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INCREASING ALONG RAYS VECTOR FUNCTIONS

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Abstract: The class of increasing along rays functions is generalized to consider vector valued functions. A general approach through scalarization is used and minimal properties for the scalarization are given. The class of vector increasing along rays functions introduced is compared with the scalar one to prove similar properties hold. The relation with convex and generalized convex functions is preserved for the vector valued counterpart.

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1. Introduction

The notion of increasing along rays function has been one of the latest generalizations of convexity. This class of scalar functions has been popularized by

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the work of Rubinov and by his recent monograph on abstract convexity (see [19] and the references therein).

The increasing along rays functions prove to enjoy nice properties with regard to optimization as well as variational inequalities, which longly motivates the interest of researchers. A first attempt to extend the scalar results to the case of vector valued function has been made in [7] where the problem is faced by means of a nonlinear scalarization which allows to study a vector function through a family of scalar functions. It is proved that, supposed at least one of the functions in this family is increasing along rays, some properties are inherited by the vector function as well. Moreover, the relations between the class of vector increasing along rays functions and those of C- convex and C-quasiconvex functions are investigated.

In this paper we generalize the approach used in [7] by considering a more general scheme for scalarization and by relating the notion of vector increasing along rays function to the chosen scalarization. In Section 2 we briefly recall the notion and the properties of scalar increasing along rays functions, as a guideline for the following generalization while Section 3 is devoted to the approach used in [7] which is generalized in Section 4, where the minimal properties for a scalarization to define vector increasing along rays functions are listed. Examples are provided to support the results.

2. Increasing Along Rays Scalar Functions

The aim of this section is to present some results on increasing along rays scalar functions, results which will be generalized for vector functions in the remaining of the paper.

First, we recall, the notion of increasing along rays scalar function; this notion arises mainly in the study of abstract convexity [19] and can be viewed as a generalization of the concept of quasiconvex function.

Let K be a nonempty star-shaped (for short st-sh) set: its kernel is the set

$$\ker K := \left\{ x \in K \,:\, x' + t\left(x' - x\right) \in K \;\; \forall x' \in K;\, \forall t > 0 \right\}.$$

It is known (see e.g. [19]) that the set $\ker K$ is convex for an arbitrary st-sh set K.

Definition 1. Let $K \subseteq R^n$ be a st-sh set and $x^0 \in \ker K$. A function defined on $K, f : R^n \to R$, is called increasing along rays (for short IAR) at x^0, if its restriction on the segment $R_{x^0,x} \cap K$ is increasing, for each $x \in K$[a function g of one real variable is increasing on the interval I if $t_2 \geq t_1, t_1, t_2 \in I$ implies $g(t_2) \geq g(t_1)$].

If f fulfils Definition 1, we write $f \in IAR(K, x^0)$.

If $K = R^n$, then f is said increasing along ray starting at the point x^0.

Definition 2. A function f defined on $R^n, f : R^n \to R$, is called increasing along rays at the point x^0 (for short, $f \in IAR\,(x^0)$) if the restriction of this function on the ray starting at x^0, i.e. $R_{x^*,x} = \left\{ x^0 + \alpha x / \alpha \geq 0 \right\}$, is an increasing function of α, for each $x^0 \in R^n$.

We now define, with regard to optimization problems, some basic properties for scalar functions increasing along rays, which can be considered extensions of analogous properties holding for convex functions.

First we need to consider the (scalar) minimization problem

$$P(f, K) \quad \min f(x), \quad x \in K \subseteq R^n \,.$$

A point $x^0 \in K$ is a (global) solution of $P(f, K)$ when $f(x) - f(x^0) \geq 0,\ \forall x \in K$. The solution is strict if $f(x) - f(x^0) > 0,\ \forall x \in K \backslash \{x^0\}$. We denote by $\arg\min\,(f, K)$ the set of solutions of $P(f, K)$.

Proposition 1. *Let $K \subseteq R^n$ be a st-sh set, $x^0 \in \ker K$ and $f \in IAR\,(K, x^0)$. Then:*

i) x^0 is a solution of $P(f, K)$.

ii) No point $x \in K, x \neq x^0$, can be a strict local solution of $P(f, K)$.

iii) $x^0 \in \ker\ \arg\min\,(f, K)$.

