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ESTIMATES FOR TURBULENCE MODELS

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Abstract: The present results are based upon the assumption that the three–dimensional Navier-Stokes equations possess unique regular solutions and can be adopted as the basis for a Reynolds decomposition and the introduction of turbulence models. These Reynolds averaged equations, for specific turbulence models, are studied using energy methods. It is found that there is also a norm closure problem to address, in addition to the classical closure of the Reynolds decomposition, which again results from the nonlinearities in the equations. Relationships between model constants, required to obtain norm convergence for selected turbulence models, are found.

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1. Background

Turbulence models form an integral part of many computational studies of fluid motion since numerical solutions of the instantaneous Navier-Stokes equations are not economically viable. Results for selected turbulence models are obtained below using standard energy methods; uniqueness results will be reported elsewhere. Foias et al [1] have examined some properties of fluid turbulence using functional methods and a mean value operator applied to the instantaneous equations. Such estimates are not directly related to the present study which concerns turbulence models. It can be noted that the large eddy simulation method has been studied in considerable detail (see Berselli et al [2] for example)

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but that work is also distinct from the present study. The present estimates speak to the stability of the dynamical system generated by a given turbulence model rather than the ability of the model to predict flow features. The latter is a separate issue that can only be resolved by comparison of computed results with experimental data. The estimates herein are compared with the standard energy estimate for the instantaneous Navier-Stokes equations as reproduced in Section 2.

Energy estimates work with global norms, such as that for the instantaneous velocity $|\mathbf{v}|_g^2 = \int_{\mathcal{D}} \langle \mathbf{v}, \mathbf{v} \rangle \, dV$, whose components must be square integrable on domain for the analysis to proceed. The inner product of vectors in $\mathbb{R}^3$, $\langle \mathbf{a}, \mathbf{c} \rangle = \Sigma_i a_i c_i$, provide a corresponding local norm $|\mathbf{z}|^2 = \langle \mathbf{z}, \mathbf{z} \rangle$. The regularity assumption appears at all stages of the development as the need for norm quantities to have a finite upper bound. In what follows, regularity of the Navier-Stokes equations is assumed and spatial gradients of the equations taken. This assumption is unsatisfactory from the mathematical viewpoint but does allow construction of engineering estimates. Since the Navier-Stokes equations are nonlinear there is an unclosed hierarchy of norm equations. This happenstance directly parallels the situation in the Reynolds decomposition with its classical "closure problem". In the development below, the norm and the Reynolds decomposition closure problems are interlaced.

The equations of interest for *constant density motion* can be written (with $\mathbf{x} \in \mathbb{R}^3$ the space variable and $t \in \mathbb{R}$ the time) as:

$$d\mathbf{v}/dt = \nu \nabla^2 \mathbf{v} - \nabla(P) + \mathbf{f}; \qquad \mathrm{div}\,(\mathbf{v}) = 0\,, \tag{1}$$

$d(\cdot)/dt \equiv \partial(\cdot)/\partial t + \mathbf{v} \cdot \partial(\cdot)/\partial \mathbf{x}$ represents the material derivative and P denotes the fluid pressure normalized by the constant density ρ. The kinematic viscosity coefficient is written as ν. Any smooth body force present is denoted by $\mathbf{f}(\mathbf{x})$ (per unit mass) and is assumed to be a given function of the spatial variables but independent of time t. The velocity gradient, $\mathbf{L} = \partial \mathbf{v}/\partial \mathbf{x}$ is assumed to exist and has the evolution:

$$d\mathbf{L}/dt + \mathbf{L}\mathbf{L} = \nu \nabla^2 \mathbf{L} - \mathbf{P} + \mathbf{F}; \qquad \mathrm{trace}\,(\mathbf{L}) = 0\,. \tag{2}$$

It is assumed here that the velocity and pressure fields are sufficiently smooth for equation (2) to be well defined: in particular the velocity gradient norm, $\|\mathbf{L}\|$, is assumed to exist and be bounded for all $\mathbf{x}$ and t. The velocity gradient $\mathbf{L} \in \mathbb{M}_3$ ($\mathbb{M}_3$ being the space of all linear operators on $\mathbb{R}^3$) is not, in general, symmetric. $\mathbf{P} = \partial^2 P/\partial x_i \partial x_j$ in equation (2) represents the pressure Hessian and $\mathbf{F} = \nabla(\mathbf{f})$ is the body force gradient with $\|\mathbf{F}\|$ assumed to be bounded. Introduce the local inner product $\langle\!\langle \mathbf{A}, \mathbf{B} \rangle\!\rangle = \mathrm{trace}\,(\mathbf{A}\mathbf{B}^T)$ and the induced local norm $\|\mathbf{A}\|^2 = \langle\!\langle \mathbf{A}, \mathbf{A} \rangle\!\rangle$ on $\mathbb{M}_3$. This local norm implies a global norm given as $\|\mathbf{A}\|_g^2 = \int_{\mathcal{D}} \|\mathbf{A}\|^2 \, dV$ provided that each component of $\mathbf{A}(\mathbf{x})$ is a square integrable function over the domain $\mathcal{D}$. Any $\mathbf{A} \in \mathbb{M}_3$ has a natural decomposition which may be illustrated for the velocity gradient $\mathbf{L}$ as $\mathbf{L} = \mathbf{D} + \mathbf{W}$ with symmetric part $\mathbf{D}$, *the stretching*, and skew part $\mathbf{W}$ *the spin* whose axial vector is half the vorticity $\zeta = \mathrm{curl}\,(\mathbf{v})$.

