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APPROXIMATION PROPERTIES OF CERTAIN NONLINEAR SUMMATION OPERATORS

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Abstract: We investigate a class of nonlinear summation operators L_n^* defined by a modification of some positive linear operators L_n in polynomial weighted spaces. The operators $L_{n;r}$ and $L_{n;r}^*$ for r-times defferentiable functions will be examined also.

We give direct approximation theorems for considered operators.

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1. Introduction

1.1. M. Becker in the paper [2] has examined approximation properties of the Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \in \mathbb{R}_0\,,\ n \in \mathbb{N},$$

($\mathbb{R}_0 = [0,\infty)$, $\mathbb{N} = \{1,2,\ldots\}$) in polynomial weighted spaces $C_p \equiv C_p(\mathbb{R}_0)$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here the space C_p is connected with the weighted function w_p,

$$w_0(x) := 1, \qquad w_p(x) := (1+x^p)^{-1} \qquad \text{if} \quad p \geq 1, \tag{1}$$

for $x \in \mathbb{R}_0$, and it is the set of all functions $f : \mathbb{R}_0 \to \mathbb{R}$ for which $w_p f$ is

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uniformly continuous and bounded on $\mathbb{R}_0$. The norm in C_p is defined by the formula

$$\|f\|_p := \sup_{x\in\mathbb{R}_0} w_p(x)|f(x)|. \tag{2}$$

1.2. In [10] and many papers (e.g. [6-9]) approximation properties of modified Szász-Mirakyan operators

$$S_{n;r}(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\sum_{j=0}^{r}\frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x-\frac{k}{n}\right)^j, \quad x\in\mathbb{R}_0, \quad n,r\in\mathbb{N},$$

and other modified operators for functions f belonging to the space $C^r(\mathbb{R}_0)$ of all r-times differentiable functions $f\in C_r(\mathbb{R}_0)$ which derivatives $f^{(k)}\in C_{r-k}(\mathbb{R}_0)$, $k=1,2,\ldots,r$ were examined. The norm of $f\in C^r$, $r\in\mathbb{N}$, is defined by $\|f\|_r$.

In the above papers it was shown that the order of approximation of $f\in C^r$ by $S_{n;r}(f)$ (or other correspondent operators $L_{n;r}(f)$ of the type $S_{n;r}(f)$) is dependent on r and it improves if r grows.

1.3. Recently, C. Bardaro and I. Mantellini in the paper [1] (Section 6) examined the Voronovskaya type theorem for nonlinear Szász-Mirakyan operators defined by the following formula:

$$S_n^*(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}G_n\left(f\left(\frac{k}{n}\right)\right), \quad x\in\mathbb{R}_0, \quad n\in\mathbb{N}, \tag{3}$$

where $(G_n)_1^\infty$ is a sequence of nonlinear functions $G_n:\mathbb{R}\to\mathbb{R}$ satisfying certain conditions, in particular: for some $\alpha>0$ there holds

$$\lim_{n\to\infty} n^\alpha\left(G_n(u)-u\right)=0 \quad \text{at every} \quad u\in\mathbb{R}.$$

1.4. In this paper we shall extend the Bardaro-Mantellini definition (3) to certain class of operators L_n and $L_{n;r}$ considered in polynomial weighted spaces $C_p(\mathbb{R}_0)$ and $C^r(\mathbb{R}_0)$.

In Section 2 we shall give definitions of operators L_n, $L_{n;r}$, L_n^* and $L_{n;r}^*$ and some auxiliary results. The approximation theorems will be given in Section 3.

In this paper we shall denote by $M_k(a,b)$, $k\in\mathbb{N}$, suitable positive constants depending only on indicated parameters a, b.

