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PEST MANAGEMENT ABOUT OMNIVORA WITH CONTINUOUS BIOLOGICAL CONTROL

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Abstract: This paper develops a predator-prey system with modified Leslie-Gower and Holling-type II schemes, by introducing continuous releasing for predator. It is shown that the model admits a globally asymptotically stable pest-eradication equilibrium under some appropriate conditions. Further, in order to establish a procedure to maintain the pests at an acceptably low level, we educe the global stability of the positive equilibrium.

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1. Introduction

In order to avoid the serious ecological and economic problems caused by pest outbreak, people develop a wide range of pest strategies. Chemical control usually means the use of pesticides. Pesticides are effectual because they can quickly kill a significant portion of the pest population. For example, the American state of Utah once was invaded by a great deal of Mormon crickets and grasshoppers, which caused severe damage to the local crop. The federal gov-

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ernment settled this problem mostly by spraying pesticides. However, pesticide pollution is also recognized as a major health hazard to human being and beneficial insects. Furthermore, it would increase the cost of pest control. So we should be cautious to its use. Another important method is biological control. It generally means man's use of a suitably chosen living organism, referred as the biocontrol agent, to control another. Biocontrol agents can be predators, pathogens or parasites of the organism to be controlled that either kill the harmful organism or interfere with its biological processes [?], [?], [?], [?]. For example, the orangewoods in California were damaged by some kind of aphid population in 1888. Then the government imported its natural enemies – ladybird beetle from Australia. And before long the pests were well controlled and orangewoods were primely protected.

Recently, a lot of people research the pest management and many models are established [?], [?], [?], [?], [?], but there is only a little investigation about the omnivora [?], [?]. In this paper we discuss the systems in which the growth of the pest is of Holling-type II and its natural enemy is of modified Leslie-Gower function. It means that the predator eat not only the pest but also other food, which is common in nature. We control the pest in the way of continuously releasing its natural enemy. When the predator's releasing reaches a certain quantity, the pest is controlled and maintains at an acceptably low level in the long term. With the increasing of the releasing, the pest is extinct. The main purpose of this paper is to construct a realistic model of biocontrol for pest management, investigate their dynamics and compare the results obtained for the ordinary differential model. This paper is arranged as follows: In the following section, we formulate the model for predator-prey system with the continuous release. In Section 3, we give the conditions in which the pest exists permanently or becomes extinct. Then we investigate the local and global stability of the equilibrium in above two cases. At last the biologic significance is given.

2. Model Formulation

In 1948, Leslie [?] introduced a predator-prey model where the carrying capacity of the predator's environment is proportional to the number of prey. It is $\frac{dy}{dt} = r_2 y(1 - \frac{y}{\alpha x})$, in which the growth of the predator is logistic. The term $\frac{y}{\alpha x}$ of this equation is called Leslie-Gower term. It measures the loss in the predator population due to the rarity (per capita y/x) of its favorite food. If its most favorite food (x) is not available in abundance, the predator (y) can switch over to other populations but its growth will be limited. This situation can be taken care of by adding a positive constant to the denominator. Then the equation becomes $\frac{dy}{dt} = r_2 y(1 - \frac{y}{\alpha x + d})$. M.A. Aziz and M. Daher [**?**] proposed the following model

$$\begin{cases} \dfrac{dx}{dt} = (r_1 - b_1 x - \dfrac{a_1 y}{x + k_1})x\,, \\ \dfrac{dy}{dt} = (r_2 - \dfrac{a_2 y}{x + k_2})y\,. \end{cases} \tag{2.1}$$

Here $x(t)$ and $y(t)$ denote the population densities of prey and predator at time t. All parameters are positive constants. r_1 is the intrinsic birth rate of prey x , b_1 measures the competitive strength among individuals of species x. a_1 is the maximum value which per capita reduction rate of x can attain, and a_2 has the similar meaning to a_1. k_1 and k_2 respectively measure the extent to which environment provides protection to prey x and to predator y.

The results they obtained are boundedness of the solutions, existence of an attracting set and global stability of the coexisting interior equilibrium.

