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BOUNDS OF THE NUMBER OF IMP-SETS IN EDGE-COLOURED GRAPHS

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Abstract: We call the graph G an edge-m-coloured if its edges are coloured with m colours. A path is called monochromatic if all its edges are coloured alike. A subset $S \subset V(G)$ is independent by monochromatic paths if for every pair of different vertices from S there is no monochromatic paths between them. We consider the number $NI_{mp}(G)$ of independent by monochromatic paths sets. We present several lower and upper bounds for $NI_{mp}(G)$ in terms of order, size or independence by monochromatic paths number.

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1. Introduction

Let G be a finite graph of order $n = |V(G)$ and size $m = |E(G)|$. For a vertex $x \in V(G)$ let $\deg_G x$ denote its degree. By P_n, for $n \geq 2$ we mean graph with the vertex set $V(P_n) = \{x_1, ..., x_n\}$ and the edge set $E(P_n) = \{x_i x_{i+1}; i = 1, ..., n-1\}$. Moreover P_1 is a graph with only one vertex. A graph is said to be claw-free if it does not contain the star $K_{1,3}$ as an induced subgraph. Let $X \subset V(G) \cup E(G)$. The notation $G \setminus X$ means the graph obtained from G by deleting the set X. If $X \subset V(G)$ then $G[X]$ denotes a subgraph of G induced by X. Let $e \in E(G)$. Then $G - e$ is called *an edge-deleted subgraph of* G and we write also $G - xy$ if $e = xy$. If x and y are nonadjacent vertices of G then $G + xy$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{x, y\}$. A graph

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G is said to be an edge m-coloured if its edges are coloured with m colours and we say that 1-coloured graph is monochromatic. For $e \in E(G)$ the colour of the edge e we denote by $c_G(e)$ and the colour of a monochromatic subgraph Q of the graph G we denote by $c_G(Q)$. By a path from a vertex x_1 to a vertex x_n, $n \geq 2$ we mean a sequence of vertices $x_1, ..., x_n$ and edges $x_i x_{i+1} \in E(G)$, for $i = 1, ..., n-1$ and for simplicity we denote it by $x_1 ... x_n$. A cycle is a path with $x_1 = x_n$. A path (cycle) is called monochromatic if all its edges are coloured alike. A set S is said to be independent by monochromatic paths of the edge-coloured graph G if for any two different vertices $x, y \in S$ there is no monochromatic path between them. In addition a subset containing only one vertex and the empty set also are independent by monochromatic paths set of G. For convenience throughout this paper we will write an imp-set instead of an independent by monochromatic paths set of G. Every imp-set of G is an independent set of G in the classical sense. For the proper edge-colouring of the graph G an independent set of G is an imp-set of G. The independence by monochromatic paths number of G, denoted by $\alpha_{mp}(G)$ is the cardinalities of a largest imp-set in G. The concept of independence by monochromatic paths generalize independence in the classical sense. Imp-sets in graphs were studied for instance in [2, 5, 7, 6].

The number of imp-sets in G is denoted by $NI_{mp}(G)$. For a graph G on $V(G) = \emptyset$ we put $NI_{mp}(G) = 1$. Let x be an arbitrary vertex of $V(G)$. By $\mathcal{F}_x$ (respectively $\mathcal{F}_{-x}$) we denote the family of imp-sets S of G such that $x \in S$ (respectively $x \notin S$). Of course $\mathcal{F} = \mathcal{F}_x \cup \mathcal{F}_{-x}$ is the family of all imp-sets in G and $NI_{mp}(G) = |\mathcal{F}| = |\mathcal{F}_x| + |\mathcal{F}_{-x}|$. For the proper edge-colouring of the graph G, $NI_{mp}(G) = NI(G)$, where $NI(G)$ is the number of independent sets in G. Prodinger and Tichy in [4] gave impetus to the study of the number $NI(G)$ of independent sets in a graph G. They proved:

Fact 1. *$NI(P_n) = F_{n+1}$ and $NI(C_n) = F_n + F_{n-2}$, for $n \geq 3$, where F_n is the n-th Fibonacci number defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$.*

The literature includes many paper dealing with the theory of counting of independent sets in graph, see for instance [1, 3, 4]. In the chemical literature the graph parameter $NI(G)$ is referred to as the Merrifield-Simmons index. The number $NI_{mp}(G)$ of imp-sets in special edge-coloured graphs were studied in [7]. In this paper we present several lower and upper bounds for $NI_{mp}(G)$ in terms of order, size or independence by monochromatic paths number and we generalize some results for the number $NI(G)$.

