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FRACTIONAL DERIVATIVE AND FORMAL POWER SERIES

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Abstract: The new method of summation of divergent series, given in [9, 10] for calculation of fractional derivatives of function of binomial type is presented. A new indentity for hypergeometric functions is used to justify this calculus.

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1. Introduction

Several authors have considered different methods for calculating fractional derivatives of a given function (see e.g. [1], [3]-[10]). Different approaches and definitions lead to a variety of different results, even if the most elementary functions are considered.

In [6] Thomas J. Osler gave a definition of fractional differentiation by generalizing the Cauchy integral formula

$$D_z^N f(z) = \frac{N!}{2\pi i} \oint_C f(t)(t-z)^{-N-1} dt, \tag{1}$$

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where the contour C is a simple closed curve enclosing z in the positive sense and containing inside only regular points of $f(t)$. Namely the fractional derivative of order p of $f(z)$, by Osler is

$$D^p_{z-a} f(z) = \frac{\Gamma(p+1)}{2\pi i} \int_a^{(z+)} f(\xi)(\xi - z)^{-p-1} d\xi \quad (p \notin \mathbf{Z}^-)\,, \tag{2}$$

where he made a branch cut from z to a, and integral curve is an open contour which starts at a and encloses z in the positive sense and returns to a.

Using this definition, Osler (see [6]) proves that

$$D^p_z z^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - p + 1)} z^{\alpha - p}. \tag{3}$$

In this paper we shall consider fractional derivative of a function $(1+x)^\alpha$, of order p, for which the following formulas exist in literature. The first one is given by T.J. Osler (see [4], Table on p. 171):

$$D^p_z (z+a)^\alpha = \frac{a^\alpha z^{-p}}{\Gamma(1-p)} {}_2F_1(1, -a, 1-p, -\frac{z}{a}). \tag{4}$$

Using the Weyl fractional derivatives

$$D^p_{z-\infty} f(z) = \frac{(-1)^{-p}}{\Gamma(-p)} \int_x^\infty f(t)(t-z)^{-p-1} dt,$$

J.L. Lavoie, T.J. Osler and R. Tremblay in [1] obtained

$$D^p_{z-\infty} (z+a)^a = \frac{\Gamma(p-\alpha)}{\Gamma(-\alpha)} (z+a)^{\alpha - p}. \tag{5}$$

S.G. Samko, A.A. Kilbas and O.I. Marichev verified Euler formula:

$$D^p_{a^+} (x-a)^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - p + 1)} (x-a)^{\alpha - p}, \qquad -1 < \alpha < 0, \quad 0 < p < 1. \tag{6}$$

Here $D^p_{a^+}$ denotes Riemann-Liouville fractional derivative:

$$(D^p_{a^+} f)(x) = \frac{1}{\Gamma(1-p)} \frac{d}{dx} \int_a^x \frac{f\,(t)}{(x-t)^p} dt, \qquad 0 < \ p < 1.$$

(see [7], p. 43). For $q \in \mathbf{C}$ and $\operatorname{Re}\beta > 0$, it follows from from (6) that (see [7], p. 47):

$$D^{-q}_{a^+} (x-a)^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(q+\beta)} (x-a)^{q+\beta-1}, \qquad \operatorname{Re} q < 0.$$

If $\beta - 1 = \alpha$ and $-q = p$, it follows that (6) is valid for $-1 < \operatorname{Re}\alpha < 0$ and $0 < \operatorname{Re} p < 1$.

An old idea for more than 170 years, known by the name of J. Liouville and B. Riemann, is to use power series and to apply the fractional derivatives to each summand. In this paper we apply the idea given in [8, 9, 10]. We represent a function by formal power series $\sum_{i=-\infty}^{+\infty} a_i x^i$. Operators of usual (integer) derivative and integral applies to this series term by term.

Fractional derivative of order p of formal power series is defined by [9]:

$$\left(\sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}\right)^{(p)} = \sum_{i=-\infty}^{\infty} a_i \frac{x^{i-p}}{(i-p)!}, \tag{7}$$

where $x! = \Gamma(x+1)$. Note that $(-1)! = (-2)! = ... = \pm\infty$ and hence $\frac{x^i}{i!} = 0$ for $i \in \mathbf{Z}$, $i \leq 0$ but these zeros summands of f have important role for the derivatives of order p, because $\frac{x^{i-p}}{(i-p)!} \neq 0$ if p is a non-integer number.

In this paper we illustrate this method for calculation the fractional derivatives of function of binomial type. A new identity for Gauss hypergeometric functions is used to justify method of calculation with divergent series, which can be very useful in fractional calculus.

2. Main Results

Using the summation method of divergent series of [8, 9, 10], we give the following analogous result of (6).