In the system, the prey is the favorite food of the predator. Actually, nature abounds in systems which exemplify this model, such as insect pest-spider food chain (see [**?**]).

Now we adapt the model (2.1) and construct the following system to describe the pest management:

$$\begin{cases} \dfrac{dx}{dt} = (r_1 - b_1 x - \dfrac{a_1 y}{x + k_1})x\,, \\ \dfrac{dy}{dt} = (r_2 - \dfrac{a_2 y}{x + k_2})y + u\,. \end{cases} \tag{2.2}$$

The constant u is the release amount of the predator. The other coefficients have the same biological significance as in the model (2.1).

The problem we need to solve is how to release the predator (natural enemy) y such that the prey (pest) x reduces to an acceptably low level to avoid economic loss. So the target is $x(t) < c$, as $t \to \infty$, where c is a given constant.

3. Qualitative Analysis for Model (2.2)

In this section, we investigate the dynamics of the ordinary differential system (2.2) by means of stability analysis and apply the subsequently obtained stability results to study the control problem. For biological reason, we restrict our discussion to the feasible region $R^2_+ = \{(x, y) \mid x \geq 0, y \geq 0\}$.

The following lemma is obvious.

Lemma 1. *Let $z(t) = (x(t), y(t))$ be the solution of system (2.2) with $z(0) \geq 0$, then $z(t) \geq 0$, for all $t \geq 0$. And further $z(t) > 0$, for all $t \geq 0$, if $z(0) > 0$.*

Theorem 2. *Let $(x(t), y(t))$ be the solution of system (2.2) and S be the set*

$$S = \left\{(x, y) \in R_+^2 \mid 0 \leq x \leq \frac{r_1}{b_1}, 0 \leq x + y \leq M\right\},$$

where

$$M = \frac{1}{4a_2 b_1}[a_2 r_1(4 + r_1) + (1 + r_2)^2(r_1 + b_1 k_2)] + u\,.$$

Then $(x(0), y(0)) \in R_+^2$ implies $(x(t), y(t)) \to S$, as $t \to \infty$.

Proof. Let $(x(0), y(0)) \in R_+^2$. Then $(x(t), y(t))$ remain nonnegative by Lemma 1. It is obvious that $x(t) \leq u(t)$, where $u(t)$ satisfies $\frac{du}{dt} = u(t)(r_1 - b_1 u(t)), u(0) = x(0) > 0$. Thus we have $u(t) = \frac{r_1 u(0)}{b_1 u(0) + (r_1 - b_1 u(0))e^{-r_1 t}}$. So there exist a $T_1 > 0$, such that $x(t) \leq \frac{r_1}{b_1}$, for all $t > T_1$.

We define the function $V(t) = x(t) + y(t)$ and calculate the derivative of $V(t)$ along the solution of system (2.2). Then we have

$$\begin{aligned}\dot{V}(t) = \dot{x}(t) + \dot{y}(t) &= (r_1 - b_1 x - \frac{a_1 y}{x + k_1})x + (r_2 - \frac{a_2 y}{x + k_2})y + u \\ &\leq (r_1 - b_1 x)x + x + (1 + r_2 - \frac{a_2 y}{x + k_2})y + u - V\,.\end{aligned}$$

Because (see [?]),

$$\max_{R_+^2}(r_1 - b_1 x)x = \frac{r_1^2}{4b_1},$$

$$\max_{R_+^2}(1 + r_2 - \frac{b_1 a_2 y}{r_1 + b_1 k_2})y = \frac{(1 + r_2)^2(r_1 + b_1 k_2)}{4b_1 a_2},$$

then $\dot{V}(t) + V(t) \leq \frac{r_1^2}{4b_1} + \frac{r_1}{b_1} + \frac{(1+r_2)^2(r_1+b_1k_2)}{4b_1 a_2} + u = M$, for all $t > T_1$. From the comparison theorem, we have $V(t) \leq M + (V(0) - M)e^{-t}$. Further, for every $\epsilon > 0$, there exists $T_2 > T_1$, such that when $t > T_2$, $(V(0) - M)e^{-t} < \epsilon$. So for all $t > T_2$, we have $x(t) + y(t) < M + \epsilon$. Hence $\lim_{t \to \infty} x(t) + y(t) < M$. This completes the proof. □

From Theorem 2, we know that all solutions of system (2.2) initiating in R_+^2 are ultimately bounded and eventually enter the attracting set S.

