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# A NOTE ON INITIAL BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER SYSTEMS

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**Abstract:** We consider linear hyperbolic boundary-value problems for second-order systems in a half-space, both for operators with constant coefficients and for operators with coefficients which depend explicitly on the space-variable. Concerning the operator with constant coefficients, we prove, by means of the Fourier-Laplace analysis and the application of Hille-Yosida Theorem, that the problem with homogeneous boundary condition and divergence-free constraint is strongly well-posed in the Sobolev space $H^1(\Omega)$. Furthermore, we prove that the problem admits finite energy surface waves. Next, we discuss the boundary-value problem for a linear second-order differential operator with variable coefficients. A sufficient condition for strong well-posedness is proved, by means of Hille-Yosida Theorem.

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**Key Words:** hyperbolic second-order system, boundary-value problem

## 1. Introduction

This paper deals with linear hyperbolic boundary-value problems for second-order systems with constant and variable coefficients.

Let us denote by $\Omega$ the half-space $\Omega = \Re^{d^{d-1}} \times (0,\infty)$ and by $M_{d\times d}$ the space of $d \times d$ real matrices. In the first two sections we shall be concerned with a second-order IBVP with constant coefficients. Let $u$ be a vector field $u : \Omega \longrightarrow \Re^d$, and let $W$ be the sum of the functions $W(u, \nabla_x u) = W_1(\nabla_x u)+W_2(u, \nabla_x u)+W_3(u)$, where $W_1 : M_{d\times d} \longrightarrow \Re^d$, $W_2 : \Re^d \times M_{d\times d} \longrightarrow$

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$\mathfrak{R}^d$, $W_3 : \mathfrak{R}^d \longrightarrow \mathfrak{R}^d$; $W_1$ and $W_3$ are quadratic forms, $W_2$ is a bilinear form with constant coefficients. We consider the second-order linear differential operator $Q : H^1(\Omega)^d \times L^2(\Omega) \longrightarrow H^{-1}(\Omega)^d$, defined by

$$(Q[u,p])_j = -\sum_{\alpha=1}^{d} \partial_\alpha \left( \frac{\partial W_1}{\partial F_{\alpha j}} + \frac{\partial W_2}{\partial F_{\alpha j}} \right) + \frac{\partial W_3}{\partial F_j} + \partial_j p, \quad j = 1, \cdots, d; \tag{1}$$

and the Neumann-type boundary operator

$$B : H^1(\Omega)^d \times L^2(\Omega) \longrightarrow H^{-1/2}(\partial\Omega)^d,$$

defined by

$$(B[u,p])_j = \frac{\partial W_1}{\partial F_{dj}} + \frac{\partial W_2}{\partial F_{dj}} - p\delta_d^j, \quad j = 1, \cdots, d. \tag{2}$$

We shall study below the following IBVP with homogeneous boundary condition and divergence-free constraint

$$\begin{cases} \partial_t{}^2 u + Q[u,p] = f, & x \in \Omega,\, t \in \mathfrak{R}^d, \\ \mathbf{div}\, u = 0, & x \in \Omega,\, t \in \mathfrak{R}^d, \\ B[u,p] = 0, & x \in \partial\Omega,\, t \in \mathfrak{R}^d; \end{cases} \tag{3}$$

where the function $f = f(x,t)$ is given.

The problem (3) can be written in abstract form by introducing the following notations. Let us denote by $U$ the vector $U = (\partial_t u, \nabla_x u, u)^T$ and by $X$ the linear differential operator $X = (Q[\cdot, p], -\partial_t \nabla_x, -\partial_t)^T$. Thus problem (3) is equivalent to

$$\frac{dU}{dt} + X(U) = F, \tag{4}$$

where $F = (f, 0, 0)^T$ and the domain of the operator $X$ is the Hilbert space $D(X) = \{u \in H^1(\Omega)^d : B[u,p] = 0\, Q[u,p] \in L^2(\Omega)^d; \mathbf{div}\, u = 0\}$. If the homogeneous initial value problem (the problem is said homogeneous in the case where the function $f$ is identically zero) is well-posed, then the operator $X$ generates a continuous semi-group of contractions. If $f$ is different from the null vector, the non-homogeneous problem can be solved by means of Duhamel's formula. Similarly to paper [2], it holds true that if $X$ is a monotone operator and problem (4) is well-posed, then the solution satisfies the a priori estimate

$$e^{-2\gamma T} \|U(T)\|^2_{L^2(\Omega)} + \frac{3}{2}\gamma \int_0^T e^{-2\gamma t} \|U(t)\|^2_{L^2(\Omega)} dt \\ \leq \|U(0)\|^2_{L^2(\Omega)} + \frac{2}{\gamma} \int_0^T e^{-2\gamma t} \|f(t)\|^2_{L^2(\Omega)} dt\,, \tag{5}$$

where $T > 0$ and $\gamma > 0$.