Attention, herein, is restricted to constant density motion, with boundary conditions periodic on a compact spatial domain $\mathcal{D}$ (with boundary $\partial\mathcal{D}$). The results presented below extend those given in Moulden [3]. The difficulties associated with the periodic boundary condition assumption in turbulent flow are noted in Davidson [4] and the extension to non–periodic boundary conditions will be addressed separately. Certainly, periodic boundary conditions have little practical application. The periodic boundary value problem does suggest, on physical grounds, that the instantaneous velocity norm must decay to zero as $t \to \infty$, when no working, $\langle \mathbf{f}, \mathbf{v} \rangle$, adds energy to the system. This condition leads to constrains on turbulence model constants.

2. Standard Estimate for the Navier-Stokes Equations

The development of a standard energy estimate for the instantaneous Navier-Stokes equations was given in Foias et al [5], for example, and starts by constructing a mechanical energy equation from equation (1). Then an integral over the compact periodic spatial domain $\mathcal{D}$, finds:

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{v}|_g^2 + \nu \|\mathbf{L}\|_g^2 = \langle \mathbf{v}, \mathbf{f} \rangle_g \,. \tag{3}$$

The mass invariance constraint has been incorporated in the formation of the divergence term which vanishes when integrated over a periodic domain. The right hand side of equation (3) can be positive or negative depending upon the relative values of the velocity and body force: it may, or may not, be dissipative. This happenstance is confounded by the Cauchy–Schwarz inequality, where there is $\langle \mathbf{v}, \mathbf{f} \rangle_g \le |\langle \mathbf{v}, \mathbf{f} \rangle|_g \le |\mathbf{v}|_g |\mathbf{f}|_g$ and the final estimate only provides an upper bound. Equation (3) also demonstrates the norm closure problem alluded to above: the evolution of $|\mathbf{v}|_g$ involves the velocity gradient norm $\|\mathbf{L}\|_g$. Norm closure is effected in equation (3) by adoption of the Poincaré inequality which requires that $|\mathbf{L}|_g^2 \ge L_v^2 |\mathbf{v}|_g^2$; $L_v > 0$ some constant. The differential equation (3) reduces to the differential inequality:

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{v}|_g^2 \le -\nu L_v^2 |\mathbf{v}|_g^2 + |\mathbf{v}|_g |\mathbf{f}|_g \le -\frac{\nu L_v^2}{2}|\mathbf{v}|_g^2 + \frac{1}{2\nu L_v^2}|\mathbf{f}\,|_g^2$$

when the Gronwall Lemma gives:

$$|\mathbf{v}|_g^2(t) \leq |\mathbf{v}|_g^2(0)\,\exp[-\nu L_v^2 t] + |\mathbf{f}\,|_g^2\{1-\exp[-\nu L_v^2 t]\}/(\nu L_v^2)^2\,. \tag{4}$$

There is the limit condition:

$$|\mathbf{v}|_g^2(t) \leq |\mathbf{f}\,|_g^2/(\nu L_v^2)^2 \quad \text{as } t \to \infty\,,$$

assuming that $|\mathbf{v}|_g(t=0)$ is bounded as a constraint on the initial data. The attractor is bounded by a ball of diameter $|\mathbf{f}|_g^2/(\nu L_v^2)$ which defines an absorbing set, $\mathbb{B}_A$, for the motion. Turbulent motion in $\mathbb{B}_A$ is not denied since the *local* velocity field, $\mathbf{v}(\mathbf{x},t)$, for large t has not been specified in estimate (4). The initial energy, $|\mathbf{v}|_g^2(t=0)$, does not influence the diameter of $\mathbb{B}_A$ which depends only on fluid properties and the given body force. In particular, diam $(\mathbb{B}_A) \to 0$ as $|\mathbf{f}|_g \to 0$ or as $\nu \to \infty$. If $|\mathbf{f}|_g \equiv 0$ then:

$$|\mathbf{v}|_g^2(t) \leq \exp[-\nu L_v^2 t]\,|\mathbf{v}|_g^2(t=0)\,,$$

as $t \to \infty$ to give a distinct attractor at the origin of phase space in accord with the physical expectation noted above. The limit $\nu \to 0$ shows that the energy, $|\mathbf{v}|_g$, is an invariant for the Euler equations if $|\mathbf{f}|_g \equiv 0$.

3. Reynolds Decomposition

The results in the previous section have provided information about solutions to the instantaneous three–dimensional Navier-Stokes equations for the initial and boundary data specified therein. As such it implies that the viscous decay of $|\mathbf{v}|_g$ is independent of the local kinematics: be they laminar or turbulent. Turbulent flows could have a larger dissipation rate than do laminar flows and still be consistent with the estimate. The results for turbulence models must be consistent with estimate (4). However, estimates for turbulence models are for the mean velocity field, $\mathbf{V}(\mathbf{x})$, and so prohibit direct comparison.

