International Journal of Pure and Applied Mathematics
Volume 43 No. 3 2008, 317-322

RICCI OPERATOR ON Diff $(S^1)/S^1$

Helene Airault
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
LAMFA UMR 6140 CNRS
Université de Picardie Jules Verne
33, rue Saint Leu, Amiens, FRANCE
e-mail: hairault@insset.u-picardie.fr

Abstract: The bracket on the Lie algebra diff (S^1) is given by $[u,v] = uv' - u'v$. Let a and b be two real numbers such that $\alpha(k) = bk^3 - ak$ is strictly positive for $k \geq 1$. The symplectic form is $\omega_{a,b}(u,v) = (1/2\pi)\int_0^{\pi}(au' + bu''')v\, d\theta$. For u, $v \in$ diff (S^1), we put $(u|v) = \omega_{a,b}(u, Jv)$, where J be the Hilbert transform on diff (S^1). Let V be the subspace of diff (S^1) generated by $\{\cos(k\theta), \sin(k\theta)\}$ with $k \geq 1$. For $v \in$ diff (S^1), the operator $\Gamma(v) : V \to V$ is defined by $2(\Gamma(v)u|w) = ([w,v]|u) + ([w,u]|v)$ and $B(v) : V \to V$ is defined by $2\,B(v)u = \pi[v,u]$, where π is the projection on V. We study $\Gamma_l(v) = \Gamma(v) + B(v)$ and $\Gamma_b(v) = \Gamma(v) - B(v)$. Then $\sum_{p\geq 1}\alpha(p)^{-1}[\Gamma_b(\cos p\theta)\Gamma_l(\cos p\theta) + \Gamma_b(\sin p\theta)\Gamma_l(\sin p\theta)]$ is a bounded operator on V.

AMS Subject Classification: 17B66, 30B50
Key Words: Ricci curvature, infinite dimensional groups

1. Introduction

The Lie algebra diff (S^1) of the group Diff (S^1) of diffeomorphisms of the circle is isomorphic to the space of Fourier series $u(\theta) = \sum_{k=0}^{+\infty} a_k\cos(k\theta) + b_k\sin(k\theta)$. For a diffeomorphism γ, let $\gamma_\epsilon(\theta) = \gamma(\theta + \epsilon u(\theta))$. On Diff (S^1), let F_θ be the evaluation functions $F_\theta(\gamma) = \gamma(\theta)$. We define the vector field X_u on Diff (S^1) by $X_u(\gamma) = \frac{d}{d\epsilon}_{|\epsilon=0}\gamma_\epsilon$. Then $X_u(\gamma)(F_\theta) = u(\theta)\gamma'(\theta)$ and $[X_u, X_v] = (uv' - u'v)\gamma'(\theta)$. As in [1], [3], [4], the Lie bracket on diff (S^1) is

$$[u,v] = uv' - u'v\,. \tag{1}$$

For (1), the Jacobi identity is verified. In the trigonometrical basis,

Received: October 8, 2007

$$2[\cos j\theta, \cos k\theta] = (j-k)\sin(j+k)\theta + (j+k)\sin(j-k)\theta\,,$$
$$2[\sin j\theta, \sin k\theta] = (k-j)\sin(j+k)\theta + (j+k)\sin(j-k)\theta\,,$$
$$2[\sin j\theta, \cos k\theta] = (k-j)\cos(j+k)\theta - (j+k)\cos(j-k)\theta\,.$$

The subspace $\mathcal{L}$ generated by $\{\sin(k\theta),\, k \geq 1\}$ is a subalgebra of diff (S^1), as well as the subspaces $\mathcal{M}_0$ of constant functions corresponding to the rotations in Diff (S^1) and for $k \geq 1$, the subspaces $\mathcal{M}_k$ generated by $\{\cos(k\theta), \sin(k\theta), 1\}$. The Hilbert transform J on diff (S^1) is given by the singular integral

$$J(u)(\theta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(\theta+t)}{\tan(\frac{t}{2})}\, dt\,. \tag{2}$$

Since $\int_0^\pi \frac{\sin(kt)}{\tan(t/2)} dt = \pi$, we have $J\cos k\theta = \sin k\theta$, $J\sin k\theta = -\cos k\theta$ if $k \geq 1$, $J1 = 0$. Moreover $J(u') = (Ju)'$ and by interverting the order of integration,

$$\int_0^{2\pi} u(\theta)\,(Jv)(\theta)\,d\theta \;=\; -\int_0^{2\pi} (Ju)(\theta)\,v(\theta)\,d\theta\,. \tag{3}$$