Due to this property, we define the strong well-posedness for problem (3) in the following way.

**Definition 1.1.** If the boundary value problem (3) satisfies an estimate of the form (5), then it is strongly well-posed.

In paper [2], we discussed the strong well-posedness of second-order hyperbolic boundary-value problems of variational type, in the case where the function $W$ is given by $W(u, \nabla_x u) = W_1(\nabla_x u) + W_2(u, \nabla_x u)$. In the same framework of second-order IBVPs of variational type studied by Serre in [3] and [4], through a Fourier-Laplace analysis, we proved that a necessary condition for strong well-posedness of the problem is the Lopatinskii condition. After discussing some properties of the space of the solutions, the main result proved in [2], through Hille-Yosida Theorem, turns out to be a sufficient condition for the well-posedness of the homogeneous BVP.

By applying analogous techniques, the first section of this paper deals with the issues treated in [2], for the more general differential operator (1).

In the second section, we shall turn our attention to the existence of special solutions of the homogeneous IBVP (3), called surface waves. The significant role played by these particular solutions, both for mathematical reasons and for physical applications, has been explained by Benzoni-Gavage and Serre in [1]. As far as the IBVP (3) is concerned, we show that it admits surface waves, which often have finite energy.

In the last part of the paper, we study the well-posedness of general IBVPs in the form (3) , in the case where the function $W$ depends explicitly on the space-variable. Thanks to Hille-Yosida Theorem, assuming that the IBVP (3) with frozen coefficients in a fixed point $x_1 \in \Omega$, is well-posed in $\Omega$ and that the coefficients of $W$ are regular and bounded, we shall prove that (3) admits a unique solution, belonging to the space $C([0, +\infty[; H^1(\Omega)^d)$.

## 2. Initial Boundary-Value Problems with Constant Coefficients

We shall study in this section the strong well-posedness of the BVP (3). As specified in Introduction, by means of Fourier-Laplace analysis, we prove first a necessary condition (Lopatinskii condition); next, we establish a sufficient condition for the strong well-posedness. Our results will be proved through the techniques applied in [2] for a variational type problem.

Let us denote by $x$ the vector of $\Re^d$, $x = (x_1, \cdots, x_d)$ and by $y$ the tangential variables $(x_1, \cdots, x_{d-1})$. If we perform a Fourier transform with respect to the tangential variable $y$ and a Laplace transform with respect to the time variable $t$, we obtain the following problems

$$\begin{cases} \tau^2 v + \hat{Q}_\eta[v,q] = 0, \\ dv_d dx_d + i\eta \cdot \omega = 0, \\ \hat{B}_\eta[v,q] = 0; \end{cases} \tag{6}$$

with $\eta \in \Re^{d^{d-1}}$, $\tau \in \mathbf{C}$; $v = (\omega, v_d)$ and $q$ are the Fourier-Laplace transforms of the functions $u$ and $p$, respectively; the transformed operators of $Q$ and $B$ are denoted by $\hat{Q}_\eta$, $\hat{B}_\eta$, respectively and have the following form

$$\hat{Q}_\eta[v,q] = -\Lambda v'' + i(A_\eta + A_\eta^T)v' + \Sigma_\eta v + iB_\eta v - \Gamma v' + +Dv + \begin{pmatrix} iq\eta \\ q' \end{pmatrix}, \tag{7}$$

where prime denotes the derivative with respect to $x_d$; the matrices $\Lambda, A_\eta, \Sigma_\eta, B_\eta, \Gamma, D$ belong to the space $M_{d\times d}$ ; $\Sigma_\eta$ is quadratic in $\eta$ ; $A_\eta$, $\beta$ are linear in $\eta$ ; $\Lambda$, $\Sigma_\eta$ are symmetric matrices. The boundary operator takes the form

$$\hat{B}_\eta[v,q] = \Lambda v'(0) - iA_\eta^T v(0) + \Gamma v(0) - q(0)e_d. \tag{8}$$