Now we prove the existence of the pest-eradication equilibrium E_1 and the coexisting equilibrium E_2 by means of geometric methods. We define

$$P(x,y) = (r_1 - b_1 x - \frac{a_1 y}{x+k_1})x, Q(x,y) = (r_2 - \frac{a_2 y}{x+k_2})y + u\,.$$

The two isoclines are l_1: $P(x,y)=0$ and l_2: $Q(x,y)=0$, where the isocline l_1 includes the line l_3: $x=0$ and the curve l_4: $r_1 - b_1 x - \frac{a_1 y}{x+k_1} = 0$. It is easy to verify that when $x = \frac{r_1 - b_1 k_1}{2b_1}$, y in the curve l_4 gets its maximum $y_{max} = \frac{(r_1 - b_1 k_1)^2}{4a_1 b_1}$. We define $E_1 = (0, y_2)$, where $y_2 = \frac{k_2 r_2 + \sqrt{k_2^2 r_2^2 + 4a_2 k_2 u}}{2a_2}$, which is the point of intersection of the curve l_2 and the line l_3; $F = (0, \frac{r_1 k_1}{a_1})$, which is the point of intersection of the line l_3 and the curve l_4.

Theorem 3. *(i) Assume* $y_2 > \frac{r_1 k_1}{a_1}$ *and* $\frac{r_2}{a_2} > \frac{r_1 - b_1 k_1}{a_1}$*, then system* (2.2) *has a unique equilibrium* $E_1(0, y_2)$*;*

(ii) Assume $y_2 < \frac{r_1 k_1}{a_1}$ *and* $b_1 k_1 > r_1$*, then system* (2.2) *has a unique positive equilibrium* $E_2(x^*, y^*)$.

Proof. (i)

$$\frac{dy}{dx}\,|_{l_2} = \frac{(r_2 y + u)^2}{a_2 r_2 y^2 + 2a_2 uy} = \frac{r_2}{a_2} + \frac{u^2}{a_2(r_2 y^2 + 2uy)} > \frac{r_2}{a_2} > 0\,, \tag{3.1}$$

$$\frac{dy}{dx}\,|_{l_4} = \frac{1}{a_1}(r_1 - b_1 k_1 - 2b_1 x) < \frac{r_1 - b_1 k_1}{a_1}\,. \tag{3.2}$$

The isocline l_2 is strictly increasing as x increases in R_+^2 due to (3.1). For the unique existence of the equilibrium E_1, l_2 and l_4 should not intersect in R_+^2. It means that $y_2 > \frac{r_1 k_1}{a_1}$ and $\frac{dy}{dx}\,|_{l_2} > \frac{dy}{dx}\,|_{l_4}$. Hence we get result of (i) (see Figure 1).

(ii) When $b_1 k_1 > r_1$, according to (3.2), we have $\frac{dy}{dx}\,|_{l_4} < 0$ in R_+^2, so l_4 is strictly decreasing as x increases in R_+^2. Then adding the condition $y_2 < \frac{r_1 k_1}{a_1}$, we conclude that l_2 and l_4 have a unique intersection $E_2(x^*, y^*)$ in R_+^2. It follows that system (2.2) has a unique positive equilibrium $E_2(x^*, y^*)$ (see Figure 2). The proof is completed.

Now we study the local stability of the equilibriums by means of Jacobian matrix analysis.