2. Definitions and Auxiliary Results

2.1. Let $A = [a_{nk}(\cdot)]$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, be an infinite matrix of functions $a_{nk} \in C_0(\mathbb{R}_0)$ satisfying the following conditions:

(i) $a_{nk}(x) \geq 0$ for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$,

(ii) $\sum\limits_{k=0}^{\infty} a_{nk}(x) = 1$ for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$,

(iii) the series $\sum\limits_{k=0}^{\infty} k^s a_{nk}(x)$ is convergent on $\mathbb{R}_0$ for $n, s \in \mathbb{N}$ and its sum is a function belonging to $C_s(\mathbb{R}_0)$ for every $n \in \mathbb{N}$,

(iv) for every $s \in \mathbb{N}$ there exists a positive constant $M_1(s)$ independent on $n \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}_0} w_{2s}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x\right)^{2s} \leq M_1(s) n^{-s} \quad \text{for} \quad n \in \mathbb{N}.$$

Using the above matrix A we define for functions $f \in C_p(\mathbb{R}_0)$, $p \in \mathbb{N}_0$, the operators

$$L_n(f;x) \equiv L_n(f;A;x) := \sum_{k=0}^{\infty} a_{nk}(x) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{4}$$

and for $f \in C^r(\mathbb{R}_0)$, $r \in \mathbb{N}$, the following operators

$$L_{n;r}(f;x) \equiv L_{n;r}(f;A;x) := \sum_{k=0}^{\infty} a_{nk}(x) F_r\left(\frac{k}{n}, x\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{5}$$

where

$$F_r(t,x) := \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j \quad \text{for} \quad t, x \in \mathbb{R}_0. \tag{6}$$

We see that L_n is a positive linear operator and $L_{n;r}$ is a linear operator but it is not positive.

We shall prove that L_n is an operator acting from the space $C_p(\mathbb{R}_0)$ to $C_p(\mathbb{R}_0)$ for every $p \in \mathbb{N}_0$ and $L_{n;r}$ is an operator from the space $C^r(\mathbb{R}_0)$ to $C_r(\mathbb{R}_0)$.

We mention that many of classical summation operators and their modifications are defined by the formulas (4)-(6) (see e.g. [1]-[6] and [9], [10]).

2.2. Denote by $C_q^*(\mathbb{R})$, with a fixed $q \in \mathbb{N}$, the set of all functions $h : \mathbb{R} \to \mathbb{R}$ for which hw_q^* with $w_q^*(u) := (1 + |u|^q)^{-1}$ is continuous and bounded on $\mathbb{R}$ and the norm is given by

$$\|h\|_q^* := \sup_{u\in\mathbb{R}} w_q^*(u)|h(u)|. \tag{7}$$

Analogously we define the space $C_q^*(I)$ for a bounded interval $I \subset \mathbb{R}$.

Let $(G_n)_1^\infty$ be a sequence of functions $G_n \in C_q^*(\mathbb{R})$ with a fixed $q \in \mathbb{N}$ and let there exist $\alpha = \text{const.} > 0$ and $M_2(q) = \text{const.} > 0$ such that

$$\sup_{u\in\mathbb{R}} w_q^*(u)\,|G_n(u) - u| \le M_2(q) n^{-\alpha} \quad \text{for} \quad n \in \mathbb{N}. \tag{8}$$

The above conditions with fixed parameters $q \in \mathbb{N}$ and $\alpha > 0$ are satisfied, for example, by $G_n(u) = u + |u|^q/n^\beta$ for $u \in \mathbb{R}$, $n \in \mathbb{N}$ and $\beta \ge \alpha$.

Using the above $(G_n)_1^\infty$ and a matrix $A = [a_{nk}]$ having the properties (i)-(iv), we can introduce for $f \in C_p(\mathbb{R}_0)$, $p \in \mathbb{N}_0$, the operators

$$L_n^*(f;x) := \sum_{k=0}^{\infty} a_{nk}(x) G_n\left(f\left(\frac{k}{n}\right)\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{9}$$

and for functions $f \in C^r(\mathbb{R}_0)$, $r \in \mathbb{N}$, the following operators

$$L_{n;r}^*(f;x) := \sum_{k=0}^{\infty} a_{nk}(x) G_n\left(F_r\left(\frac{k}{n}, x\right)\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{10}$$

where $F_r(t,x)$ is defined by (6).