Similarly to the problem studied in [2], we discuss a necessary condition for well-posedness. In accordance with Definition 1.1, let us assume that the boundary-value problem (6) is well-posed. Hence, it holds true that for every $(\tau, \eta) \in \mathbf{C} \times \Re^{d^{d-1}}$, such that either $\mathrm{Re}\tau > 0$ or $\tau = 0$ and $\eta \neq 0$, the null function is the only solution which vanishes if $x_d \to +\infty$. If $(v,q) = (\omega, v_d, q)$ provides a solution to (6) for $(\tau, \eta)$, with $\mathrm{Re}\,\tau > 0$, the pair $(V, Q)$, defined by $V(t, x_d) = e^{\tau t} v(x_d)$ and $Q(t, x_d) = e^{\tau t} q(x_d)$ turns out to be a solution of (6). If we assume that the functions $v, q$ vanish at $+\infty$, then, due to estimate (5) for $(V, Q)$, we deduce that $v$ is identically zero. Similar result we obtain in the case where $\tau = 0$ and $\eta \neq 0$, by means of the estimate (5) applied to the function $V(t, x_d) = tv(x_d)$.

A more detailed analysis of problem (6) enables us to know its space of solutions. If we denote by $Y$ the vector field $Y = \begin{pmatrix} v \\ \omega' \\ q \end{pmatrix}$, $Y \in \mathbf{C}^{2d}$, then, as we shall explain below, we can rewrite system (6) as a first-order linear system for the unknown vector $Y$. In the following proposition we extend a result proved in [2] for a variational type operator. For the sake of completeness, we outline the proof.

**Proposition 2.1.** *Let us fix $\eta \in \Re^{d^{d-1}}, \tau \in \mathbf{C}$ and consider system (6). Assume that the following conditions are satisfied:*

*(i) the matrices $B_\eta$ and $\Gamma$ are symmetric; $\Lambda$ is positive definite;*

*(ii) if $\rho \in \Re^d$, the matrix $M(\eta,\rho) = \rho^2\Lambda - (A_\eta + A_\eta^T)\rho + \Sigma_\eta + D$, is positive definite over the space $(\eta,\rho)^\perp$; in addition, $-\tau^2$ is not an eigenvalue of $M(\eta,\rho)$.*

*Then there exists a suitable matrix $N(\tau,\eta)$ in $M_{2d\times 2d}$, such that system (6) is equivalent to the first-order linear system $Y' = N(\tau,\eta)Y$, whose solution space is the direct sum of the stable and the unstable spaces associated with the matrix $N(\tau,\eta)$.*

*Proof.* The equations of (6) allow us to write the sum $\Lambda \begin{pmatrix} \omega'' \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ q' \end{pmatrix}$ as a linear function of $Y$. Since $\Lambda$ is positive definite, for every fixed vector $Y$, the vector $Y'$ can be uniquely determined in terms of $Y$ and there exists a suitable matrix $N(\tau,\eta)$ such that the linear equations (6) may be written in the form $Y' = N(\tau,\eta)Y$.

The matrix $N(\tau,\eta)$ does not admit pure imaginary eigenvalues: if $N(\tau,\eta)$ had the eigenvalue $i\rho$ for some $\rho \in \Re^d$, then the pair of functions $(v,q)(x_d) = e^{i\rho x_d}(v_0,q_0)$, with $(v_0,q_0) \in \mathbf{C}^d \times \mathbf{C}$, would provide a solution to system (6). From the equations (6), we could obtain indeed

$$\tau^2 v_0 + \rho^2\Lambda v_0 + i(A_\eta + A_\eta^T)i\rho v_0 + \Sigma_\eta v_0 + iB_\eta v_0 - i\rho\Gamma v_0 + Dv_0 + \begin{pmatrix} iq_0\eta \\ i\rho q_0 \end{pmatrix} = 0\,,$$
$$v_0 \cdot (\eta,\rho) = 0.$$

Multiplying the previous equations by $v_0^T$, we deduce $v_0^T(\tau^2 I_d + M(\eta,\rho))v_0 = 0$, since $B_\eta$ and $\Gamma$ are symmetric matrices. Thanks to the second assumption, in the case where $-\tau^2$ is not an eigenvalue of $M(\eta,\rho)$, the solution $(v,q)$ has to be identically null. □

The result of Proposition 2.1 allow us to prove a sufficient condition for the well-posedness of problem (3), by means of Hille-Yosida Theorem. The same conditions assumed to establish the result of Theorem 3.1 in [2], enable us to prove the well-posedness of the IBVP (3), according to Definition 1.1. Let us denote by $< , >$ the Hermitian product in $\mathbf{C}^d$ and by $\hat{X}$ the operator which is obtained from $X$ by performing the Fourier-Laplace transform.