For scalar functions of one real value, i.e. when K is an interval, it turns out that $f \in IAR(K, x^0)$ if and only if it is quasiconvex with a global minimum over K at x^0. Instead, when $n > 1$ the class of quasiconvex functions with a global minimizer at x^0 is a strict subset of that of IAR functions. The following result is quoted, without proof, from [4].

Proposition 2. *Let us assume:*

i) $f : R^n \to R$ is quasiconvex.

ii) $x^0 \in \ker K$ is a global minimizer for $P(f, K)$.

Then $f \in IAR(K, x^0)$.

Also functions which are not quasiconvex can indeed be IAR at x^0 over K, as the next example shows.

Example 1. Let $f(x_1, x_2) = x_1^2 x_2^2$, and $K = R^2$. Then, for $x^0 = (0\,,\,0)$ it is easily seen that $f \in IAR(K, x^0)$, but f is not quasiconvex.

We shall say that, in general, the class of functions $IAR(K, x^0)$ is broader then the class of quasiconvex ones.

The next proposition aims to present IAR functions as generalized convex, since, we recall, that level sets of quasiconvex functions are convex sets.

Proposition 3. *Let $K \subseteq X$ be a st-sh set, $x^0 \in \ker K$ and f be a function defined on K. Then $f \in IAR(K, x^0)$ if and only if for each $c \in R, c \geq f(x^0)$, we have $x^0 \in \ker lev_{\leq c} f$, where $lev_{\leq c} f = \{ x \in K : f(x) \leq c \}$.*

Some relations can be also established among IAR functions and solutions of Minty-type variational inequalities of differential type. It has been shown that IAR functions can be characterized by means of a generalized Minty variational inequality [18]. Here we consider a generalization of the well-known inequality with a gradient operator.

We recall that, for any real function f defined on an open set containing K, the lower Dini directional derivative at the point $x \in K$ in the direction $u \in X$ is defined as an element of $\bar{R} := [-\infty, +\infty]$ by

$$f'_{-}(x, u) = \liminf_{t \to +0} \frac{f(x + tu) - f(x)}{t} .$$

We say that a point $x^0 \in \ker K$ is a solution of a generalized Minty variational inequality when

$$MVI(f'_{-}, K) , \quad f'_{-}(y, x^0 - y) \leq 0 , \quad \forall \, y \in K .$$

This problem obviously reduces to the usual Minty variational inequality problem of differential type (see e.g. [18]) when f is differentiable on an open set containing K.

To study the latter problem, some (radial) continuity plays a role.

Definition 3. Let $K \subseteq X, x^0 \in \ker K$ and let f be a function defined on an open set containing K. The function f is said to be radially lower semicontinuous over K along rays starting at x^0 if, for each $x \in K$, the restriction of f on the interval $R_{x^0,x} \cap K$ is lower semicontinuous.

If f fulfils Definition 3, we write $f \in RLSC(K, x^0)$.

Theorem 1. *Let $K \subseteq X$ be a st-sh set and $x^0 \in \ker K$.*

i) If x^0 solves $MVI(f'_{-}, K)$ and $f \in RLSC(K, x^0)$, then $f \in IAR(K, x^0)$.

ii) Conversely, if $f \in IAR(K, x^0)$, then x^0 is a solution of $MVI(f'_{-}, K)$.

3. Increasing Along Rays by Means of Oriented Distance

We now move to consider the case where f is a function from R^n to R^m and we present a different definition of increasing along ray functions. The order on the space R^m is induced by a closed convex and pointed cone $C \subseteq R^m$, with $\text{int}\, C \neq \emptyset$. First we recall that, in this case, the solution of a vector minimization problem (as well of a vector variational inequality) is not uniquely defined (see e.g. [15]). We consider the vector optimization problem, defined as

$$VP(f,K) \min_C f(x)\,, \quad x \in K \subseteq R^n.$$

Solutions of the problem $VP(f,K)$ are called points of efficiency. More precisely, the point $x^0 \in K$ is an efficient solution for $VP(f,K)$ if $f(x)-f(x^0) \notin -C\setminus\{0\}$ for every $x \in K$. The point $x^0 \in K$ is a weak efficient solution for if $f(x)-f(x^0) \notin -\, int\, C$ for every $x \in K$. Further, a point $x^0 \in K$ is an ideal solution for $VP(f,K)$ if $f(x)-f(x^0) \notin -\ C$. Recall that ideal solution for $VP(f,K)$ are not similar for $VP(f,K)$.