We see that the above operators L_n^* and $L_{n;r}^*$ are not linear and positive.

We shall prove that L_n^* is an operator acting from the space $C_p(\mathbb{R}_0)$ to $C_{pq}(\mathbb{R}_0)$ for every $p \in \mathbb{N}_0$ and $L_{n;r}^*$ is an operator from $C^r(\mathbb{R}_0)$ to $C_{rq}(\mathbb{R}_0)$.

2.3. First we give some properties of spaces C_p, C^r and C_q^*. It is obvious that if $p \in \mathbb{N}_0$, $s \in \mathbb{N}$ and $p < s$, then $C_p \subset C_s$ and $\|f\|_s \le \|f\|_p$ for $f \in C_p$.

The analogous properties have the spaces C_q^* and C_s^* with $q, s \in \mathbb{N}$ and $q < s$.

From (1) there results that $w_p \in C_0$ and $\frac{1}{w_p} \in C_p$ for every $p \in \mathbb{N}_0$. Moreover, for every $p \in \mathbb{N}_0$, $s \in \mathbb{N}$ and $x \in \mathbb{R}_0$ there holds

$$(w_p(x))^s \le w_{ps}(x), \quad (w_p(x))^{-s} \le 2^s\,(w_{ps}(x))^{-1}. \tag{11}$$

By (1), (2) and (11) we deduce that if $f \in C_p$ with a fixed $p \in \mathbb{N}_0$, then $f^s \in C_{ps}$ for every $s \in \mathbb{N}$. Moreover, we have $\|f^s\|_{ps} \le 2^s\|f\|_p^s$ if $p \in \mathbb{N}$ and $\|f^s\|_0 \le \|f\|_0^s$.

We observe also that if (G_n) is a sequence of functions $G_n \in C_q^*$ satisfying the condition (8), then there exists $M_3(q) = \text{const.} > 0$ such that

$$\|G_n\|_q^* \le M_3(q) \quad \text{for} \quad n \in \mathbb{N}. \tag{12}$$

2.4. Let $x \in \mathbb{R}_0$ be a fixed point and let

$$\varphi_x(t) := t - x \quad \text{for} \quad t \in \mathbb{R}_0. \tag{13}$$

From formulas (4) and (13) and properties (i)-(iv) of a given matrix A there results that for $e_0(x) \equiv 1$

$$L_n(e_0; x) = 1 \quad \text{for} \quad x \in \mathbb{R}_0,\, n \in \mathbb{N}, \tag{14}$$

and for $s \in \mathbb{N}$

$$w_{2s}(x) L_n\left(\varphi_x^{2s}(t); x\right) \leq M_1(s) n^{-s} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{15}$$

Using the Hölder inequality and (11), (14) and (15) we get

$$w_s(x) L_n\left(|\varphi_x(t)|^s; x\right) \leq \sqrt{M_1(s)}\, n^{-s/2} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{16}$$

Moreover, from the formula (4) it follows that $|L_n(f;x)| \leq L_n(|f|;x)$ for every $f \in C_p$, $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

2.5. Now we shall prove four lemmas on introduced operators

Lemma 1. *Let $p \in \mathbb{N}_0$. Then there exists $M_4(p) = \text{const.} > 0$ such that*

$$\|L_n(1/w_p)\|_p \leq M_4(p) \quad \textit{for} \quad n \in \mathbb{N}, \tag{17}$$

and for every $f \in C_p$ there holds

$$\|L_n(f)\|_p \leq M_4(p) \|f\|_p \quad \textit{for} \quad n \in \mathbb{N}, \tag{18}$$

The formula (4) and the inequality (18) show that L_n, $n \in \mathbb{N}$, is a positive linear operator acting from the space C_p to C_p, for every $p \in \mathbb{N}_0$.

Proof. First we prove (17).