**Theorem 2.1.** *Consider the boundary value problem (3) and assume that the conditions (i) and (ii) of Proposition 2.1 are satisfied. In addition, for every $\eta \in \Re^{d-1}$, and $v \in D(\hat{X})$,*

*(i)* $$\int_0^{+\infty} \mathrm{Re}\,\big(<\Lambda v', v'> + 2 < iA_\eta v', v> + <\Sigma_\eta v, v> + < iB_\eta v, v> + <\Gamma v', v> + <Dv, v>\big)\,dx_d \geq 0;$$

*(ii) let* $p_1 : \mathbf{C}^{3d} \longrightarrow \mathbf{C}^d$ *be the projection operator and* $E_-(1,\eta)$ *be the stable space associated to the matrix* $N(1,\eta)$*; there exists a positive real constant* $\gamma_\eta$ *such that for every* $v \in p_1(E_-(1,\eta))$*,*

$$\int_0^{+\infty} \big(< \Lambda v', v' > +2\mathrm{Re} < iA_\eta^T v', v > + < \Sigma_\eta v, v > + < iB_\eta v, v > \\ + < \Gamma v, v' > + < Dv, v > + < v, v > \big)\, dx_d \geq \gamma_\eta \|v\|^2_{H^1(\mathfrak{R}^{d+})};$$

*(iii) it holds true that* $\{p_1(E_-(1,\eta))\}^\perp \bigcap \left\{\hat{B}_\eta[v,q] : v \in H^1(\mathfrak{R}^{d+})^d\right\} = \emptyset$*, for every* $\eta \in \mathfrak{R}^{d^{d-1}}$*.*

*Then the homogeneous problem (3) is strongly well-posed in* $D(X)$*.*

The proof of the previous result can be performed through the same procedure followed in [2] for Theorem 3.1. The main tool is the application of Hille-Yosida Theorem: the assumptions (i), (ii) and (iii) ensure that the operator is monotone and maximal in $D(X)$. In the case where the system (3) is not homogeneous, the IBVP is solved by means of Duhamel's formula, provided that $f$ is integrable from $(0,T)$ to $D(X)$.

## 3. Surface Waves

A surface wave of the homogeneous IBVP (1) is a non-trivial solution of the form

$$u(x,t) = e^{i(\rho t + \eta \cdot y)} v(x_d), \tag{9}$$

where $\eta \in \mathfrak{R}^{d^{d-1}}$, $\rho \in \mathfrak{R}^d$ and the function $v : \mathfrak{R}^{d+} \longrightarrow \mathbf{C}^d$ is bounded. If $v$ decays to zero at infinity, then the special solution (9) is called finite energy surface wave. Let us remark that the modulus of $u$ does not depend on time; the solution (9) turns out to be a travelling wave in the direction $\eta$ parallel to the boundary, with velocity $-\frac{\rho}{|\eta|}$. The importance of surface waves in the construction of exact solutions for homogeneous first-order hyperbolic BVPs has been explained in [1] by Benzoni-Gavage and Serre.

Concerning second-order IBVP of variational type, Serre proved in [3] and [4] that there exists a pair of surface waves for every $\eta \in \mathfrak{R}^{d^{d-1}}$, $\eta \neq 0$, and these waves are often of finite energy.

We shall be concerned below with the issue of the existence of surface waves for the homogeneous system (3). If a function of the form (9) provides a solution to (3), then $(v,q)$ solves problem (6) for $\tau^2 = -\rho^2$ and some bounded function

$q$. In view of the results proved in Proposition 2.1, the system (6) may be written in equivalent form as a first-order system $Y'(x_d) = N(i\rho, \eta)Y(x_d)$, with $Y = \begin{pmatrix} v \\ \omega' \\ q \end{pmatrix}$, $Y \in \mathbf{C}^{2d}$.