We list some obvious facts:

Fact 2. *For an arbitrary edge-coloured graph G we have $NI_{mp}(G) \leq NI(G)$.*

Fact 3. *For an spanning proper subgraph H of an edge-coloured graph G we have $NI_{mp}(G) \leq NI_{mp}(H)$.*

Fact 4. *Let an edge-coloured graph G have components $G_1, ..., G_k$. Then $NI_{mp}(G) = \prod_{i=1}^{k} NI_{mp}(G_i)$.*

That implies that if $NI_{mp}(G)$ is a prime number then G is connected.

Fact 5. *Let G be an edge-coloured connected graph of order n. Then:*

$n + 1 \leq NI_{mp}(G) \leq 2^{n-1} + 1$,

$NI_{mp}(G) = n + 1$ *if G is a monochromatic graph of order n, and*

$NI_{mp}(G) = 2^{n-1} + 1$ *if $G = K_{1,n-1}$ with proper edge-colouring.*

2. Main Results

In this section we give bounds of the number of imp-sets in G.

Theorem 1. *Let G be any edge-coloured graph of order n with independence by monochromatic paths number $\alpha = \alpha_{mp}(G)$. Then $NI_{mp}(G) \geq 2^{\alpha} + n - \alpha$.*

$NI_{mp}(G) = 2^{\alpha} + n - \alpha$ if and only if G is constructed by joining each vertex in a $\overline{K_{\alpha}}$ with at least one vertex in edge-coloured graph $G_{n-\alpha}$ of order $n - \alpha$, with colouring of added edges, such that for every $x, y \in V(G_{n-\alpha})$ and for every $u, v \in V(\overline{K_{\alpha}})$:

(i) there is a monochromatic path $x...y$ in G,

(ii) there is monochromatic path $x...u$ in G, and

(iii) there no monochromatic path $u...v$ in G.

Proof. Let $S \subset V(G)$ denote a maximum imp-set of G. Clearly every subset of S is an imp-set of G and for every $x \in V(G) \setminus S$ the set $\{x\}$ is an imp-set of G. Consequently $NI_{mp}(G) \geq 2^{|S|} + |V(G) \setminus S| = 2^{\alpha} + n - \alpha$. Assume that $NI_{mp}(G) = 2^{\alpha} + n - \alpha$. Then any imp-set of G is either a subset of S or 1-element subset containing any vertex from $V(G) \setminus S$. This implies the following facts:

(a) for every two vertices $x, y \in V(G) \setminus S$ there is a monochromatic path $x...y$ in G and

(b) for every $x \in V(G) \setminus S$ and for every $u \in S$ there is a monochromatic path $x...u$ in G.

This implies construction of the graph G.

From (a) and (b) it follows the construction of the graph G.

Let now G is a graph constructed as in the statement of the theorem. Hence $S = V(\overline{K_\alpha})$ is a maximum imp-set of the graph G and every subset of $V(\overline{K_\alpha})$ is an imp-set of G. Clearly by (ii) there no an imp-set S^* such that $S^* \cap V(\overline{K_\alpha}) \neq \emptyset$ and $S^* \cap V(G_{n-\alpha}) \neq \emptyset$. Moreover by (ii) it follows that only $n-\alpha$ subsets $\{x\}$, where $x \in V(G_{n-\alpha})$ are imp-sets of G. Hence $NI_{mp}(G) = 2^\alpha + n - \alpha$ what completes the proof. □

For the proper edge-colouring of the graph G we obtain result from [3].