If $p = 0$, then by (1), (2) and (14) we have $\|L_n(1/w_0)\|_0 = 1$ for $n \in \mathbb{N}$. If $p \in \mathbb{N}$, then by (1), (4), (13) and (14) we get

$$L_n\left(1/w_p(t); x\right) = 1 + L_n\left(t^p; x\right) \leq 1 + 2^p x^p + 2^p L_n\left(|\varphi_x(t)|^p; x\right),$$

which by (16) implies that

$$w_p(x) L_n\left(1/w_p(t); x\right) \leq 2^p + \sqrt{M_1(p)}\, n^{-p/2} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}.$$

From this and (2) follows (17).

If $f \in C_p$, then by (4) and (2) we have

$$\|L_n(f)\|_p \leq \|f\|_p \, \|L_n(1/w_p)\|_p, \quad n \in \mathbb{N},$$

and next by (17) we obtain the desired inequality (18). □

From Lemma 1 we can derive the following

Corollary 1. *Let $f \in C_p$, $p \in \mathbb{N}_0$, and let $s \in \mathbb{N}$. Then there exists*

$M_5(p,s) = \text{const.} > 0$ *such that*

$$\|L_n(f^s)\|_{ps} \le \|L_n(|f|^s)_{ps} \le M_5(p,s)\|f\|_p^s, \quad n \in \mathbb{N}.$$

Lemma 2. *Suppose that* $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ *and* $(G_n)_1^\infty$ *is a sequence of* $G_n \in C_q^*$ *satisfying the condition (8). Then there exists* $M_6(p,q) = \text{const.} > 0$ *such that for every* $f \in C_p$ *there holds*

$$\|L_n^*(f)\|_{pq} \le M_6(p,q)\left(1 + \|f\|_p^q\right) \quad \textit{for} \quad n \in \mathbb{N}. \tag{19}$$

The formula (9) and the inequality (19) show that L_n^* *is a nonlinear operator acting from the space* C_p *to* C_{pq}.

Proof. Let $f \in C_p$ with $p \in \mathbb{N}$. Then by (9), (4), (7), (12) and (14) we have

$$\begin{aligned} |L_n^*(f;x)| &\le \sum_{k=0}^{\infty} a_{nk}(x)\left|G_n\left(f\left(\frac{k}{n}\right)\right)\right| \\ &\le \|G_n\|_q^* \sum_{k=0}^{\infty} a_{nk}(x)\left(1 + \left|f\left(\frac{k}{n}\right)\right|^q\right) \\ &\le M_3(q)\left(1 + L_n(|f|^q;x)\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \end{aligned}$$

which by (1), (2) and Corollary 1 yields the inequality (19) for $p \in \mathbb{N}$.

The proof for $f \in C_0(\mathbb{R}_0)$ is analogous. □

Lemma 3. *Let* $r \in \mathbb{N}$ *and let* $L_{n;r}$ *be defined by (5) and (6). Then there exists* $M_7(r) = \text{const.} > 0$ *such that for every* $f \in C^r$ *there holds*

$$\|L_{n;r}(f)\|_r \le M_7(r)\left(\|f\|_r + \|f^{(r)}\|_0\right), \quad n \in \mathbb{N}. \tag{20}$$

The formulas (5) and (6) and the inequality (20) show that $L_{n;r}$ *is a linear operator acting from the space* C^r *to* C_r.

Proof. We choose $r \in \mathbb{N}$, $f \in C^r$ and $t \in \mathbb{R}_0$. Then, using the modified Taylor formula, we can write

$$f(x) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j + I_r(t,x), \quad x \in \mathbb{R}_0, \tag{21}$$

where

$$I_r(t,x) := \frac{(x-t)^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\left[f^{(r)}(t+u(x-t)) - f^{(r)}(t)\right] du. \tag{22}$$

By formulas (6) and (21) it follows that

$$F_r(t,x) = f(x) - I_r(t,x) \quad \text{for} \quad t,x \in \mathbb{R}_0. \tag{23}$$