Therefore, the profile $v$ has to be a solution in the form of a sum of exponentials.

Let us look for a function $Y$, such that $Y(x_d) = Y(0)\, e^{\xi x_d}$ for some $\xi \in \mathbf{C}$, that we shall determine below in such a way to obtain a solution of (6). Let us fix $\eta \in \Re^{d^{d-1}}$, $\eta \neq 0$, and substitute $Y(x_d)$ in (6). We obtain

$$\begin{cases} -\rho^2 v_0 - \xi^2 \Lambda v_0 + i(A_\eta + A_\eta^T)\xi v_0 + \Sigma_\eta v_0 + iB_\eta v_0 - \xi\Gamma v_0 + Dv_0 \\ \qquad\qquad + \begin{pmatrix} iq_0\eta \\ \xi q_0 \end{pmatrix} = 0, \\ \xi v_{0d} + i\eta\omega_0 = 0, \\ \Lambda\xi v_0 - iA_\eta^T v_0 - q_0 e_d + \Gamma v_0 = 0; \end{cases} \tag{10}$$

where $v_{0d} = v_0 \cdot e_d$. If $v_{0d}$ is not null, we get $\xi = -i\eta \cdot \omega_0 v_{0d}^{-1}$ and $\rho^2 = v_{0d}^{-1}(iA_\eta \xi v_0 + \Sigma_\eta v_0 + iB_\eta v_0 + Dv_0) \cdot e_d$. If the vector $v_0 \in \mathbf{C}^d$ is fixed in such a way that

(i) $v_{0d}^{-1} \neq 0$, and

(ii)$(iA_\eta \xi v_0 + \Sigma_\eta v_0 + iB_\eta v_0 + Dv_0) \cdot e_d \in \Re^{d^+}$, then for every direction $\eta \neq 0$, there exists a pair of surface waves. Furthermore, if we request that $\mathrm{Re}\,(-i\eta \cdot \omega_0 v_{0d}^{-1}) < 0$, then the surface wave decays exponentially at infinity and its finite energy is given by $\int_0^{+\infty} |v(x_d)|^2\, dx_d$.

## 4. Initial Boundary-Value Problems with Variable Coefficients

In this section we focus on the study of the well-posedness of a second-order IBVP for a system with coefficients that depend explicitly on the space variable.

Let $\Omega$ be the half-space $\Omega = \Re^{d^{d-1}} \times (0, \infty)$ and define the function

$$W(x, u, \nabla_x u) = W_1(x, \nabla_x u) + W_2(x, u, \nabla_x u) + W_3(x, u)\,,$$

where $x \in \Omega$, $u : \Omega \longrightarrow \Re^d$ ; $W_1(x, \cdot) : M_{d\times d} \longrightarrow \Re^d$, $W_3(x, \cdot) : \Re^d \longrightarrow \Re^d$; are quadratic forms for every $x \in \Omega$, $W_2(x, \cdot, \cdot) : \Re^d \times M_{d\times d} \longrightarrow \Re^d$, is a bilinear form. Let us consider the second-order differential operator $Q : \Omega \times H^1(\Omega)^d \times$

$L^2(\Omega) \longrightarrow H^{-1}(\Omega)^d$, defined by

$$(Q[x,u,p])_j = -\sum_{\alpha=1}^{d} \partial_\alpha \left( \frac{\partial W_1}{\partial F_{\alpha j}} + \frac{\partial W_2}{\partial F_{\alpha j}} \right) + \frac{\partial W_3}{\partial F_j} + \partial_j p, \;\; j = 1, \cdots, d; \quad (11)$$

with the Neumann-type boundary operator $B : \Omega \times H^1(\Omega)^d \times L^2(\Omega) \longrightarrow H^{-1/2}(\partial\Omega)^d$,

$$(B[x,u,p])_j = \frac{\partial W_1}{\partial F_{dj}} + \frac{\partial W_2}{\partial F_{dj}} - p\delta_d^j, \;\; j = 1, \cdots, d. \quad (12)$$

We shall study below the well-posedness of the following evolution boundary value problem, with homogeneous boundary condition and divergence-free constraint

$$\begin{cases} \partial_t{}^2 u + Q[x,u,p] = f, & x \in \Omega,\ t \in \Re^d, \\ \mathbf{div}\, u = 0, & x \in \Omega,\ t \in \Re^d, \\ B[x,u,p] = 0, & x \in \partial\Omega,\ t \in \Re^d; \end{cases} \quad (13)$$

where $f$ is a given function.