Besides, various notions of proper efficient solutions have been introduced. For a survey and comparisons among them, we refer to [13]. We focus in this section on a peculiar way to characterize weak efficiency. First, recall that the distance from a point $y \in R^m$ to a subset $A \subseteq R^m$ is given by the function $d_A(y) = \inf_{a\in A} \|y-a\|$. It is clear that the distance depends upon the choice of the norm in R^m. In [14] the author introduces the concept of oriented distance from y to A, given by

$$\Delta_A(y) = d_A(y) - d_{Y\setminus A}(y).$$

Observe that while $d_A(y) = 0$ when $y \in \text{cl}\, A$ (the closure of A) and positive elsewhere, $\Delta_A(y) < 0$ for $y \in \text{int}\, A$ (the interior of A), $\Delta_A(y) = 0$ for $y \in \text{bd}\, A$ (the boundary of A) and positive elsewhere.

When $A = C$ is a closed convex cone (see e.g. [10]),

$$\Delta_{-C}(y) = \max\{\langle \xi, y\rangle\,, \xi \in C' \cap S\},$$

where $C' = \{\xi \in R^m : \langle \xi, c\rangle \geq 0,\, \forall c \in C\}$ denotes the positive polar of the cone of C and $S = \{\xi \in R^m : \|\xi\| = 1\}$ is the unit sphere in R^m. In the sequel we also denote by $B = \{y \in R^m : \|y\| \leq 1\}$ the unit ball in R^m. Further properties of the oriented distance function can be found in [22].

Recently, function Δ_{-C} has been used (see e.g. [1], [3], [11], [12], [17], [3]) to scalarize the vector optimization problem $VP(f,K)$, that is solutions of $VP(f,K)$ can be characterized as solutions of suitable scalar optimization problems (for an axiomatization of this technique, see e.g. [16]).

The scalar problem we consider is

$$P(\tilde{f}, K) \min \tilde{f}(x) \ , \quad x \in K \ ,$$

where $\tilde{f}(x) = \Delta_{-C}\left(f(x) - f(x^0)\right)$, $x^0 \in K$. The relations among solutions of problem $P(\tilde{f}, K)$ and those of problem $VP(f, K)$ are investigated in [11, 22]. For the reader's convenience, we quote here the characterization of the weak efficient solutions. We stress that, although the scalarization depends on the norm, the following results remains true regardless the norm used in R^m.

Theorem 2. *The point $x^0 \in K$ is weak efficient for problem $VP(f, K)$ if and only if it is a solution of problem $P(\tilde{f}, K)$.*

From now on, if not otherwise specified, K denotes a st-sh subset of R^n. We consider the following definition of vector increasing along rays function.

Definition 4. A function $f : K \subseteq R^n \to R^m$ is vector increasing along rays starting at a point $x^0 \in \ker K$ when there exists a norm in R^m such that the (scalar) function $\tilde{f}(x) \in IAR(K, x^0)$.

If f fulfils Definition 4, we write $f \in VIAR(K, x^0)$.

The previous definition has a clear geometrical meaning and reduces to the notion of IAR function when $f : X \to R$. The $VIAR$ property is a monotonicity property, defined through the oriented distance function and not through the order introduced on Y by the cone C. However it strictly depends on the norm chosen, as the following example proves.

Example 2. Consider the function $f : R \to R^2$, defined as $f(x) = (x, g(x))$, where $g(x) = 2x$ if $x \in [0, 1]$ and $g(x) = -\frac{1}{4}x + \frac{9}{4}$ if $x \in (1, +\infty)$ and let $C = R^2_+, K = R_+$ and $x_0 = 0$. Then it is easy to show that function $f \in VIAR(K, x_0)$ if R^2 is endowed with the Euclidean norm l^2, but $f \notin VIAR(K, x_0)$ if R^2 is endowed with the norm l^∞.

The next propositions are meant to extend properties presented in Section 2 for IAR functions. For any omitted proof, we refer to [1].

