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EXISTENCE/NON-EXISTENCE OF CONFORMAL MAPS FOR A FAMILY OF FUNCTIONS

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Abstract: Existence/non-existence of conformal maps for a family of functions is shown concretely for mapping from a pentagon-like configuration to a rectangular region.

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1. Introduction

Techniques of conformal mapping cover many classical applications in mathematical physics. Under most conformal mapping, complicated physical configurations are transformed into simpler or more convenient configurations. For mapping one region to another region, it is implicitly assumed that exactly unique one-to-one correspondence is guaranteed by a mapping analytic function, i.e. the derivative of the function $\neq 0$ in the said region, see [1].

In the following, availability of a concrete family of functions for a conformal mapping is discussed, and limited usage of a parameter in the family of functions is pointed out.

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2. Analysis

2.1. Case I

Consider the mapping of a family function from z in a pentagon-like configuration to w in a rectangular region.

$$z = i + i\sin\theta \Big/ \left(\sin\frac{\pi}{2n}\right) , \tag{1}$$

$$f \equiv \frac{\sin n\theta + \epsilon \sin 5n\theta}{1+\epsilon} = \frac{w - ib}{ib}, \tag{2}$$

$$|\Re(w)| \leq a\,, \quad 0 \leq \Im(w) \leq b\,, \tag{3}$$

where $n > 1, a > 0, b > 0, \epsilon$: complex parameter. The point corresponding to $n\theta = -\pi/2$ just indicates the vertex $(z = 0)$. In case of $n =$ an odd integer > 1, f is a polynomial of z, which is similar to a kind of series approximation [2]. In case of $\epsilon = 0$, it is trivial that the set of equations constitutes a conformal mapping. For the values of θ satisfying $f_\theta = 0$, i.e. $\cos n\theta + 5\epsilon\cos 5n\theta = 0$, the sufficient condition to the existence of a conformal mapping due to equations (1)-(3) means that $(f+1)bi$ should not lie inside of the said rectangular region. That is,

$$|\Re\{(f+1)ib\}| \geq a\,,$$

or

$$|\Re\{(f+1)ib\}| < a \quad \& \quad |\Im\{(f+1)ib\} - b/2| \geq b/2\ ,$$

which becomes

$$|\Im(f+1)| \geq a/b\,,$$

or

$$|\Im(f+1)| < a/b \quad \& \quad [\ \Re(f+1) \leq 0 \text{ or } \Re(f+1) \geq 1]\ .$$

For a critical case of a small real value of $\epsilon = \epsilon_0, (\theta = \theta_0)$, where $\Re(f+1) = 0$,

$$\epsilon_0 \approx 0.0020278\ ,$$

$$n\theta_0 \approx -0.76058 \mp 1.150313i\ ,$$

$$\Im\{f(\theta_0;\epsilon_0) + 1\} = \mp 0.77629\ .$$

Therefore for the point corresponding to the one root, assuming a small value of $\epsilon(> 0)$

$$n\theta = -i\ln\left[\sqrt{\frac{5-\sqrt{5-0.8/\epsilon}}{8}} - i\sqrt{\frac{3+\sqrt{5-0.8/\epsilon}}{8}}\right] , \tag{4}$$