As is classical for turbulent flow, place:

$$\mathbf{v} \mapsto \mathbf{V} + \mathbf{u} \text{ with } \mathbf{V} = \mathcal{E}(\mathbf{v});\;\; \mathcal{E}(\mathbf{u}) = \mathbf{0}\,,$$

$$P \mapsto \overline{P} + p' \text{ with } \overline{P} = \mathcal{E}(P);\;\; \mathcal{E}(p') = 0\,,$$

for some mean value operator $\mathcal{E}$. Here $\overline{P}$ denotes the mean pressure normalized by the density while p' denotes the corresponding normalized pressure fluctuations. With the above regularity assumptions:

$$\mathbf{L} \mapsto \overline{\mathbf{L}} + \mathbf{L}';\;\; \overline{\mathbf{L}} = \partial\mathbf{V}/\partial\mathbf{x} \equiv \mathcal{E}(\mathbf{L});\;\; \mathbf{L}' = \partial\mathbf{u}/\partial\mathbf{x}\,.$$

This decomposition can be placed in equation (2) to yield an evolution equation for $\|\overline{\mathbf{L}}\|_g$ as explored in Section 5. Now inequalities relate the mean and instantaneous norms since the Reynolds decomposition requires that:

$$|\mathbf{v}| \leq |\mathbf{V}| + |\mathbf{u}|; \qquad \|\mathbf{L}\| \leq \|\overline{\mathbf{L}}\| + \|\mathbf{L}'\| \tag{5a,b}$$

which serve as constraints, for all time, upon the decay of both the mean and fluctuating velocity and velocity gradients. Since $|\mathbf{v}|$ is assumed to be bounded, so are $|\mathbf{V}|$ and $|\mathbf{u}|$. Equation (5a) implies that if $|\mathbf{v}| \to 0$ as $t \to \infty$ then both $|\mathbf{V}|$ and $|\mathbf{u}|$ must vanish at that limit. $\|\mathbf{L}\|_g$ (as well as $\|\overline{\mathbf{L}}\|_g$) is unknown but evolution equations can be found for these quantities from equation (2) when regularity is assumed. However, these derived equations contain such terms as $\|\nabla\mathbf{L}\|_g$ and a closed set of norm equations is denied. This is in direct parallel with the closure problem for the Reynolds decomposition and has the same origin: Davidson [4] discusses the latter. Replacing $\|\overline{\mathbf{L}}\|_g \geq L_V|\mathbf{V}|_g$ by the Poincaré inequality in the mean motion equations is referred to as *zero order* norm closure. Replacing $\|\nabla\overline{\mathbf{L}}\|_g \geq L_L\|\overline{\mathbf{L}}\|_g$ in the evolution equation for $\|\overline{\mathbf{L}}\|_g$ is referred to as *first order* norm closure. No evolution equation for pressure exists in constant density motion and the pressure field only serves to ensure that the velocity field is solenoidal. The term $\int_{\mathcal{D}} \langle \nabla(\overline{P}), \mathbf{V}\rangle \, dV \equiv \int_{\mathcal{D}} \nabla(\overline{P}\mathbf{V})\, dV$ vanishes identically for *periodic boundary* conditions. The same is true for the pressure Hessian term in the $\overline{\mathbf{L}}$ equation but in a more subtle way since $\int_{\mathcal{D}} \operatorname{trace}(\mathbf{P}\mathbf{L}^T) dV \equiv 0$ in this case.

Now apply the mean value operator $\mathcal{E}$ and introduce the Reynolds tensor $\boldsymbol{\mathcal{R}} = \mathcal{E}(\mathbf{u}\otimes\mathbf{u})$ when the Navier-Stokes equations (1) reduce to the mean motion equations (Davidson [4]):

$$\partial\mathbf{V}/\partial t + \mathbf{V}\cdot\overline{\mathbf{L}} + \operatorname{div}(\boldsymbol{\mathcal{R}}) + \nabla(\overline{P}) = \nu\nabla^2(\mathbf{V}) + \mathbf{f}; \qquad \operatorname{div}(\mathbf{V}) = 0. \tag{6}$$

This equation implies the norm evolution of the mean motion:

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + \nu\|\overline{\mathbf{L}}\|_g^2 = \langle\!\langle \overline{\mathbf{L}}, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g + \langle \mathbf{V}, \mathbf{f}\rangle_g\,. \tag{7}$$

Further development cannot be undertaken until estimates for the terms involving the Reynolds tensor are either modeled directly or are obtained from an evolution equation for $\|\boldsymbol{\mathcal{R}}\|_g$. For historical reasons start with the zero order Reynolds closure model and also adopt zero order norm closure:

Example 1. (Simple Boussinesq Model: $\boldsymbol{\mathcal{R}} = -\epsilon\,\overline{\mathbf{D}}$) This specification of the Reynolds tensor is sufficient to close equation (6) and so provide an estimate for $|\mathbf{V}|_g$ from equation (7). Assume that the scalar eddy viscosity, ϵ, is a global positive constant so that:

$$\langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}} \rangle\!\rangle_g = -\epsilon\,\langle\!\langle \overline{\mathbf{D}}, \overline{\mathbf{L}} \rangle\!\rangle_g \equiv -\epsilon\,\|\overline{\mathbf{D}}\|_g^2 = -\epsilon\,\|\overline{\mathbf{L}}\|_g^2/2\,,$$

where the last equality is just that due to Korn for constant density motion:

see Horgan [6]. Hence the estimate for $|\mathbf{V}|_g$ becomes:

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{V}|_g^2 \leq -\epsilon_T L_V^2|\mathbf{V}|_g^2 + |\mathbf{f}|_g\,|\mathbf{V}|_g \leq -\frac{\epsilon_T L_V^2}{2}|\mathbf{V}|_g^2 + \frac{1}{2\epsilon_T L_V^2}|\mathbf{f}\,|_g^2$$

provided that $\epsilon_T \equiv \nu + \epsilon/2 > 0$. Gronwall gives the inequality:

$$|\mathbf{V}|_g^2(t) \leq |\mathbf{V}|_g^2(0)\,E(t) + |\mathbf{f}\,|_g^2[1 - E(t)]/(\epsilon_T L_V^2)^2; \quad E(t) \equiv \exp[-\epsilon_T L_V^2 t]$$

to define an absorbing set diameter as $t \to \infty$. This limit diameter defined from, $|\mathbf{V}|_g(t)|_{t\to\infty} \leq |\mathbf{f}|_g/(\epsilon_T L_V^2)$, depends upon ϵ_T as well as $|\mathbf{f}|_g$ and is smaller than the corresponding one obtained from the instantaneous motion in equation (4) since $\epsilon_T > \nu$. $|\mathbf{V}|_g$ decays faster in time than does $|\mathbf{v}|_g$.