2. Symplectic Form and Scalar Product

Let $\omega_{a,b}(u,v) = \int_0^{2\pi} (a\,u' + b\,u''')\,v\,\frac{d\theta}{\pi}$, where $\alpha(k) = bk^3 - ak$ is strictly positive for $k \geq 1$,

$$\omega_{a,b}(\cos k\theta, \sin k\theta) \;=\; \omega_{a,b}(\sin k\theta, -\cos k\theta) \;=\; bk^3 \,-\, ak\,. \tag{4}$$

If $u = 1$, then $\omega_{a,b}(1,v) = 0$ for any $v \in$ diff (S^1). The map $(u,v) \to \omega_{a,b}(u, Jv)$ is bilinear symmetric and

$$\omega_{a,b}(u, Ju) \geq 0\,. \tag{5}$$

Let $u(\theta) = \sum_{k\geq 0} a_k(u)\cos(k\theta) + \sum_{k\geq 1} b_k(u)\sin(k\theta)$, we have

$$\omega_{a,b}(u, Jv) = \sum_{k\geq 1} (bk^3 \,-\, ak)(a_k(u)a_k(v) + b_k(u)b_k(v))\,. \tag{6}$$

The derivation is antisymmetric, $(u'|v) = -(u|v')$ with respect to the pseudo-scalar product

$$(u|v) = \omega_{a,b}(u, Jv)\,. \tag{7}$$

The associated norm satisfies

$$||\cos k\theta||^2 \;=\; ||\sin k\theta||^2 = \alpha(k)\,. \tag{8}$$

The vectors $\{\cos(k\theta), \sin(k\theta)\}$ with $k \geq 1$ are orthogonal. The function $\alpha(k) = bk^3 - ak$ has very special properties, for example to verify the Jacobi identity

$$\omega_{a,b}([u,v],w) + \omega_{a,b}([v,w],u) + \omega_{a,b}([w,u],v) = 0\,, \tag{9}$$

we can integrate by parts or we can verify as in [3], its validity on the trigonometric functions, $\cos k\theta,\ \sin k\theta,\ k \geq 0$, since

$$(k-j)\alpha(j+k) - (2j+k)\alpha(k) + (2k+j)\alpha(j) = 0\,. \tag{10}$$

3. Invariance Properties

Lemma 3.1. *1) For any a, b, the form $\omega_{a,b}$ is invariant by rotation.*

2) For $k \geq 1$, the form $\omega_{a,b}$ is invariant by the action of the subgroup H_k with Lie algebra $\mathcal{M}_k$ if $a = b\,k^2$.

Proof. For 2), let $u_t(\theta)$ and $v_t(\theta)$ be two distinct solutions of $\frac{d}{dt}h_t(\theta) = [f(\theta), h_t(\theta)]$, where $f \in \mathcal{M}_k$, then

$$\frac{d}{dt}\omega_{a,b}(u_t, v_t) = \omega_{a,b}(\frac{d}{dt}u_t, v_t) + \omega_{a,b}(u_t, \frac{d}{dt}v_t) = \omega_{a,b}(f,\, [u_t, v_t])$$

and this is zero if $bf''' + af' = 0$, this gives $bk^3 - ak = 0$. □

Remark. If $a = b = 1$, then $\omega_{1,1}(u,v)$ is invariant under the subgroup of homographic transformations. This permits to define a symplectic structure on the quotient space Diff (S^1) modulo the subgroup of homographic transformations, see [2].

4. The Operators $\Gamma_l(v)$ and $\Gamma_b(v)$

Let V be the subspace of diff (S^1) generated by $\{\cos(k\theta), \sin(k\theta)\}$ with $k \geq 1$. To any $u \in$ diff (S^1), we associate the linear operators $\Gamma_l(u)$ and $\Gamma_b(u)$ on V defined by

$$\begin{cases} 2\,(\Gamma_l(u)\,v\,|\,w) \;=\; ([w,v]\,|\,u) \;+\; ([w,u]\,|\,v) \;+\; ([u,v]\,|\,w)\,, \\ 2\,(\Gamma_b(u)\,v\,|\,w) \;=\; ([w,v]\,|\,u) \;+\; ([w,u]\,|\,v) \;-\; ([u,v]\,|\,w)\,. \end{cases} \tag{11}$$

