, such that $\omega_j^i = \omega_i^j$.*

Proof. A set of one forms such that equations (17) and (18) are determined in terms of the components of these one-forms. Setting $\omega^1 = a\,dt + b\,dx$ and $\omega^2 = g\,dt + l\,dx$, it follows that $d\mathbf{e}_3 = (a\mathbf{e}_1\,dt + g\mathbf{e}_2)\,dt + (b\mathbf{e}_1 + l\mathbf{e}_2)\,dx$. This generates (17) if we take

$$\omega^1 = C\,dt + F\,dx, \quad \omega^2 = B\,dt + E\,dx. \tag{21}$$

Similarly, writing $\omega_1^2 = a_1\,dt + b_1\,dx = \omega_2^1$ and $\omega_1^3 = g_1\,dt + l_1\,dx$, we calculate

$$d\mathbf{e}_1 = \omega_1^b\mathbf{e}_b + \omega_1^3\mathbf{e}_3 = (a_1\mathbf{e}_2 + g_1\mathbf{e}_3)\,dt + (b_1\mathbf{e}_2 + l_1\mathbf{e}_3)\,dx.$$

It follows that the components must be selected as

$$\omega_1^2 = A\,dt + D\,dx = \omega_2^1, \qquad \omega_1^3 = -C\,dt - F\,dx. \tag{22}$$

Finally, calculating $d\mathbf{e}_2$ determines ω_2^3 to be

$$\omega_2^3 = B\,dt + E\,dx. \tag{23}$$

From (21)-(23), it follows that $d\omega^1 = (C_x - F_t)\,dx \wedge dt$ and $\omega_2^1 \wedge \omega^2 = (DB - AE)\,dx \wedge dt$, hence the first equation in (9) gives the first equation in (18). Moreover, $d\omega^2 = (B_x - E_t)\,dx \wedge dt$ and $\omega_1^2 \wedge \omega^1 = (DC - AF)\,dx \wedge dt$ generates the second equation of (18). Finally, $d\omega_1^2 = (A_x - D_t)\,dx \wedge dt$ and $\omega_1^3 \wedge \omega_3^2 = (CE - BF)\,dx \wedge dt$. Hence, the second equation of (9) gives the third equation in (18).

The Gauss equation for time-like surfaces will again be $d\omega_2^1 = R_{1212}\,\omega^1 \wedge \omega^2$ and the sign of K reversed. Using these forms we have $d\omega_2^1 = (A_x - D_t)\,dx \wedge dt$ as well as $\omega^1 \wedge \omega^2 = (BF - CE)\,dx \wedge dt$ and with $K = -1$, the third equation of (18) results. □

In the final case, the integrability condition gives the Gauss equation for $H^2 \subset R^{2,1}$. The relevant matrices U and V in this case are given by

$$U = \begin{pmatrix} 0 & C & B \\ C & 0 & A \\ B & -A & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & F & E \\ F & 0 & D \\ E & -D & 0 \end{pmatrix}. \tag{24}$$

Consider the orthonormal frame $\{\mathbf{p}, \mathbf{m}, \mathbf{n}\}$ such that $-\mathbf{p}^2 = \mathbf{m}^2 = \mathbf{n}^2 = 1$ and let $\mathbf{p}^2 = -1$ be the equation of $H^2 \subset R^{2,1}$. The equations generated by the matrices in (24) are given as

$$\begin{aligned} \mathbf{p}_t &= C\mathbf{m} + B\mathbf{n}, & \mathbf{p}_x &= F\mathbf{m} + E\mathbf{n}, \\ \mathbf{m}_t &= C\mathbf{p} + A\mathbf{n}, & \mathbf{m}_x &= F\mathbf{p} + D\mathbf{n}, \\ \mathbf{n}_t &= B\mathbf{p} - A\mathbf{m}, & \mathbf{n}_x &= E\mathbf{p} - D\mathbf{m}. \end{aligned} \tag{25}$$

The compatibility condition for (3) using (24) leads to the following system

$$C_x - F_t + AE - BD = 0,$$
$$B_x - E_t + CD - AF = 0, \tag{26}$$
$$A_x - D_t + CE - BF = 0.$$

Solving the first two equations of (26) for A and D, we obtain

$$A = \frac{1}{EC - FB}(C(F_t - C_x) + B(E_t - B_x)),$$
$$D = \frac{1}{EC - FB}(E(E_t - B_x) + F(F_t - C_x)). \tag{27}$$

Using these expressions in the third equation of (26), the Gauss equation of $H^2 \subset R^{2,1}$ is obtained

$$(\frac{(E_t - B_x)B + C(F_t - C_x)}{EC - FB})_x - (\frac{E(E_t - B_x) + F(F_t - C_x)}{EC - FB})_t + EC - BF = 0. \tag{28}$$

Now let us proceed in an alternative way by showing how the equations for the frame elements can be developed using a set of one forms and the fundamental equations (8).