Let G be an edge-coloured graph. Let $\mathcal{Q} = \{Q_i\}_{i\in\mathcal{I}}$ be the family of maximal (with respect to set inclusion) monochromatic connected subgraphs of G. We define uncolored simple graph $G(\mathcal{Q})$ as follows: $V(G(\mathcal{Q})) = V(G)$ and $E(G(\mathcal{Q})) = \{v_p v_q; v_p, v_q \in V(Q_i), i \in \mathcal{I}\}$ with replacing multiple edges by one edge. It has been proved:

Theorem 2. (see [7]) *For an arbitrary edge-coloured graph G, $NI_{mp}(G) = NI(G(\mathcal{Q}))$.*

Proposition 1. *Let G be an edge-coloured graph and $x, y \in V(Q_i)$ for $i \in \mathcal{I}$. Then $NI_{mp}(G) = NI_{mp}(G + xy)$ with $c_{G+xy}(xy) = c_G(Q_i)$.*

Proof. The result follows by the fact that graphs $G(\mathcal{Q})$ and $(G + xy)(\mathcal{Q})$ are isomorphic. □

Theorem 3. *Let G be an edge-coloured graph of order n and let $m(\overline{G(\mathcal{Q})})$ denote the number of edges in the graph $\overline{G(\mathcal{Q})}$. Then $NI_{mp}(G) \geq 1 + n + m(\overline{G(\mathcal{Q})})$. $NI_{mp}(G) = 1 + n + m(\overline{G(\mathcal{Q})})$ if and only if $\alpha_{mp}(G) \leq 2$ this means $\overline{G(\mathcal{Q})}$ is triangle-free.*

Proof. Clearly the empty set and every 1-element subset of $V(G)$ are imp-sets of G. Since every edge in the graph $\overline{G(\mathcal{Q})}$ corresponds to an imp-set in G, so the family of imp-sets in G contains exactly $m(\overline{G(\mathcal{Q})})$ two element imp-sets. This implies that $NI_{mp}(G) \geq 1 + n + m(\overline{G(\mathcal{Q})})$.

Assume that $NI_{mp}(G) = 1 + n + m(\overline{G(\mathcal{Q})})$. Then we deduce that every imp-set of G has at most two elements that is $\alpha_{mp}(G) \leq 2$. If $\alpha_{mp}(G) \leq 2$ then the result is obvious. □

Corollary 1. *Let G be an edge-coloured graph of order n and let t denote the number of components in $\overline{G(\mathcal{Q})}$. Then $NI_{mp}(G) \geq 2n + 1 - t$. $NI_{mp}(G) = 2n + 1 - t$ if and only if $\overline{G(\mathcal{Q})}$ is a forest.*

Proof. Since $m(\overline{G(\mathcal{Q})}) \geq n-t$ so by Theorem 3 we obtain that $NI_{mp}(G) \geq 1+n+m(\overline{G(\mathcal{Q})}) \geq 1+n+n-t = 2n+1-t$. If $NI_{mp}(G) = 2n+1-t$ then $m(\overline{G(\mathcal{Q})}) = n-t$ so $\overline{G(\mathcal{Q})}$ is a forest and consequently triangle-free. □

In [3] it has been proved:

Theorem 4. (see [3]) *If G is a connected claw-free graph of order n, then $1+n \leq NI(G) \leq F_{n+1}$. $NI(G) = n+1$ if and only if $G = K_n$ and $NI(G) = F_{n+1}$ if and only if $G = P_n$.*

Let G be an edge-coloured graph and $x \in V(G)$. The *chromaticity degree* $\text{chdeg}_G x$ of the vertex x is defined as the number of colours using for colouring edges incident with the vertex x in the graph G. Note that for the proper edge-colouring of the graph G, $\text{chdeg}_G x = \deg_G x$. The vertex $x \in V(G)$ is an end vertex of monochromatic subgraph if it belongs to at least two monochromatic subgraphs in the graph G. The subset of end vertices in G we denote by $V_e(G)$.

Theorem 5. *Let G be an edge-coloured graph of order n such that for every $x \in V_e(G)$, $\text{chdeg}_G x = 2$. Then $NI_{mp}(G) \leq F_{n+1}$.*

$NI_{mp}(G) = F_n$ if G is a proper edge-colouring graph P_n.

Proof. By Theorem 2 we have that $NI_{mp}(G) = NI(G(\mathcal{Q}))$. To prove the Theorem firstly we will prove the following claim.