The method of defining an operator with a triple sum as above goes back to [7]. It has been developped in the present context in [1], [3], [4]. The two operators Γ_l and Γ_b differ by the bracket,

$$(\Gamma_l(u)\,v\,|\,w) \;=\; (\Gamma_b(u)\,v \;+\; [u,v]\,|\,w)\,. \tag{12}$$

It is easy to verify that $\Gamma_l(1)(u) = u'$ and $\Gamma_b(1)(u) = 0$. We put $c_p = \cos p\theta$ and $s_p = \sin p\theta$, we have (13)-(14) for Γ_l and (15)-(16) for Γ_b,

$$\begin{cases} 2\Gamma_l(c_p)(s_k) = (p+k)\, 1_{k\geq p+1}\, c_{k-p} + \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)}\, c_{p+k}\,, \\ 2\Gamma_l(c_p)(c_k) = -(p+k)\, 1_{k\geq p+1} s_{k-p} - \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)} s_{p+k}\,, \end{cases} \tag{13}$$

$$\begin{cases} 2\Gamma_l(s_p)(s_k) = -(p+k)\, 1_{k\geq p+1}\, s_{k-p} + \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)} s_{p+k}\,, \\ 2\Gamma_l(s_p)(c_k) = -(p+k)\, 1_{k\geq p+1}\, c_{k-p} + \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)} c_{p+k}\,, \end{cases} \tag{14}$$

$$\begin{cases} 2\Gamma_b(c_p)(s_k) = -(p+k)\, 1_{p\geq k+1}\, c_{p-k} + \dfrac{(p+2k)\alpha(p)}{\alpha(p+k)} c_{p+k}\,, \\ 2\Gamma_b(c_p)(c_k) = -(p+k)\, 1_{p\geq k+1}\, s_{p-k} - \dfrac{(p+2k)\alpha(p)}{\alpha(p+k)} s_{p+k}\,, \end{cases} \tag{15}$$

$$\begin{cases} 2\Gamma_b(s_p)(s_k) = -(p+k)\, 1_{p\geq k+1}\, s_{p-k} + \dfrac{(p+2k)\alpha(p)}{\alpha(p+k)} s_{p+k}\,, \\ 2\Gamma_b(s_p)(c_k) = (p+k)\, 1_{p\geq k+1}\, c_{p-k} + \dfrac{(p+2k)\alpha(p)}{\alpha(p+k)} c_{p+k}\,. \end{cases} \tag{16}$$

5. The Ricci Type Operator $\boldsymbol{\Delta(\Gamma_b, \Gamma_l)}$

Definition 5.1. Let the series of operators be

$$\begin{cases} \Delta(\Gamma_b, \Gamma_l) = \sum_{p\geq 1} \frac{1}{\alpha(p)} \left[\Gamma_b(c_p)\Gamma_l(c_p) + \Gamma_b(s_p)\Gamma_l(s_p) \right], \\ \Delta(\Gamma_l, \Gamma_b) = \sum_{p\geq 1} \frac{1}{\alpha(p)} \left[\Gamma_l(c_p)\Gamma_b(c_p) + \Gamma_l(s_p)\Gamma_b(s_p) \right]. \end{cases} \tag{17}$$

Definition 5.2. Let $p \geq 1$, p integer, the linear operators τ_p and τ_{-p} on V are given by

$$\begin{cases} \tau_p(c_k) = c_{k+p}\,, \\ \tau_p(s_k) = s_{k+p}\,, \end{cases} \quad \text{and} \quad \begin{cases} \tau_{-p}(c_k) = 1_{k\geq p+1}\, c_{k-p}\,, \\ \tau_{-p}(s_k) = 1_{k\geq p+1}\, s_{k-p}\,. \end{cases} \tag{18}$$

Lemma 5.3. *The operators $A_p = \Gamma_b(c_p)\Gamma_l(c_p) + \Gamma_b(s_p)\Gamma_l(s_p)$ and $B_p = \Gamma_l(c_p)\Gamma_b(c_p) + \Gamma_l(s_p)\Gamma_b(s_p)$ are diagonal in the trigonometrical basis $\{c_k, s_k\}_{k\geq 1}$*

$$\tau_{-p}\, o\, A_p\, o\, \tau_p = B_p\,. \tag{19}$$

Theorem 5.4. *Let $\alpha(k) = bk^3 - ak$ as in Sections 2 and 4. If $b \neq 0$, the first series $\Delta(\Gamma_b, \Gamma_l)$ converge and the operator $\Delta(\Gamma_b, \Gamma_l)$ is bounded, the second series $\Delta(\Gamma_l, \Gamma_b)$ diverge. If $b = 0$, both series diverge.*