Theorem 4. *For the space H^2, the ω^i_j in the fundamental equations (9) generate (25) and (26) provided that these one-forms are given by $\omega^1 = C\,dt + F\,dx$, $\omega^2 = B\,dt + E\,dx$, $\omega^2_1 = A\,dt + D\,dx$, $\omega^3_1 = C\,dt + F\,dx$ and $\omega^3_2 = B\,dt + E\,dx$ such that $\omega^i_j = -\omega^j_i$.*

Proof. Since the equations for $\mathbf{p}$ are the same as the previous case, ω^1 and ω^2 are the same as well

$$\omega^1 = C\,dt + F\,dx, \qquad \omega^2 = B\,dt + E\,dx. \tag{29}$$

Writing $\omega^2_1 = a_1\,dt + b_1\,dx = -\omega^1_2$ and $\omega^3_1 = g_1\,dt + l_1\,dx$, then calculating $d\mathbf{e}_1$ we find

$$d\mathbf{e}_1 = \mathbf{m}_t\,dt + \mathbf{m}_x\,dx = (a_1\mathbf{e}_2 + g_1\mathbf{e}_3)\,dt + (b_1\mathbf{e}_2 + l_1\mathbf{e}_3)\,dx.$$

It follows that (25) will obtain if

$$\omega^2_1 = A\,dt + D\,dx, \qquad \omega^3_1 = C\,dt + F\,dx. \tag{30}$$

Finally, calculating $d\mathbf{e}_2$ gives

$$\omega^3_2 = B\,dt + E\,dx. \tag{31}$$

Again, the equations $d\omega^1 + \omega^1_2 \wedge \omega^2 = 0$ and $d\omega^2 + \omega^2_1 \wedge \omega^1 = 0$ generate the first two equations of (26). The third equation in (26) is produced by the second equation of (9). Again with $K = -1$ and using these forms determined here, the Gauss equation for the time like surface reduces to the third equation in (26). Let us summarize these results as follows.

Theorem 5. *A nonlinear PDE which admits an $SO(2,1)$ Lax pair with $EC - BF \neq 0$ is equation (20) or (28) or a subequation. Here (20) and (28) are the Gauss equation for $S^{1,1} \subset R^{2,1}$ or $H^2 \subset R^{2,1}$, respectively, and B, C, E, F are given functions of φ and the partial derivatives of φ up to a certain order.*

4. Nonlinear PDEs of Second Order Admitting $SO(3)$ Lax Pairs

A method can be written down to identify whether a given PDE (1) of second order admits a Lax pair of class zero. The $SO(3)$ case will be treated here. Similar results can be developed for $SO(2,1)$, but will not be presented here. Given an irreducible $SO(3)$ Lax pair with $BE - CF \neq 0$, where B, C, E and F are functions of φ and the derivatives of φ up to order r, it is said that this Lax pair is of class r. In this case, the functions B, C, E and F depend on the function φ. If a prime denotes differentiation with respect to φ, then (12) can be written in the form of the following second order nonlinear PDE

$$\begin{aligned}
&(\frac{E'B+F'C}{EC-BF}\varphi_t)_x - (\frac{B'B+C'C}{EC-BF}\varphi_x)_x \\
&\qquad - (\frac{E'E+F'F}{EC-BF}\varphi_t)_t + (\frac{B'E+C'F}{EC-BF}\varphi_x)_t + BF - CE \\
&= -(\frac{B'B+C'C}{EC-BF})\varphi_{xx} + (\frac{E'B+B'E+F'C+C'F}{EC-BF})\varphi_{xt} - (\frac{E'E+F'F}{EC-BF})\varphi_{tt} \\
&\quad - (\frac{B'B+C'C}{EC-BF})'\varphi_x^2 + (\frac{E'B+F'C+B'E+C'F}{EC-BF})'\varphi_x\varphi_t - (\frac{E'E+F'F}{EC-BF})'\varphi_t^2 \\
&\qquad + BF - CE = 0. \qquad (32)
\end{aligned}$$

This equation is of second order. With the obvious identification of the coefficients of the derivatives of φ, it takes the form

$$-\alpha\varphi_{xx} + \beta\varphi_{xt} - \gamma\varphi_{tt} - \alpha_1\varphi_x^2 + \beta_1\varphi_x\varphi_t - \gamma_1\varphi_t^2 + \delta = 0, \quad \delta \neq 0. \qquad (33)$$

Here α, β, γ, α_1, β_1, γ_1 and δ are functions of the variable φ. If an equation of the form (33) admits an $SO(3)$ Lax pair of class zero, then there must exist a function λ such that the coefficients of the derivatives of φ as well as δ in (33) can be matched to the corresponding coefficients in (32),

$$\lambda\alpha = \frac{BB'+CC'}{EC-BF}, \quad \lambda\alpha_1 = (\frac{BB'+CC'}{EC-BF})',$$

$$\lambda\beta = \frac{E'B + B'E + CF' + C'F}{EC - BF}, \quad \lambda\beta_1 = (\frac{E'B + B'E + CF' + C'F}{EC - BF})',$$
$$\lambda\gamma = \frac{EE' + FF'}{EC - BF}, \quad \lambda\gamma_1 = (\frac{EE' + FF'}{EC - BF})',$$
$$\lambda\delta = BF - CE. \tag{34}$$