Using (23) to (5), and by (14) and (4) we have

$$L_{n;r}(f;x) = L_n(F_r(t,x);x) = f(x) - L_n(I_r(t,x);x)$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, which implies that

$$\|L_{n;r}(f)\|_r \le \|f\|_r + \|L_n(I_r)\|_r, \quad n \in \mathbb{N}.$$

Since $f^{(r)} \in C_0$ if $f \in C^r$, we get from (22)

$$|I_r(t,x)| \le \frac{2}{r!}\|f^{(r)}\|_0\,|x-t|^r, \tag{24}$$

and next by (13) and (16) we deduce that

$$\begin{aligned}\|L_n(I_r)\|_r &\le \frac{2}{r!}\|f^{(r)}\|_0\,\|L_n\left(|\varphi_x|^r\right)\|_r \\ &\le \frac{2}{r!}\|f^{(r)}\|_0\sqrt{M_1(r)}\,n^{-r/2}, \quad n \in \mathbb{N}.\end{aligned}$$

Combining the above, we obtain the desired inequality (20). □

Lemma 4. *Let q, $r \in \mathbb{N}$ and let $L^*_{n;r}$ be defined by (10). Then there exists $M_8(q,r) = \text{const.} > 0$ such that for every $f \in C^r$ there holds*

$$\|L^*_{n;r}(f)\|_{rq} \le M_8(q,r)\left(1 + \|f\|_r^q + \|f^{(r)}\|_0^q\right), \quad \textit{for} \quad n \in \mathbb{N}. \tag{25}$$

The formulas (10) and (6) and the inequality (25) show that $L^*_{n;r}$ is a nonlinear operator acting from the space C^r to C_{rq}.

Proof. Arguing as in the proof of Lemma 3 and using (7), (12), (23) and (24), we get for $f \in C^r$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$:

$$\begin{aligned}\left|L^*_{n;r}(f;x)\right| &\le \sum_{k=0}^{\infty} a_{nk}(x)\left|G_n\left(F_r\left(\frac{k}{n},x\right)\right)\right| \\ &\le \|G_n\|_q^*\sum_{k=0}^{\infty} a_{nk}(x)\left(1+|F_r(t,x)|^q\right) \\ &\le M_3(q)\left[1 + 2^q|f(x)|^q + 2^q L_n\left(|I_r|^q;x\right)\right] \\ &\le M_3(q)\left[1 + 2^q|f(x)|^q + \left(\frac{4}{r!}\right)^q\|f^{(r)}\|_0^q\,L_n\left(|\varphi_x(t)|^{rq};x\right)\right].\end{aligned}$$

Now, using (1), (2) and (16), we immediately obtain (25). □

3. Theorems

Here we shall give theorems on the order of approximation of functions $f \in C_p$ and $f \in C^r$ by operators $L_n(f)$, $L^*_n(f)$ and $L_{n;r}(f)$, $L^*_{n;r}(f)$, correspondingly. We shall use the modulus of continuity of $f \in C_p$, i.e.

$$\omega(f;t)_p := \sup_{0\le h\le t}\|\Delta_h f(\cdot)\|_p, \quad t \ge 0, \tag{26}$$

where $\Delta_h f(x) = f(x+h) - f(x)$ for $x \in \mathbb{R}_0$.

3.1. Arguing similarly as in [2], we shall prove two theorems.

Theorem 1. *For every $p \in \mathbb{N}_0$ there exists $M_9(p) = \text{const.} > 0$ such that for every function $f \in C_p$ having the first derivative $f' \in C_p$ there holds*

$$w_p(x)\,|L_n(f;x) - f(x)| \le M_9(p)\|f'\|_p \frac{x+1}{\sqrt{n}} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{27}$$

Proof. Choose f satisfying the above assumptions and $x \in \mathbb{R}_0$. Then we have

$$f(t) - f(x) = \int_x^t f'(u)\,du$$

and

$$\left|\int_x^t f'(u)\,du\right| \le \|f'\|_p \left(\frac{1}{w_p(x)} + \frac{1}{w_p(t)}\right)|t-x|, \quad t \in \mathbb{R}_0. \tag{28}$$