Proof. For $\Delta(\Gamma_b, \Gamma_l)$, we have

$$2[\Gamma_b(c_p)\Gamma_l(c_p) + \Gamma_b(s_p)\Gamma_l(s_p)]\, c_k = -(p+k)1_{k\geq p+1}\frac{(2k-p)\alpha(p)}{\alpha(k)}c_k\,,$$

$$2[\Gamma_b(c_p)\Gamma_l(c_p) + \Gamma_b(s_p)\Gamma_l(s_p)]\, s_k = -(p+k)1_{k\geq p+1}\frac{(2k-p)\alpha(p)}{\alpha(k)}s_k\,.$$

Since $\sum_{\begin{cases} p, \\ 1\leq p\leq k-1\end{cases}} -(p+k)(2k-p) = -\frac{(k-1)(13k^2+k)}{6}$, we deduce that $\Delta(\Gamma_b, \Gamma_l)$ is a bounded operator if $b \neq 0$. For $\Delta(\Gamma_l, \Gamma_b)$,

$$2[\Gamma_l(c_p)\Gamma_b(c_p) + \Gamma_l(s_p)\Gamma_b(s_p)]\, c_k = -\frac{(p+2k)(2p+k)\alpha(p)}{\alpha(p+k)}\, c_k\,,$$

$$2[\Gamma_l(c_p)\Gamma_b(c_p) + \Gamma_l(s_p)\Gamma_b(s_p)]\, s_k = -\frac{(p+2k)(2p+k)\alpha(p)}{\alpha(p+k)}\, s_k\,.$$

Since

$$\sum_{p\geq 1}\frac{(p+2k)(2p+k)}{\alpha(p+k)} = +\infty,$$

the series $\Delta(\Gamma_l, \Gamma_b)$ diverge. □

Remark. $\Delta(\Gamma_b, \Gamma_l)$ is a Ricci type operator on diff $(S^1)/S^1$. It is obtained as a trace of operators on V. In [5], [6], [8], [9], in relation with string theory, other bounded Ricci operators have been obtained as Ricci curvature on the infinite dimensional manifolds which are the quotient of Diff (S^1) by the subgroup of rotation or by the subgroup of homographic transformations.

References

[1] H. Airault, P. Malliavin, Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry, *J. Funct. Anal.*, **241** (2006), 99-142.

[2] H. Airault, P. Malliavin, A. Thalmaier, Support of Virasoro unitarizing measures, *C.R.Ac.Sc. Paris*, **335** (2002), 621-626.

[3] H. Airault, Riemannian connections and curvatures on the universal Teichmuller space, *C.R.Ac.Sc. Paris* (2005).

[4] H. Airault, Geometry and integration by parts on $H\backslash$Diff (S^1), In: *Mathematical Analysis of Random Phenomena, Proceedings of the International Conference Hammamet, Tunisia'2005*, Word Scientific Publ. (2007).

[5] M.J. Bowick, S.G. Rajeev, The holomorphic geometry of closed bosonic string theory and Diff $(S^1)/S^1$, *Nuclear Physics*, **B293** (1987), 348-384.

[6] M.J. Bowick, A. Lahiri, The Ricci curvature of Diff $(S^1)/SL(2,R)$, *J. Math. Phys.*, **29**, No. 9 (1988), 1979-1980.

[7] A.B. Cruzeiro, P. Malliavin, Renormalized differential geometry on path space: Structural equation, curvature, *J. Funct. Anal.*, **139** (1996), 119-180.

[8] A.A. Kirillov, D.V. Yur'ev, Kähler geometry of the infinite-dimensional homogeneous space $M = \mathrm{diff}_+(S^1)/\mathrm{rot}\,(S^1)$, *Funktsional. Anal. i Prilozhen*, **21**, No. 4 (1987), 35-46.

[9] D.K. Hong, S.G. Rajeev, Universal Teichmuller space and Diff S^1/S^1, *Comm. Math. Phys.*, **135**, No. 2 (1991), 401-411.