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A SEMILINEAR ELLIPTIC DIRICHLET BOUNDARY VALUE PROBLEM WITH EXPONENTIAL NONLINEARITY

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Abstract: In this paper we investigate the structure of solutions set of a class of semilinear elliptic equation of exponential type, including Gel'fand equation, which arises from the study of thermal self-ignition of a chemically active mixture of gases in a vessel.

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1. Introduction

I.M. Gel'fand [2] considered the problem of thermal self-ignition of a chemically active mixture of gases in a vessel. The course of the reaction is described by a system of equations of heat conduction and diffusion in the variables temperature T and concentration n of one of the component of the misture, i.e.

$$\begin{cases} \dfrac{\partial T}{\partial t} & = k\Delta T + qf(T,n)\,, \\ \dfrac{\partial n}{\partial t} & = D\Delta n - \epsilon f(T,n)\,. \end{cases}$$

In the case where ϵ is sufficienty small, one can neglect the burning up of the combustible component, on a certain interval of time, assuming $n = n_0$ constant

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and so it is equivalent to consider the equation

$$\frac{1}{k}\frac{\partial T}{\partial t} = \Delta T + \frac{q}{k} f(T, n_0).$$

Now for large value of k, we get the semilinear elliptic equation

$$\Delta T + g(T, n_0) = 0\,,$$

where $g(T, n_0) = \lim_{k\to+\infty} \frac{q}{k} f(T, n_0)$. The elliptic problem depends on two parameters having both the dimension of a length: the first parameter $l = \sqrt{\frac{k}{q\epsilon}}$ (where k denotes the conductitivity and q, ϵ denote respectively the rates of diffusion and of reaction) and the second parameter R is the characteristic dimension of the vessel. Denoting by $\lambda = \frac{R}{l}$ and, using dimensionless variables and assuming exponential growth for nonlinearity g, I.M. Gel'fand considered the following semilinear Dirichlet boundary value problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in} \quad \overline{\Omega}\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \\ u > 0 & \text{on} \quad \Omega\,. \end{cases} \tag{1.0}$$

In this paper we investigate the following class of semilinear Dirichlet boundary value problem

$$\begin{cases} -\Delta u = \lambda |x|^\alpha e^u & \text{in} \quad \overline{\Omega}\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \\ u > 0 & \text{on} \quad \Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain contained in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\alpha \geq 0$. The class of boundary value problem considered reduces to the Gel'fand problem, when $\alpha = 0$. For each $\lambda > 0$, the solution set S_λ is defined by

$$S_\lambda = \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u \quad \text{solves} \quad (1.1)\}.$$

The set of the eigenvalues of problem (1.1) is defined by

$$Sp = \{\lambda > 0 : S_\lambda \neq \emptyset\}.$$

In this paper we prove that $Sp \neq \emptyset$, i.e. more precisely, the following results hold.

Theorem 1. *There exists $\lambda_* > 0$, depending only on Ω, such that:*

$$\begin{cases} S_\lambda \neq \emptyset & \text{if} \quad 0 < \lambda \leq \lambda_*\,, \\ S_\lambda = \emptyset & \text{if} \quad \lambda > \lambda_*. \end{cases}$$

Theorem 2. *If $S_\lambda \neq \emptyset$ then S_λ has a minimal element u_λ, such that if $v \in S_\lambda$ then for any $x \in \Omega$, $v(x) \geq u(x)$.*

Moreover we prove also a molteplicity result about the maximum number of ordered solutions for any fixed $\lambda \in Sp$, using the ordered Banach space of continuous functions defined on $\overline{\Omega}$, with the cone of all positive functions. We shall prove the following

Theorem 3. *It is impossible that* $\{u_1, u_2, u_3\} \subset S_\lambda$ *with* $u_1 \le u_2 \le u_3$ *and* $u_1 \ne u_2 \ne u_3$.

2. Preliminary Results

We recall some theorems of nonlinear analysis that we shall use for the proofs in the next section.

Theorem A. (Implicit Function) *Let* E, F *and* G *be three Banach spaces,* A *an open subset of* $E \times F$ *and* $f : A \to G$ *a continuously differentiable mapping. Let* $(x_0, y_0) \in A$ *such that* $f(x_0, y_0) = 0$ *and such that*

$$D_y f(x_0, y_0) : F \to G$$

is an invertible continous linear operator. Then there exists an open neighborhood U_0 *of* x_0 *such that for any open convex neighborhood* U *contained in* U_0 *there exists a unique continuous mapping* $u : F \to G$ *such that*

$$\begin{cases} u(x_0) = y_0\,, \\ (x, u(x)) \in A\,, \\ f(x, u(x)) = 0, \quad x \in U. \end{cases}$$

Moreover u *is a continuously differentiable in* U *with derivative given by*

$$u'(x) = D_y f(x, u(x))^{-1} \circ D_x f(x, u(x)).$$

Now we recall the definitions of super-subsolution and the classical Perron's principle.