Comparing the right to the left column in (34), it follows that

$$(\lambda\alpha)' = \lambda\alpha_1, \qquad (\lambda\beta)' = \lambda\beta_1, \qquad (\lambda\gamma)' = \lambda\gamma_1. \tag{35}$$

Upon expanding the derivative in each of these, this system can be put in the form

$$-\lambda' = \frac{\alpha' - \alpha_1}{\alpha} = \frac{\beta' - \beta_1}{\beta} = \frac{\gamma' - \gamma_1}{\gamma}. \tag{36}$$

Thus (36) implies that λ can be written in the following form

$$\lambda = k \int_{\varphi_0}^{\varphi} \frac{\alpha_1 - \alpha'}{\alpha} \, d\tilde{\varphi} = k\sigma, \tag{37}$$

for some constant $k \neq 0$. Thus, we have

$$\begin{aligned} &BF - CE = \sigma k\delta, \\ &\frac{1}{2}(B^2 + C^2)' = -\sigma^2 k^2 \alpha\delta, \\ &\frac{1}{2}(E^2 + F^2)' = -\sigma^2 k^2 \gamma\delta, \\ &(EB + CF)' = -\sigma^2 k^2 \beta\delta. \end{aligned} \tag{38}$$

Let us define L, M and N to be

$$\begin{aligned} L(\varphi, k_1) &= -2k^2 \int_{\varphi_0}^{\varphi} \sigma^2 \alpha\delta \, d\varphi + K_1, \\ M(\varphi, K_2) &= -k^2 \int_{\varphi_0}^{\varphi} \sigma^2 \beta\delta \, d\varphi + K_2, \\ N(\varphi, K_3) &= -2k^2 \int_{\varphi_0}^{\varphi} \sigma^2 \gamma\delta \, d\varphi + K_3, \end{aligned} \tag{39}$$

for some constants K_1, K_2 and K_3. Comparing (38) and (39), we can write explicitly

$$B^2 + C^2 = L(\varphi, K_1), \quad EB + FC = M(\varphi, K_2), \quad E^2 + F^2 = N(\varphi, K_3). \tag{40}$$

From the elementary Lagrange identity, we have

$$\sigma^2 k^2 \delta^2 = L(\varphi, K_1) N(\varphi, K_3) - M^2(\varphi, K_2). \tag{41}$$

In fact, (36) and (41) are the necessary conditions for (33) to admit an $SO(3)$ Lax pair of class zero. Conversely, if (36) and (39) are satisfied, then from (40), we can solve for B, C, E, F and we can write (33) in the form (32). These results can be summarized in the following theorem.

Theorem 6. *A PDE admits an $SO(3)$ integrable system of class zero if and only if:*

(i) it has the structure (33),

(ii) the relation (36) holds, (iii) there exist three constants K_1, K_2, K_3 and $k \neq 0$ such that Lagrange identity (41) holds.

As an example of applying this theorem, consider determining $\delta(\varphi)$ for the equation

$$\varphi_{tt} - \varphi_{xx} + \delta(\varphi) = 0, \tag{42}$$

such that it admits an $SO(3)$ Lax pair. In this case, we have $\gamma = -1$, $\alpha = 1$, $\beta = 0$ and $\alpha_1 = \beta_1 = \gamma_1 = 0$. Thus (36) holds and we can take $\lambda = 1$ with L, M, N given as follows

$$L = -2k^2 \int_{\varphi_0}^{\varphi} \delta(\varphi)\, d\varphi + K_1 = -2k^2 H + K_1,$$

$$M = K_2,$$

$$N = 2k^2 \int_{\varphi_0}^{\varphi} \delta(\varphi)\, d\varphi + K_3 = 2k^2 H + K_3,$$

where $H(\varphi) = \int_{\varphi_0}^{\varphi} \delta(\varphi)\, d\varphi$. The Lagrange identity becomes

$$k^2 (H')^2 = (-2k^2 H + K_1)(2k^2 H + K_3) - K_2^2. \tag{43}$$

The function

$$\delta(\varphi) = \eta \sin(K\varphi + \zeta) \tag{44}$$

is a solution of the first order equation (42) and $H(\varphi)$ is determined from $\delta(\varphi)$ to be

$$H(\varphi) = \int_{\varphi_0}^{\varphi} \eta \sin(K\varphi + \zeta)\, d\varphi = -\frac{\eta}{K} \cos(K\varphi + \zeta) + c.$$

This will be a solution to (42) provided that the following system of equations can be satisfied by choosing the arbitrary constants to satisfy the equations

$$(2k - K)(2k + K) = 0,$$

$$2k^2 c - K_1 + c + K_3 = 0,$$

$$-2k^2 c^2 - 2ck^2 K_3 + cK_1 + K_1 K_3 - K_2^2 - k^2 \eta^2 = 0.$$

The first equation implies that $K = \pm 2k$ and the remaining arbitrary constants are selected such that the last two equations are satisfied. Consequently, it is concluded that equation (42) admits an $SO(3)$ Lax pair of class zero if and only if it is of the sine-Gordon form.

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