Proposition 4. *Let $K \subseteq X$ be st-sh and $x^0 \in \ker K$. If $f \in VIAR(K, x^0)$, then*

$$f(x^0 + t_2(x - x^0)) - f(x^0 + t_1(x - x^0)) \notin -\operatorname{int} C, \quad \forall t_2 \geq t_1 > 0 \text{ and } \forall x \in K. \quad (1)$$

The reversal of Proposition 4 does not hold, as the next example shows.

Example 3. Let $X = R, Y = R^2$ endowed with the l^∞ norm, $K = R_+, C = R^2_+$ and $x^0 = 0$. Consider the function $f : K \to R^2$, defined as $f(x) = (2x - x^2, x^2 - 2x)$. Then f fulfils condition (1), but $f \notin VIAR(R_+, x^0)$.

Proposition 5. *Let $K \subseteq X$ be st-sh and $x^0 \in \ker K$. Then $f \in VIAR(K, x^0)$ if and only if for every $x \in K$ and $\varepsilon > 0$ such that $f(x) \in$*

$f(x^0) - C + \varepsilon B$, it holds $f(x^0 + t(x - x^0)) \in f(x^0) - C + \varepsilon B$, for every $t \in [0,1]$.

Proposition 6. *Let $x^0 \in \ker K$ and $f \in VIAR(K, x^0)$. Then:*

i) x^0 is a w-minimizer of f over K.

ii) The set $f^{-1}(f(x^0))$ is st-sh with $x^0 \in \ker f^{-1}(f(x^0))$.

Proof. i) Since $\tilde{f}(\cdot) \in IAR(K, x^0)$, then $x^0 \in \arg\min(\tilde{f}, K)$ and hence it is a weak efficient solution of $VP(f, K)$.

ii) the set $f^{-1}(f(x^0))$ is the set of global minimizers of $\tilde{f}$ and hence the result follows from Proposition 1 in [4].

We pointed out that the interest for IAR functions comes from the fact they can be regarded as generalized convex functions. When we try to extend Proposition 2 to $VIAR$ functions, some differences arise. Namely, let x^0 be w-minimizer for a function $f : R^n \to R^m$; we show in [7] that if f is C-convex, then $f \in VIAR(K, x^0)$, whatever the norm we choose on the space R^m. On the contrary, if f is C-quasiconvex, then it possesses the $VIAR$ property only for suitable choices of the norm in R^m.

Definition 5. (see [15]) Let K be a convex subset of R^n.

i) The function $f : R^n \to R^m$ is C-convex on K if for every $x^1, x^2 \in K$ and for every $t \in [0,1]$ it holds

$$f((1-t)x^1 + tx^2) - (1-t)f(x^1) - tf(x^2) \in -C$$

ii) The function $f : R^n \to R^m$ is C-quasiconvex on K if for every $y \in R^m$, the (level) set $\{x \in K : f(x) \in y - C\}$ is convex.

Remark 1. We wish to recall that the function f is C-convex if and only if the scalar function $\langle \xi, f \rangle$ is convex for every $\xi \in C'$. The same result does not hold for C-quasiconvex functions [15]. Anyway, we remind that when $C = R^n_+$, then f is C-quasiconvex if and only if every component of f is quasiconvex. Furthermore, when C is a polyhedral cone generated by n linearly independent vectors, then also C' is generated by n linearly independent vectors $\xi^1, \ldots, \xi^n$ and f is C-quasiconvex if and only if $\langle \xi^i, f \rangle$ is quasiconvex for every $i = 1, \ldots, n$.

Theorem 3. *Let K be a convex subset of X and let f be C-convex on K. If x^0 is weak efficient for f over K, then $f \in VIAR(K, x^0)$. Moreover, $\tilde{f}(\cdot) \in IAR(K, x^0)$ for every norm in R^m.*

Proof. Function f is C-convex if and only if the scalar function $\langle \xi, f(x) \rangle$, is convex for every $\xi \in C'$. Hence, whatever the norm in R^m, by (1), the function $\tilde{f}(\cdot)$ is the maximum of convex functions, hence convex with a (global) minimizer at x^0. Proposition 2 concludes the proof. □

Now we turn our attention to C-quasiconvex functions. In [7] it has been shown that the following construction leads to a norm in R^m.