Theorem 1. *The following formula is valid for* $-1 < Re\,\alpha < 0$ *and* $0 < Re\,p < 1$*:*

$$[(1+x)^\alpha]^{(p)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-p)}(1+x)^{\alpha-p}. \tag{8}$$

Proof. Using the expansion

$$(1+x)^\alpha$$
$$= ...+\frac{\alpha!}{(\alpha+2)!}\frac{x^{-2}}{(-2)!}+\frac{\alpha!}{(\alpha+1)!}\frac{x^{-1}}{(-1)!}+\frac{\alpha!}{\alpha!}\frac{x^0}{0!}+\frac{\alpha!}{(\alpha-1)!}\frac{x^1}{1!}+\frac{\alpha!}{(\alpha-2)!}\frac{x^2}{2!}+...$$

we obtain, by (1)

$$[(1+x)^\alpha]^{(p)} = ... + \frac{\alpha!}{(\alpha+2)!}\frac{x^{-2-p}}{(-2-p)!} + \frac{\alpha!}{(\alpha+1)!}\frac{x^{-1-p}}{(-1-p)!}$$
$$+\frac{\alpha!}{\alpha!}\frac{x^{-p}}{(-p)!} + \frac{\alpha!}{(\alpha-1)!}\frac{x^{1-p}}{(1-p)!} + \frac{\alpha!}{(\alpha-2)!}\frac{x^{2-p}}{(2-p)!} + ...$$

and hence the theorem will be proved if we prove the identity:

$$\begin{aligned} &... + \frac{1}{(\alpha+2)!}\frac{x^{-2-p}}{(-2-p)!} + \frac{1}{(\alpha+1)!}\frac{x^{-1-p}}{(-1-p)!} + \frac{1}{\alpha!}\frac{x^{-p}}{(-p)!} \\ &+\frac{1}{(\alpha-1)!}\frac{x^{1-p}}{(1-p)!} + \frac{1}{(\alpha-2)!}\frac{x^{2-p}}{(2-p)!} + ... \\ &= (1+x)^{\alpha-p}\cdot\ \frac{1}{\Gamma(\alpha+1-p)}\,. \end{aligned} \tag{9}$$

Let L be the left side of (9). Then,

$$\begin{aligned} \frac{dL}{dx} &= ... + \frac{1}{(\alpha+2)!}\frac{x^{-3-p}}{(-3-p)!} + \frac{1}{(\alpha+1)!}\frac{x^{-2-p}}{(-2-p)!} + \frac{1}{\alpha!}\frac{x^{-p-1}}{(-p-1)!} \\ &+\frac{1}{(\alpha-1)!}\frac{x^{-p}}{(-p)!} + \frac{1}{(\alpha-2)!}\frac{x^{-p+1}}{(1-p)!} + ...\end{aligned}$$

Multiplying by $x^{p-\alpha-1}$, we get

$$\begin{aligned} x^{p-\alpha-1}\frac{dL}{dx} &= ... + \frac{1}{(\alpha+2)!}\frac{x^{-4-\alpha}}{(-3-p)!} + \frac{1}{(\alpha+1)!}\frac{x^{-3-\alpha}}{(-2-p)!} + \frac{1}{\alpha!}\frac{x^{-2-\alpha}}{(-p-1)!} \\ &+\frac{1}{(\alpha-1)!}\frac{x^{-1-\alpha}}{(-p)!} + \frac{1}{(\alpha-2)!}\frac{x^{-\alpha}}{(1-p)!} + ...\end{aligned}$$

Hence,

$$\begin{aligned} \int_0^x t^{p-\alpha-1}L'(t)dt &= ... - \frac{1}{(\alpha+3)!}\frac{x^{-3-\alpha}}{(-3-p)!} - \frac{1}{(\alpha+2)!}\frac{x^{-2-\alpha}}{(-2-p)!} \\ &-\frac{1}{(\alpha+1)!}\frac{x^{-1-\alpha}}{(-1-p)!} - \frac{1}{\alpha!}\frac{x^{-\alpha}}{(-p)!} - \frac{x^{-\alpha+1}}{(\alpha-1)!(1-p)!} - ....\end{aligned}$$

Multiplying the last equality by $x^{-p+\alpha}$, we obtain

$$\begin{aligned} x^{-p+\alpha}\int_0^x t^{p-\alpha-1}L'(t)dt &= ... - \frac{1}{(\alpha+3)!}\frac{x^{-3-p}}{(-3-p)!} - \frac{1}{(\alpha+2)!}\frac{x^{-2-p}}{(-2-p)!} \\ &-\frac{1}{(\alpha+1)!}\frac{x^{-1-p}}{(-1-p)!} - \frac{1}{\alpha!}\frac{x^{-p}}{(-p)!} - \frac{x^{-p+1}}{(\alpha-1)!(1-p)!} - ....\end{aligned}$$

Since the right side of this equality is $-L$, we have the following integro-differential equation

$$x^{-p+\alpha}\int_0^x t^{p-\alpha-1}L'(t)dt = -L\,,$$

i.e.

$$x^{p-\alpha-1}L'(x)=-\frac{L'x^{-p+\alpha}-L(\alpha-p)x^{\alpha-p-1}}{x^{2\alpha-2p}}.$$

Hence we obtain the following ordinary differential equation

$$L'(1+x)=(\alpha-p)L.$$

The solution of the last differential equation is given by $L=C(1+x)^{\alpha-p}$. Thus it is sufficient to prove that $C=\frac{1}{\Gamma(\alpha+1-p)}$. Hence it is sufficient to prove (6) for $x=1$, i.e.