Using the operator L_n and (14), we get

$$L_n\left(f(t);x\right) - f(x) = L_n\left(\int_x^t f'(u)\,du; x\right), \quad n \in \mathbb{N},$$

which by (28), (13), (11) and the Hölder inequality implies that

$$\begin{aligned} w_p(x)\,|L_n(f;x) - f(x)| &\le w_p(x) L_n\left(\left|\int_x^t f'(u)\,du\right|; x\right) \\ &\le \|f'\|_p \left(L_n\left(|\varphi_x(t)|;x\right) + w_p(x) L_n\left(|\varphi_x(t)|/w_p(t);x\right)\right) \\ &\le \|f'\|_p \left(L_n\left(\varphi_x^2(t);x\right)\right)^{1/2}\left(1 + 2\|L_n\left(1/w_{2p}\right)\|_{2p}^{1/2}\right). \end{aligned}$$

Now using (15) and (17), we obtain the desired estimation (27). □

Theorem 2. *Let $p \in \mathbb{N}_0$. Then there exists $M_{10}(p) = \text{const.} > 0$ such that for every $f \in C_p$ we have*

$$w_p(x)\,|L_n(f;x) - f(x)| \le M_{10}(p)\omega\left(f; \frac{x+1}{\sqrt{n}}\right)_p, \tag{29}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. Similary to [2] we use the Steklov function f_h of $f \in C_p$, i.e.

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)\,dt, \quad x \in \mathbb{R}_0, h > 0.$$

For this f_h we have: $f_h \in C_p$, $f'_h \in C_p$ and

$$\|f - f_h\|_p \le \omega(f;h)_p, \tag{30}$$

$$\|f_h'\|_p \le h^{-1}\omega(f;h)_p, \quad \text{for} \quad h>0. \tag{31}$$

Hence for $L_n(f)$, $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $h > 0$ we can write

$$|L_n(f;x) - f(x)| \le |L_n(f - f_h;x)| + |L_n(f_h;x) - f_h(x)| + |f_h(x) - f(x)|.$$

Next, using (18), Theorem 1 for f_h, (30) and (31), we deduce that

$$\begin{aligned} w_p(x)\,|L_n(f;x) - f(x)| &\le (M_4(p)+1)\,\|f - f_h\|_p + M_9(p)\|f_h'\|_p \frac{x+1}{\sqrt{n}} \\ &\le (1 + M_4(p) + M_9(p))\,\omega(f;h)_p \left(1 + h^{-1}\frac{x+1}{\sqrt{n}}\right), \end{aligned}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $h > 0$. Setting $h = \frac{x+1}{\sqrt{n}}$, we obtain the desired inequality (29). □

Theorem 3. *Let $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ and let L_n^* be operators connected with functions $G_n \in C_q^*$ satisfying the condition (8) with a fixed $\alpha > 0$. Then there exists $M_{11}(p,q) = \mathrm{const.} > 0$ such that for every $f \in C_p$ there holds*

$$\begin{aligned} &w_{pq}(x)\,|L_n^*(f;x) - f(x)| \\ &\le \quad M_{11}(p,q)\left(\left(1 + \|f\|_p^q\right) n^{-\alpha} + \omega\left(f; \frac{x+1}{\sqrt{n}}\right)_p\right), \end{aligned} \tag{32}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. The formulas (9) and (4) and properties (i)-(iv) of a matrix A imply that

$$\begin{aligned} L_n^*(f;x) - f(x) = &\sum_{k=0}^{\infty} a_{nk}(x)\left[G_n\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right] \\ &+ [L_n(f;x) - f(x)] := T_n(x) + V_n(x), \end{aligned}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Now applying the inequality (8) and (14) and Corollary 1, we get

$$\begin{aligned} w_{pq}(x)|T_n(x)| &\le w_{pq}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left|G_n\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right| \\ &\le \quad M_2(q)n^{-\alpha} w_{pq}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left(1 + \left|f\left(\frac{k}{n}\right)\right|^q\right) \\ &\le \quad M_2(q)n^{-\alpha}\,(1 + \|L_n(|f|^q)\|_{pq}) \\ &\le \quad M_2(q)n^{-\alpha}\left(1 + M_5(p,q)\|f\|_p^q\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \end{aligned}$$