Let $k \in \operatorname{int} C$, $\alpha > 0$ and consider the hyperplane $H_\alpha = \{y \in Y : \langle k, y\rangle = \alpha\}$. The compact base $G_\alpha = H_\alpha \cap C'$ of C' (see [15]) allows to define the compact convex set $\tilde{B}_\alpha = \operatorname{conv}\{G_\alpha \cup (-G_\alpha)\}$ (here $\operatorname{conv} A$ denotes the convex hull of the set A).

Proposition 7. (see [7]) *The function $\gamma : R^m \to R_+$, defined as $\gamma(y) = \alpha$ if and only if $y \in bd$, is a norm on R^m.*

We denote the norm defined by function γ as $\|\bullet\|_{C,k}$, to stress the dependence on both the ordering cone and the given $k \in \operatorname{int} C$.

Theorem 4. *Let K be a convex set, let f be C-quasiconvex on K and let x^0 be a weak efficient solution of $VP(f, K)$. Then, whatever $k \in \operatorname{int} C$, if R^m is endowed with the norm $\|\bullet\|_{C,k}$, then $f \in VIAR(K, x^0)$.*

To prove Theorem 4, in [7], we show that $\forall k \in \operatorname{int} C$, the functions $\tilde{f}(\cdot)$ are IAR. Indeed, as the next example shows for C-quasiconvex functions, not every norm in R^m makes $\tilde{f}(\bullet)$ IAR.

Example 4. Consider the function $f : R_+ \to R^2, f(x) = (f_1(x), f_2(x))$, with $f_1(x) = 0$, for $x \in [0, \frac{1}{3}]$ and $f_1(x) = x - \frac{1}{3}$, for $x \in (\frac{1}{3}, +\infty)$, while $f_2(x) = x$, for $x \in [0, \frac{1}{3}]$ and $f_2(x) = \frac{1}{3}$, for $x \in (\frac{1}{3}, +\infty)$.

Let $K = R_+, x^0 = 0$ and $C = R^2_+$. Further, consider the cone $C_1 = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0, \frac{1}{2}x_1 \leq x_2 \leq 2x_1\}$ and $k = (1, 1) \in \operatorname{int} C_1$. Then f is C-quasiconvex and x^0 is w-minimizer for f over K. If R^2_+ is endowed with a norm $l^p, 1 \leq p \leq +\infty$, then $f \in VIAR(K, x^0)$. But if R^2_+ is endowed with the norm $\|\cdot\|_{C_1,k}$, then $f \notin VIAR(K, x^0)$.

Here we state Theorem 4 for the Pareto case, which allows the use of the most common l^p norms.

Proposition 8. *Let $f : R^n \to R^m, C = R^n_+$ and assume f is C-quasiconvex. If x^0 is w-minimizer for f over K (K convex) and R^m is endowed with a l^p norm $(1 \leq p \leq +\infty)$, then $f \in VIAR(K, x^0)$.*

Proof. We start noting that, given a point $y = (y_1, \ldots, y_n) \in R^m \backslash (-R^m_+)$, if R^m is endowed with a l^p norm, we have $\Delta(y, -R^m_+) = (\sum_{\sum} (i \in I_+(y))^{\frac{1}{p}}$, for $1 \leq p < +\infty$ and $\Delta(y, -R^m_+) = \max\{y_i, i \in I_+(y)\}$, for $p = +\infty$, where $I_+(y) = \{i = 1, \ldots, n : y_i > 0\}$. Now, without loss of generality assume that $f(x^0) = 0$, take any $x \in K$, consider the ray $R_{x^0,x}$ and observe that the restriction of every function f_i (a component of f) on this ray is quasiconvex.

To show that $f \in VIAR(K, x^0)$, consider any two numbers $t_1, t_2 > 0$, with $t_2 > t_1$. If $f_i(x + t_1(x - x^0)) > 0$, then it cannot be $f_i(x^0 + t_2(x - x^0)) < f_i(x+t_1(x-x^0))$. In fact, if this is the case, the level set, $\{t : f_i(x^0+t(x-x^0)) \leq \max\{0, f_i(x^0+t_2(x-x^0))\}\}$ would not be convex, since t_1 does not belong to it. Form these considerations, it follows readily that the function $\tilde{f} \in IAR(K, x^0)$, which completes the proof. □

An easy exercise would be to extend the previous result to the case where C' is a polyhedral cone generated by n linearly independent vectors $\xi^1, \ldots, \xi^n$.