By relating $\boldsymbol{\mathcal{R}}$ directly to $\overline{\mathbf{D}}$, the Reynolds decomposition is, in essence, occluded: nothing is added beyond a trivial augmentation of the Lamé constant as $\nu \mapsto \nu + \epsilon/2$. This example, like the model, is of little interest.

It can be noted that the uniqueness result in Gurtin [7] for equation (1) can be extended to equation (6) for the mean motion. In this case the mean velocity field $\mathbf{V}(\mathbf{x})$ is unique provided that the tensor $\boldsymbol{\mathcal{R}}+\overline{P}\mathbf{I}$ is non–unique to the extent of an additive divergence free tensor–valued gauge function $\mathbf{A}(\mathbf{x},t) \in \mathbb{M}_3$. The result again depends upon a regularity assumption.

Further developments must include more properties of the Reynolds tensor. To this end consider the *second moment equation* in the form (see Davidson [4] for example):

$$\frac{\partial \mathcal{R}_{ij}}{\partial t} + V_k\frac{\partial \mathcal{R}_{ij}}{\partial x_k} + \mathcal{R}_{jk}\frac{\partial V_i}{\partial x_k} + \mathcal{R}_{ik}\frac{\partial V_j}{\partial x_k} - \nu\frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2} + \Psi_{ij} = 0 \tag{8}$$

as an evolution equation for the Reynolds tensor $\boldsymbol{\mathcal{R}}$. The notation:

$$\Psi_{ij} = 2\nu\mathcal{E}\left[\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right] + \frac{\partial}{\partial x_k}\mathcal{E}(u_i u_j u_k) + \psi_{ij} \equiv \mathbf{E}_T|_{ij} + \mathbf{D}_F|_{ij} + \psi_{ij}$$

has been introduced with $\mathbf{E}_T$ the turbulence dissipation and $\mathbf{D}_F$ the velocity diffusion term. Here ψ_{ij} is given as the linear combination:

$$\psi_{ij} = \left[\frac{\partial}{\partial x_i}\mathcal{E}(u_j p) + \frac{\partial}{\partial x_j}\mathcal{E}(u_i p))\right] - \mathcal{E}\left[p\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right] \equiv \mathbf{D}_P|_{ij} + \mathbf{S}_P|_{ij}$$

of the pressure strain and pressure diffusion. Hence $\boldsymbol{\Psi} = \mathbf{E_T} + \mathbf{D_F} + \mathbf{D_P} + \mathbf{S_P}$ are the terms requiring a turbulence model: that is, any statement of the form: $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\boldsymbol{\mathcal{R}}, \overline{\mathbf{k'_T}})$. $\boldsymbol{\Psi}$ is estimated in example (2) of Section 4 below.

The trace of (8) produces the *turbulence kinetic energy equation*:

$$\frac{\partial \overline{k'_T}}{\partial t} + V_k\frac{\partial \overline{k'_T}}{\partial x_k} + \mathcal{R}_{ik}\frac{\partial V_i}{\partial x_k} - \nu\frac{\partial^2 \overline{k'_T}}{\partial x_k \partial x_k} + \frac{\psi}{2} = 0\,, \tag{9}$$

where $\psi \equiv \Psi_{ii}$ is the trace of $\boldsymbol{\Psi}$ with:

$$\psi/2 \equiv \nu\left[\left(\frac{\partial u_i}{\partial x_k}\right)^2\right] + \frac{1}{2}\frac{\partial}{\partial x_k}\mathcal{E}[u_i^2 u_k] + \frac{\partial}{\partial x_i}\mathcal{E}(pu_i)$$

a combination of diffusion and dissipation. Here, $\overline{k}_T = \langle \mathbf{V}, \mathbf{V}\rangle/2$ is the mean motion kinetic energy while $k'_T = \langle \mathbf{u}, \mathbf{u}\rangle/2$ denotes the kinetic energy of the velocity fluctuations. The definition of $\boldsymbol{\mathcal{R}}$, and the realizability constraints, in Schumann [8] for example, show that:

$$0 \leq \|\boldsymbol{\mathcal{R}}\|^2 \leq 4|\overline{k'_T}|^2; \qquad |\overline{k'_T}| = \mathcal{E}(\langle \mathbf{u}, \mathbf{u}\rangle/2) = \text{trace}\,(\boldsymbol{\mathcal{R}})/2 \tag{10}$$

and $\|\boldsymbol{\mathcal{R}}\|$ is bounded above by $2|\overline{k'_T}|$ with a similar statement for the global norm $\|\boldsymbol{\mathcal{R}}\|_g \leq 2|\overline{k'_T}|_g$. Extract from equation (8) the evolution for $\|\boldsymbol{\mathcal{R}}\|_g$:

