International Journal of Pure and Applied Mathematics
Volume 43 No. 3 2008, 367-369

COHERENT SYSTEMS OF TYPE $(n, d, n + 1)$

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a smooth genus g curve. Fix integers $d \geq n > 0$ such that $(n+1)d - ng - n(n+1) \geq 0$. Here we use a paper by L. Brambila-Paz to show the existence of a coherent system (E, V) of type $(n, d, n+1)$ on X with E semistable, V spanning a rank n subsheaf of E and which is α-stable for all $\alpha > 0$.

AMS Subject Classification: 14H60
Key Words: vector bundle on curves, coherent systems on curves, Brill-Noether theory for vector bundles

*

This note is an afterthought after rereading [2]. We fix an integer $g \geq 2$ and only look at smooth and connected genus g curves defined over an algebraically closed field with $\mathrm{char}(\mathbb{K}) = 0$. For all integers n, d, k set $\beta_g(n, d, k) := n^2(g-1) + 1 - k(k - d + n(g-1))$. The integer $\beta_g(n, d, k)$ is the Brill-Noether number for rank n vector bundles with degree n and at least k sections on a genus g curve. For the main definitions and foundational results on coherent system on smooth projective curves, see [1].

Theorem 1. *Let X be a smooth genus g curve. Fix integers $d \geq n > 0$ such that $\beta_g(n, d, n+1) \geq 0$. Then there is a coherent system (E, V) of type $(n, d, n+1)$ on X with E semistable, V spanning a rank n subsheaf of E and which is α-stable for all $\alpha > 0$.*

Received: December 12, 2007

Proof. Let Y be a general genus g curve. Tthen the result is true (and sharp) with the word "stable" instead of the word "semistable" ([2], part (1) of Theorem 2). A standard degeneration argument prove the existence of a coherent system (E, V) of type $(n, d, n+1)$ on X such that E is semistable and (E, V) is α-semistable for all $\alpha \gg 0$. We claim that V generically spans E and that (E, V) is α-stable and not only α-semistable for $\alpha \gg 0$. The claim is a well-known consequence of the α-semistability of (E, V) for all $\alpha \gg 0$. Indeed, if V spans a subsheaf F of E with $\mathrm{rank}(F) < n$, then the coherent subsystem (F, V) shows that (E, V) is not α-semistable for $\alpha \gg 0$. Assume $\alpha \gg 0$ and that (E, V) is α-semistable, but not α-stable. Take a coherent subsystem (F, W) of (E, V) wth $r := \mathrm{rank}(F) \leq n - 1$ and $\mu_\alpha(F, W) = \mu_\alpha(E, V)$. Since E is semistable, $\mu(F) \leq \mu(E)$. Hence $\dim(W)/r \geq (n + 1)/n$. Since $r < n$, we get $\dim(W) \geq r + 2$. Two equalities $\mu_\beta(F, W) = \mu_\beta(E, V)$ and $\mu_\gamma(F, W) = \mu_\gamma(E, V)$ with $\beta \neq \gamma$ give a contradiction. We also got that $\dim(V \cap H^0(X, G)) \leq \mathrm{rank}(G)$ for every subsheaf G of E with rank at most $n - 1$. Since E is semistable, the inequality just checked shows that the coherent system (E, V) is α-stable for all small positive α's. The connectedness of the set of all real numbers β such that a coherent system is β-stable gives that (E, V) is α-stable for all $\alpha > 0$. □

Remark 1. If $\beta_g(n, d, k) > 0$ and X has general moduli, there there exists (E, V) as above with V spanning E. If $n = 1$ this is not true for curves with low gonality (e.g. on a hyperelliptic curve any spanned special line bundle has even gonality). For something similar when $n \geq 2$, see [3].

Remark 2. Fix an integer $n \geq 2$. Let X be a smooth genus g curve such that $W^r_d(X) = \emptyset$ for all integers r, d such that $1 \leq r \leq n-1$ and $\beta_g(1, d, r+1) < 0$. Then all the results of [2], §3, on coherent systems (E, V) on X are true and with the same proof if $\mathrm{rank}(E) \leq n$. Indeed, we may apply [2], Lemma 3.1, to all subsheaves of E with smaller rank. Fix an integer d such that $\beta_g(1, d, n+1) \geq 0$. If we also assume that $\dim(W^n_t(X)) = \beta_g(1, t, n + 1)$ for all $t \leq d$, then also [2], Proposition 4.1 and part (1) of Theorem 4.3, are true with exactly the same proof. Moreover, [2], Theorem 4.8, except perhaps the last sentence, i.e it is true if either $\beta < g$ or $\beta = g$, $g \equiv 0 \pmod{n}$ and $(g, n) \neq (2, 2)$, or $g \leq n^2 - 2$.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] S.B. Bradlow, O. García-Prada, V. Muñoz, P.E. Newstead, Coherent systems and Brill-Noether theory, *Internat. J. Math.*, **14**, No. 7 (2003), 683-733.

[2] L. Brambila-Paz, Non-emptiness of moduli spaces of coherent system, *ArXiv:* math/0412285; *Internat. J. Math.*, To Appear.

[3] V. Mercat, Clifford's Theorem and higher rank vector bundles, *Internat. J. Math.*, **13** (2002), 785-796.