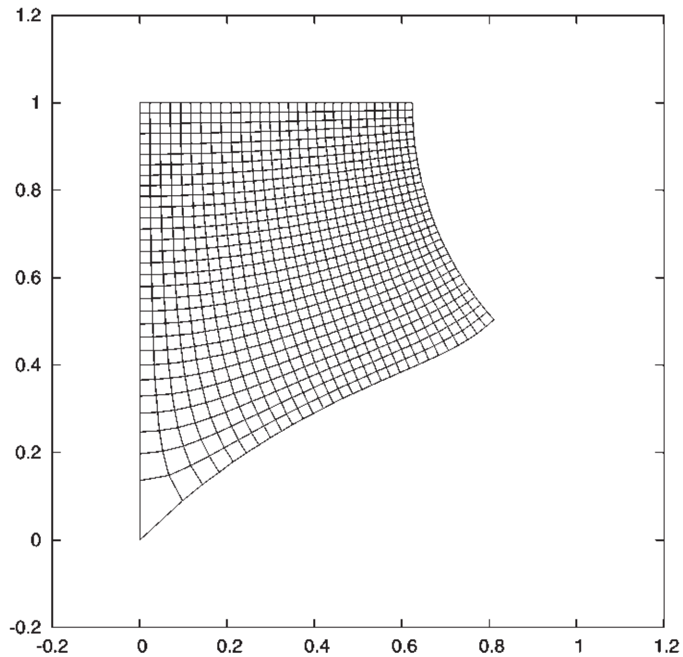

Figure 1: Isolines

where $\sqrt{\ }$ stands for a principal value. The function f is analytic with respect to ϵ at $\epsilon = \epsilon_0$, so that

$$\begin{aligned} f+1 &= f(\theta_0;\epsilon_0)+1+\frac{\sin 5n\theta_0 - f(\theta_0;\epsilon_0)}{1+\epsilon_0}(\epsilon-\epsilon_0) \\ &\quad +O\left\{(\epsilon-\epsilon_0)^2\right\} \quad (|\epsilon-\epsilon_0|\to 0)\ . \end{aligned} \tag{5}$$

Thus for moderate values of a/b and for small values of $|\epsilon - \epsilon_0|$, existence of conformal mapping becomes

$$\Re(\epsilon-\epsilon_0)+1.2796|\Im(\epsilon-\epsilon_0)| \lesssim 0\ . \tag{6}$$

Figure 1 shows isolines of $\Re w$ or $\Im w$, i.e. $\Re w =$ constant or $\Im w =$ constant for the case of producing a conformal map at $\epsilon = 0.001, n = 2, b/a = 1, z \equiv x+iy$ for $\Re w \geq 0$.

Figure 2 shows isolines of $\Re w$ or $\Im w$, i.e. $\Re w =$ constant or $\Im w =$ constant for the case of not producing a conformal map at $\epsilon = 0.006, n = 2, b/a = 1$ for $\Re w \geq 0$.

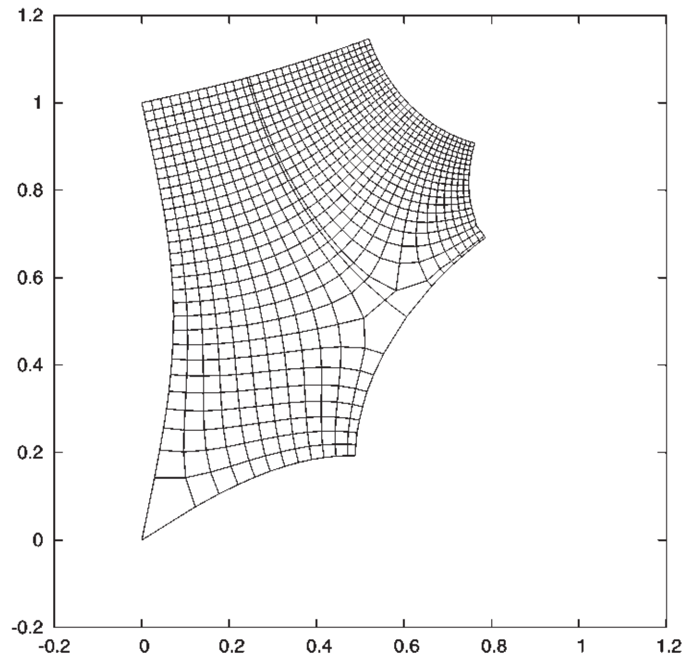

Figure 2: Isolines

2.2. Case II

Consider the mapping of a family function from z in a pentagon-like configuration to w in a rectangular region.