Claim 1. *$G(\mathcal{Q})$ is a claw-free graph.*

Assume that $G(\mathcal{Q})$ is not a claw-free graph. This means that $G(\mathcal{Q})$ contains $K_{1,3}$ as induced subgraph. Let $V(G(\mathcal{Q})) \supseteq V(K_{1,3}) = \{y, x_1, x_2, x_3\}$ where y is the center of this star. Clearly by the definition of $G(\mathcal{Q})$ we have that $y \in V_e(G)$ and no two vertices from the set $\{x_1, x_2, x_3\}$ belong to the same monochromatic subgraph in the graph G. This means that $c_G(yx_1) \neq c_G(yx_2) \neq c_G(yx_3)$ what implies $\text{chdeg}_G \geq 3$, contradiction.

Therefore using Theorem 4 the result immediately follows. □

Theorem 6. *Let G be an edge-coloured simple graph and $x \in V_e(G)$. Let $\mathcal{Q}(x) = \{Q_i; i \in \mathcal{I}\}$ be the family of all monochromatic subgraphs containing the vertex x and for every $i, j \in \mathcal{I}$, $Q_i \cap Q_j = \{x\}$ and $\text{chdeg}_G x = |\mathcal{I}|$. Then $NI_{mp}(G) \leq 2NI_{mp}(G-x) - \sum_{i \in \mathcal{I}} |V(Q_i) \setminus \{x\}|$.*

$NI_{mp}(G) = 2NI_{mp}(G-x) - \sum_{i \in \mathcal{I}} |V(Q_i) \setminus \{x\}|$ if and only if for each vertex belonging to $\bigcup_{i \in \mathcal{I}} Q_i$ different from x there is a monochromatic path to every vertex of G.

Proof. For $Q_i \in \mathcal{Q}(x)$ let $V(Q_i) = \{x, y_1^i, ..., y_{t_i}^i\}$, $t_i \geq 1$. Since $x \in V_e(G)$ and $\text{chdeg}_G x = |\mathcal{I}|$, hence $|\mathcal{F}_{-x}| = NI_{mp}(G \setminus \{x\})$. Moreover for every imp-set $S \in \mathcal{F}_x$ the set $S \setminus \{x\} \in \mathcal{F}_{-x}$. This implies $|\mathcal{F}_x| \leq |\mathcal{F}_{-x}|$. Also $|V(Q_i) \setminus \{x\}|$ singletons $\{y_p^i\}$, for all $i \in \mathcal{I}$, $1 \leq p \leq t_i$ belong to $\mathcal{F}_{-x}$ and corresponds to no set $S \setminus \{x\}$ with $S \in \mathcal{F}_x$. Thus $|\mathcal{F}_{-x}| - \sum_{i \in \mathcal{I}} |V(Q_i) \setminus \{x\}| \geq |\mathcal{F}_x|$ which implies $NI_{mp}(G) = |\mathcal{F}_x| + |\mathcal{F}_{-x}| \leq 2|\mathcal{F}_{-x}| - \sum_{i \in \mathcal{I}} |V(Q_i) \setminus \{x\}|$ and completes the proof of the inequality.

If $NI_{mp}(G) = 2NI_{mp}(G - x) - \sum_{i \in \mathcal{I}} | \bigcup_{i \in \mathcal{I}} Q_i \setminus \{x\}|$, then there is a monochromatic path $y_r^i ... z$ for each $z \in V(G) \setminus \{x, y_r^i\}$ for every $i \in \mathcal{I}$ since if no exist monochromatic path $y_r^i ... z$ in G then $\{z, y_r^i\} \in \mathcal{F}_{-x}$ and $\{z, y_r^i, x\} \in \mathcal{F}_x$ which would imply $|\mathcal{F}_x| < |\mathcal{F}_{-x}| - | \bigcup_{i \in \mathcal{I}} Q_i \setminus \{x\}|$. Hence $NI_{mp}(G) < 2NI_{mp}(G - x) - \sum_{i \in \mathcal{I}} |V(Q_i) \setminus \{x\}|$, a contradiction. The converse just is obvious and proof is completed. □

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