Let us write problem (13) in abstract form as follows. We denote by $U$ the vector $U = (\partial_t u, \nabla_x u, u)^T$ and by $A$ the linear differential operator $A = (Q[x,\cdot,p], -\partial_t \nabla_x, -\partial_t)^T$. Hence, if $F = (f,0,0)^T$, problem (13) can be written in the form $\frac{dU}{dt} + A(U) = F$, and the domain of the operator $A$ is the linear space $D(A) = \left\{ u \in H^1(\Omega)^d : B[x,u,p] = 0,\ Q[x,u,p] \in L^2(\Omega)^d; \mathbf{div}\, u = 0 \right\}$. We assume that this space is not trivial.

In the following theorem we prove a sufficient condition for the existence of the solution of problem (13), by applying Hille-Yosida Theorem similarly to the constant-coefficient case. As pointed out in Introduction and in [3] and [4], the well-posedness of the IBVP (13) in the half-space $\Omega$ relies on that of IBVPs with frozen coefficients. We assume indeed that problem (13), in the case where the coefficients of the operator are fixed in a point $x_1 \in \Omega$, is well-posed.

**Theorem 4.1.** *Consider the IBVP (13) and assume the following conditions are satisfied:*

*(i) there exist* $x_1 \in \Omega$, *and* $\gamma > 0$ *such that for every* $u \in D(A)$, $\int_\Omega W(x_1, u, \nabla_x u) dx \geq \gamma \|u\|^2_{H^1(\Omega)^d}$;

*(ii) the function* $x \rightarrow W(x,\cdot,\cdot) \in C^1(\Omega)$, *with bounded derivatives;*

*(iii) there exists a positive real constant* $\lambda$, *with* $\gamma > 2\lambda(d^4 + d^3 + d^2)$, *such that the coefficients of the forms* $W_1$, $W_2$ *and* $W_3$ *are bounded in* $\Omega$ *by the constant* $\lambda$.

*Then the evolution boundary value problem (13) admits a unique solution* *in* $D(A)$.

*Proof.* As a first step, we prove that $A$ is a monotone operator in $D(A)$. Let $u$ be a vector field in $D(A)$; we get, integrating by parts,

$$\int_\Omega (Q[x,u,p],u)dx = \int_\Omega \left(-\sum_{\alpha=1}^{d} \partial_\alpha \left(\frac{\partial W_1}{\partial F_{\alpha j}} + \frac{\partial W_2}{\partial F_{\alpha j}}\right) + \frac{\partial W_3}{\partial F_j} + \partial_j p, u\right) dx$$
$$= \int_\Omega \sum_{j=1}^{d} \left(\sum_{\alpha=1}^{d} \left(\frac{\partial W_1}{\partial F_{\alpha j}} + \frac{\partial W_2}{\partial F_{\alpha j}}\right) \partial_\alpha u_j + \frac{\partial W_3}{\partial F_j} u_j - p\partial_j u_j\right) dx$$
$$- \int_{\Re^{d\,d-1}} (B[x,u,p],u)dy = \int_\Omega W(x,u,\nabla_x u)dx. \quad (14)$$

Thanks to the assumptions,

$$\int_\Omega (Q[x,u,p],u)dx = \int_\Omega W(x,u,\nabla_x u)dx$$
$$\geq \int_\Omega (W(x,u,\nabla_x u) - W(x_1,u,\nabla_x u))dx + \gamma\|u\|^2_{H^1(\Omega)^d}$$
$$\geq (-2\lambda(d^4+d^3+d^2)+\gamma)\|u\|^2_{H^1(\Omega)^d} \geq 0. \quad (15)$$

Hence, $A$ is a monotone operator. We prove now that $A$ is maximal.