$$\frac{\partial}{\partial t}\|\boldsymbol{\mathcal{R}}\|_g^2 + 2\nu\|\nabla\boldsymbol{\mathcal{R}}\|_g^2 + 4\langle\!\langle \overline{\mathbf{L}}\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}}\rangle\!\rangle_g + 2\langle\!\langle \boldsymbol{\Psi}, \boldsymbol{\mathcal{R}}\rangle\!\rangle_{\mathbf{g}} = \mathbf{0} \tag{11}$$

again for periodic boundary conditions. Similarly from equation (9):

$$\frac{\partial}{\partial t}|\overline{k'_T}|_g^2 + 2\nu|\nabla(\overline{k'_T})|_g^2 + \langle \psi, \overline{k'_T}\rangle_g + 2\langle\!\langle \overline{k'_T}\boldsymbol{\mathcal{R}}, \overline{\mathbf{L}}\rangle\!\rangle_g = 0 \tag{12}$$

Equations (11) and (12) can only be closed if the functions $\boldsymbol{\Psi}$ and ψ are given by turbulence models.

4. Reynolds Stress Model with Order Zero Norm Closure

To put the above theory into perspective consider a simple Reynolds stress turbulence model which allows equations (7), (11) and (12) to be integrated directly. Section 5 extends this example to first order norm closure.

Example 2. (A Reynolds Stress Model) Select an invariant model (of the form developed in Lewellen [9] for example). All the model constants, $\alpha_{(i)}$, $\beta_{(i)}$, introduced herein are assumed to be positive (and are *not defined in the same way* as those in the stated reference) and, in general, are dimensional. From the definitions associated with equation (8) the following model will be adopted for illustration:

a) The dissipation: $\mathbf{E}_T|_{ij} = 2\nu\mathcal{E}(L'_{ik}L'_{jk})$ can be modeled as a linear function of $\boldsymbol{\mathcal{R}}$. So, for simplicity, take the linear combination $\mathbf{E}_T = \alpha_e\boldsymbol{\mathcal{R}} + \alpha_i\,\text{trace}\,(\boldsymbol{\mathcal{R}})\mathbf{I}$ when there is:

$$\langle\!\langle \mathbf{E}_T, \boldsymbol{\mathcal{R}}\rangle\!\rangle_g = \alpha_e\|\boldsymbol{\mathcal{R}}\|_g^2 + 4\alpha_i|\overline{k'_T}|_g^2 \geq (\alpha_e + \alpha_i)\|\boldsymbol{\mathcal{R}}\|_g^2\,.$$

b) For the velocity diffusion: $\mathbf{D}_F|_{ij} = \partial\mathcal{E}(u_i u_j u_k)/\partial x_k$ adopt the simple algebraic diffusive model: $\mathbf{D}_F = \alpha_f \boldsymbol{\mathcal{R}}$ which implies:

$$\langle\!\langle \mathbf{D}_F, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g = \alpha_f \|\boldsymbol{\mathcal{R}}\|_g{}^2 .$$

c) Model the pressure strain by means of the Reynolds anisotropy tensor:

$$\mathbf{S}_P = -2\mathcal{E}(p'\mathbf{D}') = \alpha_s \left[\boldsymbol{\mathcal{R}} - \mathrm{trace}\,(\boldsymbol{\mathcal{R}})\mathbf{I}/3\right],$$

with dimensional constant α_s. Now $\langle\!\langle \mathbf{S}_P, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g$ evaluates as:

$$\langle\!\langle \mathbf{S}_P, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g = \alpha_s \langle\!\langle (\boldsymbol{\mathcal{R}} - \mathrm{trace}\,(\boldsymbol{\mathcal{R}})\mathbf{I}/3), \boldsymbol{\mathcal{R}} \rangle\!\rangle_g = \alpha_s \|\boldsymbol{\mathcal{R}}\|_g^2 - 4\alpha_s |\overline{k'_T}|_g^2/3 .$$

Finally:

d) The pressure diffusion: $\mathbf{D}_P|_{ij} = \partial\mathcal{E}(p'u_j)/\partial x_i + \partial\mathcal{E}(p'u_i)/\partial x_j$. In the present exploratory study, the pressure diffusion will be subsumed into the velocity diffusion discussed above. Hence:

$$\begin{aligned}\langle\!\langle \boldsymbol{\Psi}, \boldsymbol{\mathcal{R}} \rangle\!\rangle_{\mathbf{g}} &= (\alpha_{\mathbf{e}} + \alpha_{\mathbf{s}} + \alpha_{\mathbf{f}}) \|\boldsymbol{\mathcal{R}}\|_{\mathbf{g}}^{\mathbf{2}} + \mathbf{4}(\alpha_{\mathbf{i}} - \alpha_{\mathbf{s}}/\mathbf{3}) |\overline{\mathbf{k'_T}}|_{\mathbf{g}}^{\mathbf{2}} \\ &\geq (\alpha_e + 2\alpha_s/3 + \alpha_f + \alpha_i) \|\boldsymbol{\mathcal{R}}\|_g^2 \end{aligned} \qquad (13)$$

on using inequality (10) and assuming that $\alpha_i > \alpha_s/3$. Physically, the latter inequality assumes that the pressure strain (controlled by constant α_s) is a small contribution to the model for $\boldsymbol{\Psi}$ relative to the isotropic dissipation term (controlled by the parameter α_i). Taking the trace of equation (13) gives $\langle \psi, \overline{k'_T} \rangle_g = \beta |\overline{k'_T}|_g^2$, with $\beta = 2(\alpha_e + 3\alpha_i + \alpha_f) \geq 0$, to complete the present model. The initial data is assumed to be bounded in norm. □