The function $\underline{u}$ is called a subsolution for the problem (1.1) if satisfies

$$\begin{cases} -\Delta u \le \lambda |x|^\alpha e^u \quad \text{in} \quad \overline{\Omega}\,, \\ u \le 0 \quad \text{on} \quad \partial\Omega\,, \\ u > 0 \quad \text{on} \quad \Omega\,, \end{cases} \tag{1.1. $\le$}$$

and the function $\overline{u}$ is called supersolution if satisfies

$$\begin{cases} -\Delta u \ge \lambda |x|^\alpha e^u \quad \text{in} \quad \overline{\Omega}\,, \\ u \ge 0 \quad \text{on} \quad \partial\Omega\,, \\ u > 0 \quad \text{on} \quad \Omega\,. \end{cases} \tag{1.1. $\ge$}$$

Theorem B. (Perron's Super-Subsolution Principle) *If there exists a subsolution* $\underline{u}$ *and a supersolution* $\overline{u}$ *for problem* (1.1) *such that*

$$\underline{u} \le \overline{u}\,,$$

then there exists a solution u *of problem* (1.1) *such that*

$$\underline{u} \le u \le \overline{u}.$$

Theorem C. (Jensen's Inequality) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex function, let* $h : \Omega \to \mathbb{R}$ *be a positive weight function and* $g : \Omega \to \mathbb{R}$ *continuos function then one has*

$$f\Big(\frac{\int_\Omega h(x)g(x)dx}{\int_\Omega h(x)dx}\Big) \le \frac{\int_\Omega h(x)f(g(x))dx}{\int_\Omega h(x)dx}.$$

3. Proofs of the Basic Results on the Nonlinear Eigenvalue Problem

This section begins with

Proof of the Theorem 1. The proof is divided into three steps.

Step 1. We prove that, for λ sufficiently small, the solution set of nonzero solutions S_λ is not empty. We perform an implicit functional approach.

Let us consider the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda |x|^\alpha e^u \quad \text{in} \quad \overline{\Omega}\,, \\ u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.2}$$

and let us denote by $C^\theta(\overline{\Omega})$ $(0 < \theta < 1)$ the space of all Hölder continuous functions defined on $\overline{\Omega}$ and let

$$C^{2+\theta}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : \partial_{i,j} v \in C^\theta(\overline{\Omega}), \quad 1 \le i, j \le N\}\,.$$

Let us define

$$\phi : \mathbb{R} \times C_0^{2+\theta}(\Omega) \to C_0^\theta(\Omega), \quad 0 < \theta < 1 \tag{2.1}$$

by

$$\phi(\lambda, u) \equiv -\Delta u - \lambda |x|^\alpha e^u. \tag{2.2}$$

The problem (1.2) turn out to be, thus, formulated as an implicit function problem in this way: find $u = u(\lambda) \in C_0^{2+\theta}(\Omega)$ such that

$$\begin{cases} \phi(\lambda, u(\lambda)) = 0\,, \\ \phi(0,0) = 0, \quad 0 < \lambda < \lambda_0. \end{cases} \tag{2.3}$$

One has the following identity

$$\phi(\lambda_0, u_0 + v) - \phi(\lambda_0, u_0) = -\Delta v - \lambda_0 |x|^\alpha e^{u_0}(e^v - 1) \quad (|v| \to 0) \tag{2.4}$$

and

$$\phi(\lambda_0, u_0 + v) - \phi(\lambda_0, u_0) = -\Delta v - \lambda_0 |x|^\alpha e^{u_0}[v + o(v)], \quad (|v| \to 0). \tag{2.5}$$

The left side of (2.5) implies that ϕ admits as Fréchet derivative

$$\phi_u(\lambda_0, u_0)v = -\Delta v - \lambda_0|x|^\alpha e^{u_0}v. \tag{2.6}$$

Now, since $\phi(0,0) = 0$ and since the Fréchet derivative

$$\phi_u(0,0) = -\Delta : C_0^{2+\theta}(\Omega) \to C^\theta(\Omega)$$

is a continuous linear invertible operator, we can apply the implicit function theorem to the problem (2.3). Thus we get a continuous nonzero solution $u = u(\lambda)$ defined on the interval $(0 < \lambda < \lambda_0)$, for suitable $\lambda_0 > 0$. We note that we have the local uniqueness too. The minimum principle shows that the previous solution satisfy the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda|x|^\alpha e^u \quad \text{in} \quad \overline{\Omega}\,, \\ u = 0 \quad \text{on} \quad \partial\Omega\,, \\ u > 0 \quad \text{on} \quad \Omega. \end{cases} \tag{1.1}$$