By Theorem 2 and (1) we have

$$w_{pq}(x)\,|V_n(x)| \le 2M_{10}(p)\omega\left(f; \frac{x+1}{\sqrt{n}}\right)_p, \quad x \in \mathbb{R}_0, n \in \mathbb{N}.$$

Summarizing the above, we obtain (32). □

From Theorem 2 and Theorem 3 we obtain the following result.

Corollary 2. *If assumptions of Theorem 2 and Theorem 3 are satisfied, then for every $f \in C_p$, $p \in \mathbb{N}_0$, there holds*

$$\lim_{n\to\infty} L_n(f;x) = f(x) = \lim_{n\to\infty} L_n^*(f;x) \quad at \quad x \in \mathbb{R}_0.$$

This convergence is uniform on every interval $[x_1, x_2]$, $x_1 \geq 0$.

Moreover, if the functions G_n associated with operators L_n^* satisfy the condition (8) with $\alpha > \frac{1}{2}$, then orders of approximation of $f \in C_p$ by $L_n(f)$ and $L_n^*(f)$, given in Theorem 2 and Theorem 3, are identical.

3.2. Here we give analogies of estimations (29) and (32) for $L_{n;r}$ and $L_{n;r}^*$.

Theorem 4. *For each $r \in \mathbb{N}$ there exists $M_{12}(r) = \text{const.} > 0$ such that for every function $f \in C^r$ we have*

$$w_r(x)\left|L_{n;r}(f;x) - f(x)\right| \leq M_{12}(r) n^{-r/2} \omega\left(f^{(r)}; \frac{x+1}{\sqrt{n}}\right)_0, \tag{33}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. Let $x \in \mathbb{R}_0$ and $n, r \in \mathbb{N}$ be fixed. Then by (5), (6), (14) and (23) we have for $f \in C^r$:

$$|L_{n;r}(f;x) - f(x)| = |L_n(F_r(t,x) - f(x); x)| \leq L_n(|I_r(t,x)|; x),$$

where $I_r(t,x)$ is defined by (22). Next by (26) and the inequality $\omega(g;\lambda t)_0 \leq (1+\lambda)\omega(g;t)_0$ for $g \in C_0$ and $\lambda, t > 0$, we get from (22):

$$\begin{aligned} |I_r(t,x)| &\leq \frac{|t-x|^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\omega\left(f^{(r)}; u|t-x|\right)_0 du \\ &\leq \frac{|t-x|^r}{r!}\omega\left(f^{(r)}; |t-x|\right)_0 \\ &\leq \frac{1}{r!}\omega\left(f^{(r)};\lambda\right)_0 \left(|t-x|^r + \lambda^{-1}|t-x|^{r+1}\right), \end{aligned}$$

for a fixed $\lambda > 0$. Consequently, using (13), (11) and the Hölder inequality, we get

$$\begin{aligned} & w_r(x)\left|L_{n;r}(f;x) - f(x)\right| \\ \leq\ & \frac{w_r(x)}{r!}\omega\left(f^{(r)};\lambda\right)_0 \left\{L_n\left(|\varphi_x(t)|^r; x\right) + \lambda^{-1} L_n\left(|\varphi_x(t)|^{r+1}; x\right)\right\} \\ \leq\ & \frac{1}{r!}\omega\left(f^{(r)};\lambda\right)_0 \left(w_{2r}(x) L_n\left(\varphi_x^{2r}(t);x\right)\right)^{1/2}\left\{1 + \lambda^{-1}\left(L_n\left(\varphi_x^2(t);x\right)\right)^{1/2}\right\}. \end{aligned}$$

Now, applying the inequality (15) and setting $\lambda = \frac{x+1}{\sqrt{n}}$, we obtain the estimation