Finally we try to extend Theorem 1, that is to relate $VIAR$ property at $x^0 \in K$ with solutions of Minty variational inequality problem. The study of vector Minty-type variational inequalities has been first presented in [9] and has been deepened in [20]. This approach is based on a vector-valued variational inequality. However, other researches, see e.g. [6], suggest a different approach which relates the scalar generalized Minty variational inequality $MVI(\tilde{f}_-, K)$ of Section 1 directly to vector optimization problems. Basically we refer to the Dini derivative of the function $\tilde{f}$. As for IAR functions we need to assume some radial continuity.

Definition 6. i) Let f be a function defined on a set $K \subseteq X$. We say that f is C- continuous at $\bar{x}$ when for every neighbourhood U of $\bar{x} \in X$, there exists a neighbourhood V of $f(\bar{x}) \in Y$, such that

$$f(x) \in V + C\,, \quad \forall x \in U \cap K\,.$$

ii) We say that f is C- continuous on K, when f is C-continuous at any point of K.

iii) Let K be a st-sh set with $x^0 \in \ker K$ and let f be a function defined on an open set containing K. The function f is said to be C-radially continuous in K along the rays starting at x^0 (for short, $f \in C - RC(K, x^0)$), if for every $x \in K$, the restriction of f on the interval $R_{x^0,x} \cap K$ is C-continuous.

Proposition 9. *Let $f \in C - RC(K, x^0)$. Then $\tilde{f}(x)$ is radially lower semicontinuous in K along the rays starting at x^0($\tilde{f} \in RLSC(K, x^0)$).*

Proposition 10. *Let K be a st-sh set and $x^0 \in \ker K$. Assume that f is a function defined on an open set containing K.*

i) Let $f \in C-RC(K, x^0)$. If x^0 solves $MVI(\tilde{f}_-, K)$, then $f \in VIAR(K, x^0)$.

ii) Conversely, if $f \in VIAR(K, x^0)$, then x^0 solves $MVI(\tilde{f}_-, K)$.

Proof. The proof follows recalling Proposition 9 and Theorem 1. □

Similarly to the scalar case, the assumption $f \in C - RC(K, x^0)$ appears in only one of the two opposite implications.

4. A General Scheme

The class of $VIAR$ functions introduced in the previous section contains some arbitrary elements. For instance the scalarization used could be replaced by other technique. Moreover the choice to assume in Propositions 2 and 6 that "x^0 weakly efficient" generalizes assumption ii) in Proposition 2 could be replaced by other solution concept of $VP(f,K)$. These arguments motivates a further investigation to make as general as possible the approach of Section 3.

First we focus on the notion of scalarization itself. Following the axiomatic approach presented in [16] we denote a scalarization by means of a scalar function $\gamma : R^m \to R$ and its composition with $f : R^n \to R^m$, namely $\gamma\left(f \leq ft(\cdot) - f\left(x^0\right)\right)$. We define the set $\Gamma := \{\gamma : R^m \to R\}$ of all considered scalarizing functions. For instance, throughout this section we may consider as guideline examples the following sets:

$$\Gamma_1 := \left\{\gamma_{\|\cdot\|}(y) = \sup\left\{\langle\xi, y\rangle\,,\ \xi \in C' \cap S_{\|\cdot\|}\right\}, \|\cdot\| \in I\right\}\,,$$

where $S_{\|\cdot\|}$ is the unit sphere of R^m, with respect to the norm $\|\cdot\|$, and I is the set all the possible norms over R^m, which define the nonlinear scalarization of Section 3, and:

$$\Gamma_2 := \left\{\gamma_\xi(y) = \langle\xi, y\rangle\,,\ \xi \in C' \backslash \{0\}\right\}\,,$$

which denotes linear scalarization:

Given a parameter $x^0 \in \ker K$, the family of scalarized functions depending on Γ is denoted by

$$\Phi(\Gamma, x^0) := \left\{f_\gamma(x) = \gamma\left(f(x) - f(x^0)\right), \gamma \in \Gamma\right\}.$$