Now $\langle\!\langle \boldsymbol{\Psi}, \boldsymbol{\mathcal{R}} \rangle\!\rangle_{\mathbf{g}}$ from (13) can be placed in equation (11) to give:

$$\frac{\partial \|\boldsymbol{\mathcal{R}}\|_g^2}{\partial t} + 2\nu \|\nabla \boldsymbol{\mathcal{R}}\|_g^2 + 2(\alpha_e + 2\alpha_s/3 + \alpha_f + \alpha_i) \|\boldsymbol{\mathcal{R}}\|_g^2 + 4 \langle\!\langle \overline{\mathbf{L}} \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g \leq 0 \qquad (14)$$

which is closed when the estimate $\|\overline{\mathbf{L}}\|_g \leq \overline{L} \;\Rightarrow\; \langle\!\langle \overline{\mathbf{L}} \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g \leq \overline{L}\, \|\boldsymbol{\mathcal{R}}\|_g^2$ enacts zero order norm closure. For completeness:

$$\frac{\partial |\overline{k'_T}|_g^2}{\partial t} + [2\nu L_k^2 + \beta] |\overline{k'_T}|_g^2 \leq 4\overline{L} |\overline{k'_T}|_g^2$$

as the estimate for $|\overline{k'_T}|_g$ when inequality (10) is included in equation (12). This inequality integrates directly from Gronwall to give:

$$|\overline{k'_T}|_g^2(t) \leq \exp[-(\Gamma - 4\overline{L})t]\, |\overline{k'_T}|_g^2(0) ,$$

which decays to zero provided that $\Gamma - 4\overline{L} > 0$. Also if $\langle\!\langle \overline{\mathbf{L}}, \boldsymbol{\mathcal{R}} \rangle\!\rangle_g \leq \overline{L} \|\boldsymbol{\mathcal{R}}\|_g$ in equation (7) the model is now represented by the pair of inequalities:

$$\frac{\partial |\mathbf{V}|_g^2}{\partial t} + 2\nu L_V^2 |\mathbf{V}|_g^2 \leq 2\overline{L}\, \|\boldsymbol{\mathcal{R}}\|_g + 2|\mathbf{f}|_g |\mathbf{V}|_g \qquad (15)$$

and

$$\frac{\partial \|\boldsymbol{\mathcal{R}}\|_g^2}{\partial t} + 2\Lambda \|\boldsymbol{\mathcal{R}}\|_g^2 - 4\overline{L}\|\boldsymbol{\mathcal{R}}\|_g^2 \leq 0\,. \tag{16}$$

The definitions:

$$\Lambda = [\nu L_R^2 + 2\alpha_s/3 + \alpha_e + \alpha_f + \alpha_i] \geq 0; \qquad \Gamma = 2\nu L_k^2 + \beta \geq 0$$

have been adopted. Inequality (16) integrates to give:

$$\|\boldsymbol{\mathcal{R}}\|_g(t) \leq \exp[-(\Lambda - 2\overline{L})t]\, \|\boldsymbol{\mathcal{R}}\|_g(0)\,, \tag{17}$$

and again converges to zero if $\Lambda - 2\overline{L} > 0$. With estimate (17) inequality (15) has the form:

$$\frac{\partial |\mathbf{V}|_g^2}{\partial t} + \nu L_V^2 |\mathbf{V}|_g^2 \leq 2\overline{L}\, \exp[-(\Lambda - 2\overline{L})t]\, \|\boldsymbol{\mathcal{R}}\|_g(0) + |\mathbf{f}\,|_g^2/(\nu L_V^2)\,. \tag{18}$$

Integration gives:

$$|\mathbf{V}|_g^2(t) \leq P\, \exp[-\nu L_V^2 t] - Q\, \exp[-(\Lambda - 2\overline{L})t] + |\mathbf{f}\,|_g^2/(\nu L_V^2)^2\,.$$

Here:

$$P = |\mathbf{V}|_g^2(0) - |\mathbf{f}\,|_g^2/(\nu L_V^2)^2 + Q; \qquad Q = 2\overline{L}\|\boldsymbol{\mathcal{R}}\|_g(0)/(\Lambda - 2\overline{L} - \nu L_V^2)\,.$$

Both results assume that $\Lambda - 2\overline{L} > 0$ and that $\Lambda \neq 2\overline{L} + \nu L_V^2$. *Except* on subsets of measure zero in parameter space defined by the conditions $\Gamma = 4\overline{L}$ and ($\Lambda = 2\overline{L}$ or $\Lambda = 2\overline{L} + \nu L_V^2$) the fixed points for the system are:

a) No body force:

$$|\overline{k_T'}|_g|_{FP} = 0; \qquad \|\boldsymbol{\mathcal{R}}\|_g|_{FP} = 0; \qquad |\mathbf{V}|_g|_{FP} = 0\,.$$

This fixed point is linearly stable provided that $\Gamma > 4\overline{L}$ for the $|\overline{k_T'}|_g$ equation and $\Lambda - 2\overline{L} > 0$ for the $\|\boldsymbol{\mathcal{R}}\|_g$ and $|\mathbf{V}|_g^2$ equations. These constraints imply that $\overline{L}$ cannot be too large and so restricts the class of flow for which the turbulence model is viable.