$$z = i + i \sin\theta \Big/ \left(\sin\frac{\pi}{2n}\right) \ , \tag{7}$$

$$f \equiv \frac{\sin n\theta + \epsilon \sin 3n\theta}{1-\epsilon} = \frac{w - ib}{ib} \ , \tag{8}$$

$$|\Re(w)| \le a \ , \ 0 \le \Im(w) \le b \ , \tag{9}$$

where $n > 1, a > 0, b > 0, \epsilon$: complex parameter. For the values of θ satisfying $f_\theta = 0$, i.e. $\cos n\theta + 3\epsilon \cos 3n\theta = 0$, the sufficient condition to the existence of a conformal mapping due to equations (7)-(9) is the same as before. For a critical case of a small pure imaginary value of $\epsilon = \epsilon_0, (\theta = \theta_0)$, where $\Re(f+1) = 0$,

$$\epsilon_0 \approx \mp 0.0233816i \ ,$$

$$n\theta_0 \approx -0.75072 \mp 1.33223i \ ,$$

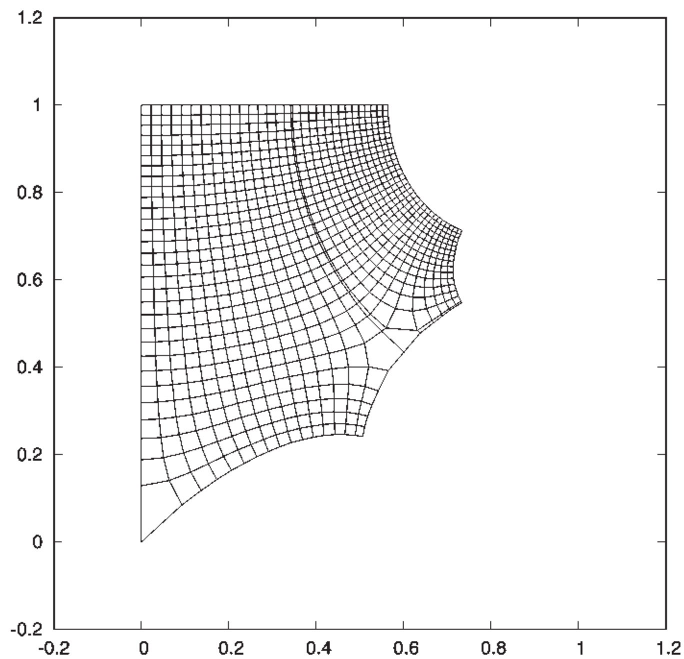

Figure 3: Isolines

$$\Im\{f(\theta_0;\epsilon_0)+1\} = \mp 0.77127 \ .$$

Therefore for the point corresponding to the one root, assuming a small value of $\epsilon(>0)$

$$n\theta = -i\ln\left[\sqrt{3-1/(3\epsilon)}\ /2 - i\sqrt{1+1/(3\epsilon)}\ /2\right]\ , \tag{10}$$

$$\begin{aligned} f+1 &= f(\theta_0;\epsilon_0)+1+\frac{f(\theta_0;\epsilon_0)+\sin 3n\theta_0}{1-\epsilon_0}(\epsilon-\epsilon_0) \\ &\quad +O\left\{(\epsilon-\epsilon_0)^2\right\} \quad (|\epsilon-\epsilon_0|\to 0)\ . \end{aligned} \tag{11}$$

Finally for moderate values of a/b and for small values of $|\epsilon-\epsilon_0|$, existence of conformal mapping becomes

$$-0.0233816 \lesssim 1.2891\ \Re(\epsilon) - |\Im(\epsilon)|\ . \tag{12}$$

Figure 3 shows isolines of $\Re w$ or $\Im w$, i.e. $\Re w =$ constant or $\Im w =$ constant for the case of not producing a conformal map at $\epsilon = 0.055i, n = 3, b/a = 1$ for $\Re w \geq 0$.

3. Discussion

For a given family of functions, as far as the absolute values of the parameter ϵ remains small and the ratio b/a is moderate, conformal maps does not exist if equation (6) or equation (12) does not hold.

4. Conclusions

Existence and non-existence of conformal maps for a family of functions is discussed and concrete examples are shown for a map for a pentagon-like configuration.

References

[1] Peter Henrici, *Applied and Computational Complex Analysis*, Volume 1, Wiley-Interscience (1974).

[2] R. Schinzinger, P.A.A. Laura, Conformal mapping, methods and applications, *Dover* (2003), 98-99.