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SOME DYNAMICAL SYSTEMS WITH CHAOTIC STATES

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Abstract: This article continues works of Dr. Nikolaevski on chaos theory [1, 4, 5]. The detailed proof of neuron/neural network nonintegrability is being published here for the first time.

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In a completely integrable dynamical system, any trajectory lies on an invariant torus in the phase space, and any chaotic motion is impossible. On the contrary, if the system is not fully integrable, a chaotic motion occurs in an open set of the phase space.

The distinctive features of regular and chaotic motion are: A regular trajectory is dense on an invariant torus in the phase space or has a regular limiting set: a fixed point or a limit cycle. Chaotic trajectories: close trajectories diverge exponentially; at least one Lyapunov exponents is positive; each trajectory is dense in an open set of the phase space. A chaotic trajectory can have its limit in a strange limiting set and be dense in it or even contain it.

We will consider several examples of dynamical systems with chaotic states.

Example 1. In the beginning of 1970s, Serpukhov particle accelerator could not attain its project capacity, because of local overheat. Chaotic trajec-

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tories were found, attending almost every place in the ring. No overheat; the project capacity was attained and tripled. It saved about 2,000,000,000$/year. This was one of the first applications of chaos theory.

Example 2. Shchur and Nikolaevski [4] proved nonintegrability of Yang-Mills dynamics. They proved that in a low-dimensional subspace, finite contact homoclinic and heteroclinic points exist, from which non-existence of analytic integrals other than Hamiltonian follows. This results in the absence of a complete set of integrals for the Yang-Mills system and existence of chaotic states for physical vacuum.

Example 3. Using Okuda's [3] versions of neuron/neural network definitions and Smales's [5] horseshoe, Nikolaevski [2] proved that a neuron/neural network is nonintegrable. Chaotic states are manifestly built in the proof. We will consider this example more detailed.

Okuda [3], usung Green's function method, brought equation of neuron/neural networks to the form, allowing solutions X(x, t), taking only values 0 and 1. Regions, where X(x, t) = 1, are called active regions; regions, where X(x, t) = 0, are called resting regions.

Two solutions can be written explicitly:

$$X(x,t) = \{1, x > ut; 0, x < ut\} \text{ and } X(x,t) = \{0, x > ut; 1, x < ut\},$$

where u is a constant, parameter of the network, and X(x, t) can be either 0 or 1, when x = ut.

Using these solutions, we can build other solutions with almost arbitrary borders between active and resting regions, borders with arbitrary low deviation from prescribed border.

Now we will consider mostly (almost) horizontal and (almost) vertical borders. As a model of one neuron, consider a direct product of interval 0 = x = 1 per ray t = 0.

Consider the following solution, constant on interval 0 = x = 1 and depending only on t:

$$\begin{aligned} X(x,t) &= 1,\ 0 = t = 1; \\ X(x,t) &= 1,\ 1 = t = 4/3; \\ X(x,t) &= 0,\ 4/3 < t < 5/3; \\ X(x,t) &= 1,\ 5/3 = t = 2; \\ X(x,t) &= 1,\ 2 = t = 19/9; \\ X(x,t) &= 0,\ 19/9 < t < 20/9; \end{aligned}$$

$$X(x,t) = 1,\ 20/9 = t = 7/3;$$
$$X(x,t) = 0,\ 7/3 < t < 8/3;$$
$$X(x,t) = 1,\ 8/3 = t = 25/9;$$
$$X(x,t) = 0,\ 25/9 < t < 26/9;$$
$$X(x,t) = 1,\ 26/9 = t = 3;$$

and so on: for every interval n= t = n+1, we take off the open middle third of each interval of t, where X(x, t) =1, and put X(x, t) = 0 instead. In the limit of big n, we have a direct product of interval 0 = x = 1 per Cantor set in t.

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For every fixed x, the trajectory x, t, t=0, contains a strange limiting set, so it is chaotic.

This solution can be interpreted as a sequence of maps, similar to those, used in building of Smale's horseshoe. Such a construction would allow to show that Lyapunov exponent in horizontal direction is greater than 1 and in vertical direction less than 1.

We can also define trajectories by any continuous sequence of maps of interval 0<x<1, fixed t, on the same interval at other t. Trajectories will be more complex, but also chaotic.

Real neurons, of course, cannot change the state infinitely often, but frequent and seemly disordered change of state can destroy any tending to a regular limiting set (fixed point or limit cycle). It gives relaxation from such a limiting set.

It is very easy to build chaotic states, in which many neurons take part, and which give relaxation from arbitrary regular limiting set, for example, a limit cycle, in which many neurons take part.

The brain of a person, suffering from epilepsy, is often cycling around the same items and cannot refresh itself. The parts of the brain, involved into the cycling, become very tired ("overheated"), which can provoke seizures. Chaotic states can help against "overheat" of local parts of the brain. It furthered progress in understanding of the mechanism of epilepsy.

Example 4. A common belief exists that chaotic behavior is impossible to predict, because close trajectories diverge exponentially in time. Gurvitz and Nikolaevski [1] developed and successfully implemented in Government Projects two new methods that allow to forecast chaotic behavior with error growing only as square root from time. One method is based on semi-Markov chains: probabilities of transfers are calculated by all available similar life histories; predictive modeling of the target history allows to forecast its future. The error, as usual, grows as square root from time.

The other method is based on time series forecasting. In database, we find a life history(ies), the most similar to the target history (with time shift). The difference between a selected and target history locally meets a linear equation, so it can grow exponentially. It is not a chaos yet, only its precondition; the chaos begins, when trajectories become entangled. So let us consider the most complex case, when trajectories diverge exponentially; we take logarithm from their difference. This series grows only linearly. From the theory of quasilinear equations, we have the presentation of this logarithm as at + w(t), where a is a Lyapunov exponent and w(t) is a stationary time series. We can forecast it by many different known methods; the error does not grow infinitely with time. We must stop our forecasting, if:

(a) the trajectory selected for comparison finishes: we cannot trace it further;

(b) the behavior of the difference between target/predicted and selected trajectories becomes far from exponential growth (entangled trajectories);

(c) the difference between target/predicted and selected trajectories becomes too big.

In these cases, we return to the source series, find the closest histories to the predicted one, and repeat the procedure. The difference between trajectories starts with a new level. In case (c), if we have enough many histories, we stop forecasting for the primary selection only when we found a secondary selection with starting difference less than the final difference for primary selection.

This process is equivalent to conditional Monte Carlo, and the error grows as square root from time.

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