Theorem 4. *(i) If* $y_2 > \frac{r_1 k_1}{a_1}$*, then* $E_1(0, y_2)$ *is locally asymptotically stable.*

(ii) If $y_2 < \frac{r_1 k_1}{a_1}$ *and* $b_1 k_1 > r_1$*, then* E_1 *is unstable, while* $E_2(x^*, y^*)$ *is locally asymptotically stable.*

Proof. (i) Linearizing the model (2.2) at the point E_1, we easily get the eigenvalues $\lambda_1 = r_1 - \frac{a_1 y_2}{k_1}$ and $\lambda_2 = r_2 - \frac{2a_2 y_2}{k_2}$. For the local stability of E_1, we need $\lambda_1 < 0$ and $\lambda_2 < 0$, which implies $y_2 > \frac{r_1 k_1}{a_1}$ and $y_2 > \frac{r_2 k_2}{2a_2}$. The last

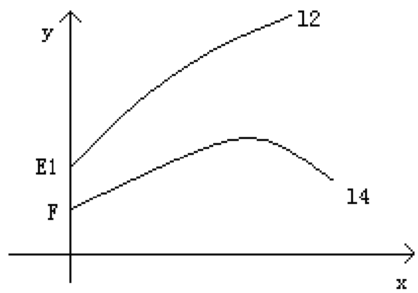

Figure 1: No positive equilibrium

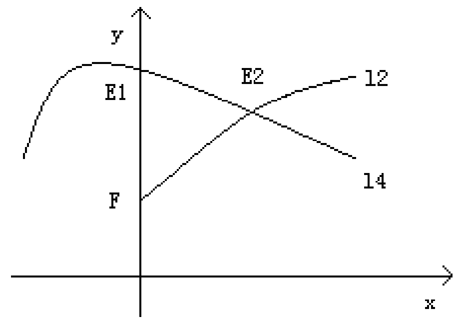

Figure 2: A unique positive equilibrium

inequality holds spontaneously. So we get the result of (i).

(ii) The positive equilibrium $E_2(x^*, y^*)$ satisfies

$$r_1 - b_1x^* - \frac{a_1y^*}{x^*+k_1} = 0, (r_2 - \frac{a_2y^*}{x^*+k_2})y^* + u = 0\,.$$

The Jacobian matrix of the system (2.2) at $E_2(x^*, y^*)$ is given by

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial x} \end{pmatrix}_{E_2(x^*,y^*)} = \begin{pmatrix} x^*(-b_1 + \frac{(r_1-b_1x^*)^2}{a_1y^*}) & -\frac{x^*}{y^*}(r_1 - b_1x^*) \\ \frac{(r_2y^*+u)^2}{a_2y^{*2}} & -r_2 - \frac{2u}{y^*} \end{pmatrix}.$$

Its eigenvalues λ_1, λ_2 satisfy

$$\lambda_1 + \lambda_2 = x^*(-b_1 + \frac{(r_1 - b_1x^*)^2}{a_1y^*}) + (-r_2 - \frac{2u}{y^*})$$
$$= -\frac{x^*}{a_1y^*}(a_1b_1y^* - (r_1 - b_1x^*)^2) - r_2 - \frac{2u}{y^*}\,,$$

$$\lambda_1\lambda_2 = x^*(-b_1 + \frac{(r_1 - b_1x^*)^2}{a_1y^*})(-r_2 - \frac{2u}{y^*}) + \frac{x^*}{y^*}(r_1 - b_1x^*)\frac{(r_2y^*+u)^2}{a_2y^{*2}}$$
$$= \frac{x^*}{a_1y^{*2}}(r_2y^* + 2u)(a_1b_1y^* - (r_1 - b_1x^*)^2) + \frac{x^*(r_1 - b_1x^*)(r_2y^*+u)^2}{a_2y^{*3}}\,.$$

According to Theorem 2, we have $(x^*, y^*) \in S$. Thus $r_1 - b_1x^* > 0$. When $b_1k_1 > r_1$, owing to $y^* = \frac{1}{a_1}(r_1 - b_1x^*)(x^* + k_1)$, we have

$$a_1b_1y^* - (r_1 - b_1x^*)^2 = (r_1 - b_1x^*)(2b_1x^* + b_1k_1 - r_1) > 0$$

which implies $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$. So we have $\lambda_1 < 0$ and $\lambda_2 < 0$. This completes the proof. □

In the following, we present that under some assumptions, the steady state is globally asymptotically stable.