b) With body force ($|\mathbf{f}\,|_g^2$ given):

$$|\overline{k_T'}|_g|_{FP} = 0; \qquad \|\boldsymbol{\mathcal{R}}\|_g|_{FP} = 0; \qquad |\mathbf{V}|_g \leq |\mathbf{f}|_g/(\nu L_V^2)\,,$$

and $|\mathbf{V}|_g$ need not attain a fixed point. The turbulent fluctuations decay to zero while the mean motion is constrained to an absorbing set of diameter $|\mathbf{f}|_f/(\nu L_V^2)$ as $t \to \infty$. This finding, $|\mathbf{V}|_g \leq |\mathbf{f}|_g/(\nu L_V^2)$, is directly equivalent to inequality (4) for the instantaneous motion. The body force only influences the mean motion as $\mathbf{f} = \mathbf{f}(\mathbf{x})$ has no stochastic content. The analysis is unsatisfactory since $\overline{L}$, the global extremum of the eigenvalues of $\overline{\mathbf{D}}$, is unknown. That is, the present estimate is not an *a priori* estimate but depends upon the local flow evolution. As will be discussed in Section 5, the constant $\overline{L}$ can be replaced by $\|\overline{\mathbf{L}}\|_g$ from an evolution equation for $\|\overline{\mathbf{L}}\|_g$ if additional regularity is assumed.

5. Reynolds Stress Model with First Order Norm Closure

A more consistent estimate would include an evolution equation for $\|\overline{\mathbf{L}}\|_g$ which is only needed to study the norm convergence of $|\mathbf{V}|_g$ and is not part of the turbulence model *per se.* Evolution of $\|\overline{\mathbf{L}}\|_g$ becomes part of the computation (see equation (21)) if $\|\overline{\mathbf{L}}\|_g$ is assumed to be bounded on the initial data. More regularity is also required as the spatial gradient of $\overline{\mathbf{L}}$ is assumed to exist in equation (19). If $\overline{d}/dt$ is the material derivative based upon the mean velocity, equation (2) requires:

$$\overline{d}\,\overline{\mathbf{L}}/dt + \overline{\mathbf{L}}^2 + \mathbf{M} = \nu\,\nabla^2(\overline{\mathbf{L}}) - \overline{\mathbf{P}} + \mathbf{F}$$

for the mean velocity gradient and implies an evolution for $\|\overline{\mathbf{L}}\|_g$ given by:

$$\frac{\partial}{\partial t}\|\overline{\mathbf{L}}\|_g^2 + 2\langle\!\langle \mathbf{K}, \overline{\mathbf{L}}\rangle\!\rangle_g + 2\nu\|\nabla\overline{\mathbf{L}}\|_g^2 = 2\langle\!\langle \mathbf{F}, \overline{\mathbf{L}}\rangle\!\rangle_g\,, \tag{19}$$

where $\mathbf{K} = \overline{\mathbf{L}}^2 + \mathbf{M} \in \mathbb{M}_3$ has been defined. Here $\mathbf{M} = \nabla(\mathrm{div}\,(\boldsymbol{\mathcal{R}}))$ must be represented in terms of $\|\boldsymbol{\mathcal{R}}\|_g$ but is additional to the turbulence model under study. First order norm closure is effected in equation (19) for $\|\mathbf{L}\|_g$ by adoption of the Poincaré inequality $\|\nabla(\overline{\mathbf{L}})\|_g^2 \geq L_L^2\|\overline{\mathbf{L}}\|_g^2$. The constant $\overline{L}$ from the previous case is now replaced by L_L and an evolution equation for $\|\overline{\mathbf{L}}\|_g^2$. The only change that is required in equation (11) for estimating $\|\boldsymbol{\mathcal{R}}\|_g$ is the production term, $\langle\!\langle \overline{\mathbf{L}}\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}}\rangle\!\rangle_g$, with: $|\langle\!\langle \overline{\mathbf{L}}\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}}\rangle\!\rangle_g| \leq \|\overline{\mathbf{L}}\|_g\|\boldsymbol{\mathcal{R}}\|_g^2$. Inequality (16) and equation (19) are now replaced by:

$$\frac{\partial\|\boldsymbol{\mathcal{R}}\|_g^2}{\partial t} + 2\Lambda\|\boldsymbol{\mathcal{R}}\|_g^2 \leq 4\|\overline{\mathbf{L}}\|_g\|\boldsymbol{\mathcal{R}}\|_g^2\,,$$

$$\frac{\partial}{\partial t}\|\overline{\mathbf{L}}\|_g^2 + \nu L_L^2\|\overline{\mathbf{L}}\|_g^2 \leq \frac{1}{\nu L_L^2}|\mathbf{f}\,|_g^2 + 2\|\overline{\mathbf{L}}\|_g^3 + 4\alpha_m\|\boldsymbol{\mathcal{R}}\|_g\|\overline{\mathbf{L}}\|_g\,,$$

respectively. This result has used the above definitions for the quantity $\mathbf{K}$ and adopts the simple model $\|\mathbf{M}\|_g = 2\alpha_m\|\boldsymbol{\mathcal{R}}\|_g$ which has no physical significance. The mean motion estimate, using (7) is:

$$\frac{\partial|\mathbf{V}|_g^2}{\partial t} + \nu L_V^2|\mathbf{V}|_g^2 \leq 2\|\boldsymbol{\mathcal{R}}\|_g\|\overline{\mathbf{L}}\|_g + |\mathbf{f}\,|_g^2/(\nu L_V^2)$$

and compares with (18). From (10) and (12):