$$...+\frac{1}{(\alpha+2)!(-2-p)!}+\frac{1}{(\alpha+1)!(-1-p)!}+\frac{1}{\alpha!(-p)!}$$
$$+\frac{1}{(\alpha-1)!(1-p)!}+\frac{1}{(\alpha-2)!(2-p)!}+...=\frac{2^{\alpha-p}}{\Gamma(\alpha+1-p)}.$$

Multiplying by $\alpha!(-p)!$, we obtain

$$...-\frac{p(p+1)(p+2)}{(\alpha+3)(\alpha+2)(\alpha+1)}+\frac{p(p+1)}{(\alpha+2)(\alpha+1)}-\frac{p}{\alpha+1}$$
$$+1-\frac{\alpha}{p-1}+\frac{\alpha(\alpha-1)}{(p-2)(p-1)}-\frac{\alpha(\alpha-1)(\alpha-2)}{(p-3)(p-2)(p-1)}+...$$
$$=\frac{\Gamma(\alpha+1)\Gamma(1-p)}{\Gamma(\alpha+1-p)}2^{\alpha-p}.$$

The right side of the last equality is equal to

$$2^{\alpha-p}(\alpha+1-p)B(\alpha+1,1-p),$$

where B is the well known beta function. Since $\mathrm{Re}\,p>\mathrm{Re}\,\alpha$, the series

$$1-\frac{\alpha}{p-1}+\frac{\alpha(\alpha-1)}{(p-2)(p-1)}-\frac{\alpha(\alpha-1)(\alpha-2)}{(p-3)(p-2)(p-1)}+...$$
$$-\frac{p}{\alpha+1}+\frac{p(p+1)}{(\alpha+1)(\alpha+2)}-\frac{p(p+1)(p+2)}{(\alpha+1)(\alpha+2)(\alpha+3)}+...$$

are divergent Gauss hypergeometric functions but we shall denote them by ${}_2F_1(1,-\alpha;1-p,-1)$ and ${}_2F_1(1,p;\alpha+1,-1)-1$, respectively, just as in the case $\mathrm{Re}\,p<\mathrm{Re}\,\alpha$.

Thus if $-1<\mathrm{Re}\alpha<0$ and $0<\mathrm{Re}p<1$, we need to prove the identity:

$$ {}_2F_1(1,p;\alpha+1,-1)-1+{}_2F_1(1,-\alpha;1-p,-1)=2^{\alpha-p}(\alpha+1-p)B(\alpha+1,1-p). \quad (10)$$

This one is proved in the following lema which can be of independent interest.

Lemma 1. *It holds*

$$ {}_2F_1(1,-a,b+1;-1)+{}_2F_1(1,-b,a+1;-1)=1+2^{a+b}(a+b+1)B(a+1,b+1)\,, $$

where $-1<Re\,a<0$, $-1<Re\,b<0$.

Proof. Using Euler's integral representation [2, p. 57]

$$ {}_2F_1(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt\,, $$

where $\mathrm{Re}\,c>\mathrm{Re}\ \mathrm{b}>0$, $|\arg(1-z)|<\pi$, we obtain following identity which is valid for $-1<\mathrm{Re}\,a<0$, $-1<\mathrm{Re}\,b<0$:

$$ \begin{aligned} &{}_2F_1(-a,1,b+1;-1)+{}_2F_1(-b,1,a+1;-1)\\ &\quad=b\int_0^1(1-t)^{b-1}(1+t)^a dt+a\int_0^1(1-t)^{a-1}(1+t)^b dt. \end{aligned} $$

Using integration by parts, it follows:

$$ \begin{aligned} &b\int_0^1(1-t)^{b-1}(1+t)^a\,dt\\ &\quad=-b(1+t)^a\cdot\frac{(1-t)^b}{b}\Big|_0^1+b\int_0^1\frac{a}{b}(1+t)^{a-1}(1-t)^b dt\\ &\quad=1+a\int_0^1(1+t)^{a-1}(1-t)^b dt\\ &\quad=1+a\int_{-1}^0(1-t)^{a-1}(1+t)^b dt\,. \end{aligned} $$

Now, we have that

$$ \begin{aligned} &b\int_0^1(1-t)^{b-1}(1+t)^a dt+a\int_0^1(1-t)^{a-1}(1+t)^b dt\\ &\quad=1+a\int_{-1}^1(1+t)^{a-1}(1-t)^b dt\\ &\quad=1+a\int_0^1(2u)^{a-1}(2-2u)^b\cdot 2du\\ &\quad=1+a\cdot 2^{a+b}\int_0^1 u^{a-1}(1-u)^b du\\ &\quad=1+a\cdot 2^{a+b}B(a,b+1)\\ &\quad=1+a\cdot 2^{a+b}\frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}\\ &\quad=1+(a+b+1)2^{a+b}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \end{aligned} $$

$$= 1 + (a + b + 1)2^{a+b}B(a + 1, b + 1).$$

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