We make the following assumptions on the scalarization

i) if x^0 is a weak efficient solution of $VP(f,K)$, then $\exists\ f_\gamma \in \Phi(\Gamma, x^0)$ such that $x^0 \in \arg\min f_\gamma(x)$;

ii) if $f : R^n \to R^m$ is C-convex, then every $f_\gamma \in \Phi(\Gamma, x^0)$ is convex;

iii) if f is C-quasiconvex, then $\exists\ f_\gamma \in \Phi(\Gamma, x^0)$ such that $f_\gamma(\cdot)$ is quasiconvex;

iv) all $f_\gamma \in \Phi(\Gamma, x^0)$ are strictly order preserving, that is

$$\forall x^1, x^2 \in K,\ \ f(x^2) - f(x^1) \in -C \Rightarrow f_\gamma(x^2) \leq f_\gamma(x^1)\,,$$

$$\forall x^1, x^2 \in K,\ \ f(x^2) - f(x^1) \in -intC \Rightarrow f_\gamma(x^2) \leq f_\gamma(x^1)\,.$$

Properties *i), ii), iii)* and *iv)* are easily verified by both choices of Γ_1 and Γ_2. We underline that *iv)* ensures that minimizers of the scalar function f_γ are at least weak efficient solutions of $VP(f,K)$ (for more details on the topic we refer to [15], [16]).

Under this setting, we generalize the definition of Section 3.

Definition 7. A vector function $f : K \subseteq R^n \to R^m$, with K st-sh set is increasing along rays with respect to the family Γ at $x^0 \in \ker K$ (for short $f \in \Gamma - IAR\left(K, x^0\right)$), when $\exists\ f_\gamma \in \Phi(\Gamma, x^0)$ such that f_γ is IAR at x^0 over K.

The following properties motivate the definition.

Proposition 11. *Let $x^0 \in \ker K$ and $f \in \Gamma - IAR\left(K, x^0\right)$. Then:*

i) x^0 is a weak efficient solution of $VP(f, K)$;

ii) the set $f^{-1}\left(f\left(x^0\right)\right)$ is star-shaped with $x^0 \in \ker K$.

Proof. $i)$ Since there exists some $f_\gamma \in \Phi(\Gamma, x^0)$ such that $f_\gamma \in IAR(K, x^0)$, then x^0 is a minimizer for f_γ. Hence, by iv), we have the thesis;

ii) The set $f^{-1}\left(f\left(x^0\right)\right)$ is the set of global minimizers of a scalar IAR function, hence the result follows from Proposition 1. □

It turns out that the nature of x^0 plays a crucial role to define the relations among convex (or generalized convex) functions and $VIAR$ ones.

First note that, because of i), weak efficiency is sufficient for C- convex function.

Corollary 1. *Let f be C- convex and x^0 be a weak efficient solution of $VP(f, K)$. Then $f \in \Gamma - IAR\left(x^0, k\right)$, for all Γ which satisfies i), iv).*

Proof. By i), there exists $f_\gamma \in \Phi(\Gamma, x^0)$ such that $x^0 \in \arg\min f_\gamma(x)$ and, by *ii)*, f_γ is also convex. Hence $f_\gamma \in IAR(x^0, K)$. □

Besides the result of Corollary 1, we can have stronger result if we focus on a set Γ and we consider efficiency as related to the choice of Γ itself.

Definition 8. A vector $x^0 \in \ker K$ is a minimizer with respect to (w.r.t.) Γ for f over K when x^0 is a solution of $VP(f, K)$ such that $x^0 \in \arg\min f_\gamma(x)$ for all $f \in \Phi_\gamma(\Gamma, x^0)$.

As said in the previous section, minimizers with respect to Γ_1 are weak efficient solutions of $VP(f, K)$. On the other hand, it is an easy exercise to show that ideal solutions of $VP(f, K)$ are minimizer with respect to Γ_2.