Indeed, if u is the solution of problem (1.2), it must be

$$\begin{cases} -\Delta u \geq 0 \quad \text{on} \quad \Omega\,, \\ u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases}$$

that, in view of minimum principle, implies $u > 0 \quad \text{on} \quad \Omega.$

Step 2. We claim that the solution set $S_\lambda = \emptyset$ if λ is sufficiently large. We argue by contradiction.

Let $\phi_1 = c(1 - |x|^2)$, where $c > 0$ is such that the function $\phi_1$satisfies $\int_\Omega |x|^\alpha \phi_1(x)dx = 1$. Note that ϕ_1 satisfies

$$\begin{cases} -\Delta\phi_1 \leq |x|^\alpha \phi_1 \quad \text{in} \quad \Omega\,, \\ u \geq 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.7}$$

Let us introduce the functional $J : C^{2+\theta} \to \mathbb{R}$ defined by

$$J(u) \equiv \int_\Omega u(x)|x|^\alpha \phi_1(x)dx > 0\,, \tag{2.8}$$

where u is a (positive) solution of (1.1).

Multiplying both side of the equation (1.1) by ϕ_1 and integrating we get

$$\int_\Omega (-\Delta u(x))\phi_1(x)dx = \lambda \int_\Omega |x|^\alpha e^{u(x)}\phi_1(x)dx. \tag{2.9}$$

Hence

$$-\int_\Omega u(x)\Delta\phi_1(x)dx = \lambda \int_\Omega |x|^\alpha e^{u(x)}\phi_1(x)dx. \tag{2.10}$$

Now let f be a convex function, the following Jensen's inequality holds

$$f(\frac{\int_\Omega h(x)g(x)dx}{\int_\Omega h(x)dx}) \leq \frac{\int_\Omega h(x)f(g(x))dx}{\int_\Omega h(x)dx}\,. \tag{2.11}$$

We apply Jensen's inequality, assuming $f(x) = e^x$, $g = u$ and $h = |x|^\alpha \phi_1$ in (2.11), and deduce the following estimates

$$J(u) = \lambda \int_\Omega e^{u(x)}\phi_1(x)dx \geq \lambda e^{\int_\Omega |x|^\alpha u(x)\phi_1(x)dx}. \tag{2.12}$$

Finally we obtain the following estimate for λ

$$\lambda \leq \sup_{J>0} J(u)e^{-J(u)}. \tag{2.13}$$

Step 3. We claim that if $\lambda_1 > \lambda_2 > 0$ and $S_{\lambda_1} \neq \emptyset$ then $S_{\lambda_2} \neq \emptyset$.

We make use of the method of super-subsolutions.

Let $\overline{u}$ be a solution in S_{λ_1}, i.e.

$$-\Delta\overline{u} = \lambda_1|x|^\alpha e^{\overline{u}}. \tag{2.14}$$

Then $\overline{u}$ is a supersolution for $\lambda = \lambda_2$ indeed

$$-\Delta\overline{u} = \lambda_1|x|^\alpha e^{\overline{u}} > \lambda_2|x|^\alpha e^{\overline{u}}. \tag{2.15}$$

On the the other hand $\underline{u} = 0$ is a subsolution because

$$-\Delta\overline{u} = 0 < \lambda_2|x|^\alpha e^0 = \lambda_2|x|^\alpha\,. \tag{2.16}$$

Since $\underline{u} \leq \overline{u}$, we get that there exists a solution u of (1.1) satisfying $\underline{u} \leq u \leq \overline{u}$. ☐

Proof of the Theorem 2. Let us consider the fixed point equation

$$u = \phi(u)\,, \tag{2.17}$$

where $\phi : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ defined by

$$\phi(u)(x) = \int_\Omega G(x,y)e^{u(y)}dy\,. \tag{2.18}$$

Here G is the Green function associated to the Laplace operator on Ω. Note that the operator $\phi : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is order preserving, i.e.

$$v \leq w \to \phi(v) \leq \phi(w)\,, \tag{2.19}$$

where $v \leq w$ means $v(x) \leq w(x)$ for any $x \in \overline{\Omega}$.

The iterative sequence

$$\begin{cases} u_0 = 0\,, \\ u_{k+1} = \lambda\phi(u_k)\,. \end{cases} \tag{2.20}$$

For any $k \geq 0$ one has $u_{k+1} \geq u_k$.