Let $g \in L^2(\Omega)^d$ and for every $u, v \in D(A)$, consider the bilinear form $\int_\Omega (Q[x,u,p]+u,v)dx$. Since $u, v \in D(A)$, we obtain

$$\int_\Omega (Q[x,u,p]+u,v)dx$$
$$= \int_\Omega \sum_{j=1}^{d} \left(\sum_{\alpha=1}^{d} \left(\frac{\partial W_1}{\partial F_{\alpha j}} + \frac{\partial W_2}{\partial F_{\alpha j}}\right)_j \partial_\alpha v_j + \frac{\partial W_3}{\partial F_j} v_j + u_j v_j\right) dx. \quad (16)$$

For the coefficients of the forms $W_1, W_2$ and $W_3$ are bounded, the bilinear form turns out to be continuous. Furthermore, in view of the previous estimate (15) and the first assumption (i), the form is coercive in $D(A)$. Thanks to Lax-Milgram Theorem, there exist a unique function $\bar{u} \in D(A)$ and $\bar{p} \in L^2(\Omega)$, such that for every $v \in D(A)$, $\int_\Omega (Q[x,\bar{u},\bar{p}]+\bar{u},v)dx = \int_\Omega (g,v)dx$.

Since the function $\bar{u} \in H^1(\Omega)^d$, by applying the result of Meyers-Serrin Theorem, we can approximate $\bar{u}$ by means of a sequence of functions, that belong to $H^1(\Omega)^d \bigcap C^\infty(\Omega)^d$. Hence, let $(\bar{u}_n)_{n\in\mathbf{N}} \subset H^1(\Omega)^d \bigcap C^\infty(\Omega)^d$, such that $\lim_{n\to+\infty} \bar{u}_n = \bar{u}$, strongly in $H^1(\Omega)^d$. Let us consider a function $\phi \in C_0^\infty(\Omega)^d$. We obtain

$$\int_\Omega (Q[x,\bar u_n,\bar p]+\bar u_n,\phi)dx=\int_\Omega \sum_{j=1}^d\left(\sum_{\alpha=1}^d\left(\frac{\partial W_1}{\partial F_{\alpha j}}(x,\nabla_x\bar u_n)\right.\right.$$
$$\left.\left.+\ \frac{\partial W_2}{\partial F_{\alpha j}}(x,\bar u_n,\nabla_x\bar u_n)\right),\partial_\alpha\phi_j+\frac{\partial W_3}{\partial F_j}(x,\bar u_n)\phi_j+\bar u_{nj}\phi_j-\bar p\partial_j\phi_j\right)dx. \quad (17)$$

Therefore, for every $\phi\in C_0^\infty(\Omega)^d$,

$$\lim_{n\to\infty}\int_\Omega (Q[x,\bar u_n,\bar p]+\bar u_n,\phi)dx$$
$$=\int_\Omega \sum_{j=1}^d\left(\sum_{\alpha=1}^d\left(\frac{\partial W_1}{\partial F_{\alpha j}}(x,\nabla_x\bar u)+\frac{\partial W_2}{\partial F_{\alpha j}}(x,\bar u,\nabla_x\bar u)\right),\right.$$
$$\left.\partial_\alpha\phi_j+\frac{\partial W_3}{\partial F_j}(x,\bar u)\phi_j+\bar u_j\phi_j-\bar p\partial_j\phi_j\right)dx=\int_\Omega(Q[x,\bar u,\bar p]+\bar u,\phi)dx. \quad (18)$$

In view of the previous limit relation (18), we obtain $\lim_{n\to\infty}(Q[x,\bar u_n,\bar p]+\bar u_n)=Q[x,\bar u,\bar p]+\bar u$, weakly in $L^2(\Omega)^d$. If $v\in D(A)$, then $\int_\Omega(Q[x,\bar u,\bar p]+\bar u,v)dx=\int_\Omega(g,v)dx$. Thus, $Q[x,\bar u,\bar p]+\bar u=g$, a.e. in $\Omega$ and $A$ turns out to be a maximal operator. Thanks to Hille-Yosida Theorem, the homogeneous problem (13) is well-posed. The non-homogeneous case can be worked out by means of Duhamel's formula. □

## References


[1] S. Benzoni-Gavage, D. Serre, *Multi-Dimensional Hyperbolic Partial Differential Equations,* Oxford University Press (2007).

[2] R. Cavazzoni, On initial boundary-value problems of variational type for second-order systems (2006), To Appear.

[3] D. Serre, Second order initial boundary value problems of variational type, *J. of Functional Analysis*, **236** (2006), 409-446.

[4] D. Serre, Second order initial boundary value problems of variational type: the incompressible case, *Rend. Circolo Mat. Palermo, II*, **78** (2006), 285-312.

[5] D. Serre, *Systems of Conservation Laws 2,* Cambridge University Press (2000).