Proposition 12. *Let $K \subseteq R^n$ be convex and $f : K \to R^m$ be C-convex over K. If x^0 is a minimizer w.r.t. Γ, then $f \in \Gamma - IAR\left(K, x^0\right)$. Moreover every $f_\gamma \in \Phi(\Gamma, x^0)$ is IAR at $x^0 \in K$.*

Proof. For all $\gamma \in \Gamma$, the function $f_\gamma(\cdot)$ is convex with a minimizer at x^0. Hence every $f_\gamma \in \Phi(\Gamma, x^0)$ is IAR, which also implies $f \in \Gamma - IAR\left(K, x^0\right)$. □

The relation among the choice of Γ and the efficiency required to x^0 is crucial.

Example 5. The function $f : R \to R^2$ defined by the two criteria:

$$f_1(x) = -x\,, \qquad f_2(x) = \begin{cases} 0 & x < 0\,, \\ x & x \geq 0\,, \end{cases}$$

is $\Gamma - IAR\left(R^2, x^0\right), x^0 = 0$, with respect to both Γ_1 and Γ_2. However, although f is R^2_+-convex, since x^0 is efficient (hence weakly efficient) solution of $VP(f, K)$ but not ideal, $\Gamma - IAR$ property with respect to Γ_2 cannot be derived by Proposition 12.

We now consider the broader class of C-quasiconvex functions. As for the scalar case, quasiconvexity does not imply $VIAR$ property without further assumptions on x^0.

Proposition 13. *Let $K \subseteq R^n$ be convex and $f : K \to R^m$ be C-quasiconvex over K. If x^0 is a minimizer w.r.t. Γ, then $f \in \Gamma - IAR\left(K, x^0\right)$.*

Proof. By iii) there exists $\tilde{f}_\gamma \in \Phi(\Gamma, x^0)$ such that it is quasiconvex. Definition 8 implies that x^0 is a minimizer also for the latter $\tilde{f}_\gamma$. Hence $\tilde{f}_\gamma \in IAR(K, x^0)$, which concludes the proof. □

The choice of Γ obliges to consider only some concepts of efficiency for x^0.

Example 6. Let $f : R \to R^2$ be defined by the two criteria

$$f_1(x) = -x\,, \qquad f_2(x) = x^3.$$

The function is R^2_+-quasiconvex (not convex) and $x^0 = 0$ is a weak efficient solution of $VP(f, R)$ (not ideal). If we consider $VIAR$ property w.r.t. the set Γ_2, we easily get that $\gamma_{\hat{\xi}}$, for $\hat{\xi}(0, 1)$, proves iii) is satisfied. However one has.

$$f_y(x) = \gamma_\xi \left(f(x) - f(x^0)\right) = -\xi_1 x + \xi_2 x^3\,, \quad \forall\, \xi \in \left(R^2_+\right)' \setminus \{0\}\ ,$$

which is not IAR at x^0.

Finally we consider variational inequalities.

Proposition 14. *Let $f : K \subseteq R^n \to R^m, K$ star-shaped, be $VIAR$ at $x^0 \in \ker K$ over K w.r.t. Γ. Then $\exists\, f_\gamma \in \Phi(\Gamma, x^0)$ such that x^0 is a solution of $MVI\left((f_\gamma)', K\right)$.*

Proof. Since f is $VIAR$ w.r.t. Γ, there exists $f_\gamma \in \Phi(\Gamma, x^0)$ which is IAR at x^0 over K. Hence the thesis follows from Theorem 1. □

Corollary 2. *Let $x^0 \in \ker K, f : R^n \to R^m$ be C-convex and $VIAR$ at x^0 with respect to Γ.*

i) If x^0 is a minimizer w.r.t. Γ, then x^0 solves $MVI\left((f_\gamma)', K\right)$, $\forall\, f_\gamma \in \Phi(\Gamma, x^0)$.

ii) There exists $f_\gamma \in \Phi(\Gamma, x^0)$ such that x^0 solves $MVI\left((f_\gamma)'_-, K\right)$.

Proof. i) is an easy consequence of the previous results. As for ii), Proposition 11 guarantees that x^0 is at least weak efficient. Now apply i) and Proposition 12 to get easily the thesis. □

Remark 2. A suitable choice of Γ is the set

$$\Gamma := \{\gamma_{a,e}(y) := \min\{t \in R \,|\, y \in a + te - C\}\,;\, a \in R^m,\, e \in \operatorname{int} C\}\ ,$$

which defines the scalarization by the minimal strictly monotone map in [15].

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