$$\frac{\partial}{\partial t}|\overline{k'_T}|_g^2 + (2\nu L_k^2 + \beta)|\overline{k'_T}|_g^2 \leq 2\|\boldsymbol{\mathcal{R}}\|_g\|\overline{\mathbf{L}}\|_g \tag{20}$$

and follows when estimates for $\|\boldsymbol{\mathcal{R}}\|_g$ and $|\overline{\mathbf{L}}|_g$ are known. The first order norm closure is complete and can be expressed as the system:

$$\frac{\partial}{\partial t}\begin{bmatrix}|\mathbf{V}|_g^2\\ \|\boldsymbol{\mathcal{R}}\|_g^2\\ \|\overline{\mathbf{L}}\|_g^2\end{bmatrix} \leq \begin{bmatrix} -\nu L_V^2|\mathbf{V}|_g^2 + 2\|\boldsymbol{\mathcal{R}}\|_g\|\overline{\mathbf{L}}\|_g + |\mathbf{f}|_g/(\nu L_V^2)\\ 2(2\|\overline{\mathbf{L}}\|_g - \Lambda)\|\boldsymbol{\mathcal{R}}\|_g^2\\ -\nu L_L^2\|\overline{\mathbf{L}}\|_g^2 + 2\|\overline{\mathbf{L}}\|_g^3 + 4\alpha_m\|\boldsymbol{\mathcal{R}}\|_g\|\overline{\mathbf{L}}\|_g + \|\mathbf{F}\|_g 2/(\nu L_L^2)\end{bmatrix}, \quad (21)$$

with inequality (20) considered retrospectively. Consider the fixed points of the corresponding *differential equation* $\partial\mathbf{y}/\partial t = \mathbf{g}$ with the vector:

$$\mathbf{y} = (|\mathbf{V}|_g^2,\ \|\boldsymbol{\mathcal{R}}\|_g^2,\ \|\overline{\mathbf{L}}\|_g^2)^T\,,$$

while the vector $\mathbf{g}$ contains the elements on the right hand side of (21). A fixed point is said to be *tenable* if every $|(\cdot)|_g|_{FP}$, evaluated at the fixed point from the condition $\mathbf{g} = \mathbf{0}$, is both real and non–negative. Non–tenable fixed points are of no interest in the present context. The fixed points of the system (21), *when no body force is included*, are the following:

Case a. $\|\boldsymbol{\mathcal{R}}\|_g^2|_{FP} \equiv \mathbf{O}$. There are two fixed points to consider:

(i) $\|\overline{\mathbf{L}}\|_g|_{FP} = 0; \qquad |\mathbf{V}|_g|_{FP} = 0 \quad \Rightarrow \quad |\overline{k_T'}|_g|_{FP} = 0,$

(ii) $\|\overline{\mathbf{L}}\|_g|_{FP} = \nu L_L^2/2; \qquad |\mathbf{V}|_g|_{FP} = 0 \quad \Rightarrow \quad |\overline{k_T'}|_g|_{FP} = 0,$

in the three–dimensional $\mathbf{y}$–phase space.

Case b. $\|\overline{\mathbf{L}}\|_g|_{FP} = \Lambda/2$. Has a single fixed point given when:

$$\|\overline{\boldsymbol{\mathcal{R}}}\|_g|_{FP} = \Upsilon; \quad |\mathbf{V}|_g^2|_{FP} = \Upsilon\Lambda/(\nu L_V^2) \ \Rightarrow \ |\overline{k_T'}|_g^2|_{FP} = \Upsilon\Lambda/[(2\nu L_k^2 + \beta)],$$

where $\Upsilon \equiv \Lambda(\nu L_L^2 - \Lambda)/(8\alpha_m)$. The fixed point listed here is only tenable if the constraint $\Upsilon > 0 \Rightarrow \nu L_L^2 > \Lambda$ holds. It is found that the phase space, when $\|\overline{\mathbf{L}}\|_g$ evolution is considered, is considerably more complex than that obtained in Section 4 for zero order norm closure.

When $|\mathbf{f}\,|_g \not\equiv 0$ and $\|\boldsymbol{\mathcal{R}}\|_g^2|_{FP} = 0$ there is $|\mathbf{V}|_g^2 \to |\mathbf{f}\,|_g^2/(\nu L_V^2)^2$ as $t \to \infty$ which compares with previous estimates.

6. Final Remarks

Selected turbulence models have been studied on the basis of functional estimates. These engineering estimates determine a relationship between model constants that are required to ensure that the model gives globally stable dynamical systems. *In no way* does this ensure that the models will give good comparison with experimental data at the local level. The estimates herein are made for simplistic turbulence models and these models could certainly be improved. In addition, attention has been restricted to flows in a periodic domain. Only global estimates such as $|\mathbf{v}|_g = \int_{\mathcal{D}} \langle \mathbf{v}, \mathbf{v}\rangle dV$ have been made and these say little about the local conditions. The most unsatisfactory aspect of the analysis is the ubiquitous regularity assumption and its implied loss of generality.

Missing in functional analysis estimates is any recognition of the physical structure of turbulence. No mention is made of the energy cascade or of the different flow physics at the small and high wave number regions of the spectrum. The same is true, of course, for the basic turbulence modeling process, but in a more direct way. The constraint of Galilean invariance must be included as part of the turbulence model and is not imposed on the norm estimates.

Estimates of this form should be carried out for each turbulence model being considered in the applications in order to establish meaningful ranges for the parameters involved.

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