Theorem 5. *(i) If* $y_2 > \frac{r_1k_1}{a_1}$ *and* $\frac{r_2}{a_2} > \frac{r_1-b_1k_1}{a_1}$*, then* $E_1(0,y_2)$ *is globally asymptotically stable.*

(ii) If $y_2 < \frac{r_1k_1}{a_1}, b_1k_1 > r_1$ *and* $\frac{a_1}{k_1^2} < \frac{b_1^2a_2}{r_1^2+r_1b_1k_2}$*, then* $E_2(x^*,y^*)$ *is globally asymptotically stable.*

Proof. (i), From Theorem 3(i) and Theorem 4(i), when $y_2 > \frac{r_1k_1}{a_1}$ and $\frac{r_2}{a_2} > \frac{r_1-b_1k_1}{a_1}$ hold, $E_1(0,y_2)$ is the unique equilibrium of system (2.2) and it is locally stable. Ulteriorly, from Theorem 2, all solutions of system (2) initiating in R_+^2 are ultimately bounded. So $E_1(0,y_2)$ is globally asymptotically stable.

(ii) Because each positive solutions of system (2) is uniformly bounded and eventually enter the set S, so we just need to consider the situation in the region S. We define the Dulac function $B(x,y) = x^{-1}y^{-1}$. Then

$$D = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = -\frac{b_1}{y} + \frac{a_1}{(x+k_1)^2} - \frac{a_2}{x(x+k_2)} - \frac{u}{xy^2}$$
$$< \frac{a_1}{(x+k_1)^2} - \frac{a_2}{x(x+k_2)} < \frac{a_1}{k_1^2} - \frac{a_2}{\frac{r_1}{b_1}(\frac{r_1}{b_1}+k_2)} = \frac{a_1}{k_1^2} - \frac{b_1^2a_2}{r_1^2+r_1b_1k_2}\,.$$

We let $\frac{a_1}{k_1^2} - \frac{b_1^2a_2}{r_1^2+r_1b_1k_2} < 0$. Then we have $D < 0$. By Bendixson-Dulac Theorem, we obtain that system (2.2) does not have a limit cycle in the region S. Together with Theorem 2 which means the boundedness of all positive solutions and Theorem 4(ii) which means the local stability of $E_2(x^*,y^*)$, we get the result of Theorem 5(ii). This complete the proof. □

Remark. For the biological control strategy, we should regulate the pest's density (numbers of pests unit area) below the economic injury level (EIL). Below this level the cost of implementing the control measure is less than the loss of suffering without control action. We now let L be the economic injury level of the pest population and discuss the strategy to control target pests.

The interior equilibrium $E_2(x,y)$ satisfies

$$r_1 - b_1x - \frac{a_1y}{x+k_1} = 0, \qquad (r_2 - \frac{a_2y}{x+k_2})y + u = 0\,.$$

We easily get $u = \frac{a_2y^2}{x+k_2} - r_2y$. So the pest x decreases as the releasing number u increases. Owing to $y = \frac{1}{a_1}(r_1-b_1x)(x+k_1)$, we have

$$u = \frac{(r_1-b_1x)(x+k_1)}{a_1^2(x+k_2)}[a_2(r_1-b_1x)(x+k_1) - a_1r_2(x+k_2)] =: f(x)\,.$$

Thus, for any positive ϵ small enough, we may choose the control variable $u \geq f(L-\epsilon)$, to control the target susceptible pest population below L. Moreover, when $u > \frac{r_1k_1}{a_1^2k_2}(a_2r_1k_1 - a_1r_2k_2)$, the pest will become extinct.

4. Discussion

In this paper, we investigate a continuous control strategies for pest management about a omnivorous model. We analyze the pest-extinction solution to guarantee its global asymptotic stability, and obtain the condition for the permanence of this system. We hope our results will provide an insight to pest management practicians. There are some other interesting questions. In real world, the time each pest grows and propagates is limit, so how to control them in short time? And the releasing is always at fixed moment, so what about the impulsive perturbation on the predator? We will continue to study these problems in the future.

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