Let $v \in S_\lambda$, we get

$$v = \lambda\phi(v) \geq 0 = u_0\,, \tag{2.21}$$

and hence

$$v \geq (\lambda\phi)^k(u_0) = u_k. \tag{2.22}$$

Thus $\underline{u} = \lim_{k\to+\infty} u_k(x)$ satisfies $\underline{u} \leq v$. Applying the Monotone Convergence Theorem to the integral equation

$$u_{k+1}(x) = \lambda \int_\omega G(x,y)e^{u_k(y)}dy \tag{2.23}$$

and taking into account that the function

$$y \to G(x,y)e^{u_k(y)} \tag{2.24}$$

belongs to $L^1_y(\Omega)$, we deduce

$$\underline{u} = \lambda \int_\Omega G(x,y)e^{\underline{u}(y)}dy \in X, \quad x \in \Omega\,. \tag{2.25}$$

The function $\underline{u}$ is sufficiently regular to be a classic solution.

In view of the fact the the element v is arbitrary, hence u turn out to be the minimal element. □

Proof the Theorem 3. Assume that $u_i \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $i = 1,2,3$ satisfies

$$-\Delta u_i = \lambda |x|^\alpha e^{u_i}\,. \tag{2.26}$$

Then we get

$$\phi(x) = u_3 - u_2 \geq 0, \quad \phi(x) \not\equiv 0, \tag{2.27}$$

and

$$\psi(x) = u_2 - u_1 \geq 0, \quad \psi(x) \not\equiv 0. \tag{2.28}$$

We have

$$-\Delta\phi(x) = c(x)\phi(x) \quad \text{in} \quad \Omega \quad \phi(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.29}$$

and

$$-\Delta\psi(x) = d(x)\psi(x) \quad \text{in} \quad \Omega \quad \psi(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.30}$$

where

$$c(x) = \lambda|x|^\alpha f'((1-\gamma)u_3 + \gamma u_2) \geq 0, \quad c(x) \not\equiv 0, \tag{2.31}$$

with $0 < \gamma < 1$,

$$d(x) = \lambda f'((1-\delta)u_2 + \delta u_1) \geq 0, \quad d(x) \not\equiv 0, \tag{2.32}$$

with $0 < \delta < 1$. Now since $f(u) = e^u$ is increasing then it must be

$$d(x) \geq c(x) \quad -d(x) > -c(x) \quad \text{for} \quad x \in \Omega \tag{2.33}$$

The strong maximum principle implies that

$$\phi(x) > 0 \quad \text{and} \quad \psi(x) > 0 \quad \text{for} \quad x \in \Omega. \tag{2.34}$$

Let $a : \Omega \to \mathbb{R}$ be a continuous function. Let us introduce the linear elliptic operator

$$L_a : L^2(\Omega) \to L^2(\Omega)$$

defined by

$$L_a \equiv -\Delta - a \tag{2.35}$$

under homogeneous Dirichlet boundary condition.

If we put $a = c(x)$, then 0 is an eigenvalue of operator $L_{c(x)}$. Indeed (2.29) can be written as

$$L_{c(x)}\phi(x) = 0 = 0\phi(x)\,. \tag{2.36}$$

Thus the eigenvalue 0 is the first eigenvalue, i.e. $\lambda_1 = 0$, since the general theory assures that the first eigenvalue admits an eigenfunction of definite sign.

Analogously if we put $a = d(x)$, then we get that 0 is the first eigenvalue of operator $L_{d(x)}$. Indeed (2.30) can be written as

$$L_d(\psi(x)) = 0 = 0\psi(x). \tag{2.37}$$

To the operator linear elliptic L_a is associated the following energy inequality

$$0 = inf E_a(v) \leq \frac{1}{2}\int_\Omega (|\bigtriangledown v|^2 - a(x)|v|^2)dx\,, \tag{2.38}$$

where $v \in H_0^1(\Omega)$.

Now, taking into account (2.33) we get

$$E_d(v) > E_c(v) \quad \text{for any} \quad 0 \neq v \in H_0^1(\Omega). \tag{2.39}$$

Note that $\psi = \psi(x)$ is an eigenfunction corresponding to the first eigenvalue of the operator $L_{d(x)}$ and thus, in view of Rayleigh principle, we have

$$E_d(\psi) = \frac{1}{2}\int_\Omega (|\bigtriangledown \psi|^2 - a(x)|\psi|^2)dx = 0, \tag{2.40}$$

against (2.39) which reads $E_d(\psi) > 0$. □

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