(33). □

Theorem 5. *Let $r, q \in \mathbb{N}$ and let $L^*_{n;r}$ be operators defined by (10) and functions $G_n \in C^*_q$ satisfy the condition (8) with a fixed $\alpha > 0$. Then there exists $M_{13} \equiv M_{13}(q,r) = \text{const.} > 0$ such that for every $f \in C^r$ we have*

$$\begin{aligned} & w_{rq}(x)\left|L^*_{n;r}(f;x) - f(x)\right| \\ \leq \; & M_{13}\left\{\left(1 + \|f\|_r^q + \|f^{(r)}\|_0^q\right)n^{-\alpha} + n^{-r/2}\omega\left(f^{(r)}; \frac{x+1}{\sqrt{n}}\right)_0\right\}, \end{aligned} \tag{34}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. Choose $f \in C^r$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Using (5) and (6), we deduce from (10)

$$\begin{aligned} \left|L^*_{n;r}(f;x) - f(x)\right| \leq & \sum_{k=0}^{\infty} a_{nk}(x)\left|G_n\left(F_r\left(\frac{k}{n},x\right)\right) - F_r\left(\frac{k}{n},x\right)\right| \\ & + \left|L_{n;r}(f;x) - f(x)\right| := Y_n(x) + Z_n(x) \end{aligned} \tag{35}$$

and by (8), (4) and (14) we have

$$\begin{aligned} Y_n(x) \;\; & \leq \;\; M_2(q)n^{-\alpha}\sum_{k=0}^{\infty} a_{nk}(x)\left(1 + \left|F_r\left(\frac{k}{n},x\right)\right|^q\right) \\ & = \;\; M_2(q)n^{-\alpha}\left(1 + L_n\left(\left|F_r\left(t,x\right)\right|^q;x\right)\right). \end{aligned}$$

Next analogously to the proof of Lemma 4 we get

$$w_{rq}(x)Y_n(x) \leq M_{14}(r,q)\left(1 + \|f\|_r^q + \|f^{(r)}\|_0^q\right)n^{-\alpha}, \tag{36}$$

with suitable $M_{14}(r,q) = \text{const.} > 0$.

By (1) and Theorem 4 we have

$$w_{rq}(x)Z_n(x) \leq M_{12}(r)n^{-r/2}\omega\left(f^{(r)}; \frac{x+1}{\sqrt{n}}\right)_0. \tag{37}$$

Now the estimation (34) is obvious by (35)-(37). □

From Theorem 4 and Theorem 5 we derive the following two corollaries.

Corollary 3. *If $f \in C^r(\mathbb{R}_0)$ with a fixed $r \in \mathbb{N}$, then*

$$\lim_{n\to\infty} n^{r/2}\left(L_{n;r}(f;x) - f(x)\right) = 0$$

at every $x \in \mathbb{R}_0$. Moreover, if the condition (8) holds for $\alpha > \frac{r}{2}$, then also

$$\lim_{n\to\infty} n^{r/2}\left(L^*_{n;r}(f;x) - f(x)\right) = 0 \quad \text{at} \quad x \in \mathbb{R}_0.$$

The above convergence is uniform on every interval $[x_1, x_2]$, $x_1 \geq 0$.

Corollary 4. *The order of approximation of r-times differerntiable func-*

tion $f \in C^r(\mathbb{R}_0)$ *by* $L_{n;r}(f)$ *is dependent on* r *and it improves if* r *grows.*

The above statement there holds for operators $L^*_{n;r}(f)$ *also if the condition (8) is satisfied with* $\alpha > r/2$.

References

[1] C. Bardaro, I. Mantellini, A Voronovskaya-type theorem for a general class of discrete operators, *Rocky Mountain J. of Math.* (2007), To Appear.

[2] M. Becker, Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.*, **27**, No. 1 (1978), 127-142.

[3] R.A. De Vore, *The Approximation of Continuous Functions by Positive Linear Operators*, Springer-Verlag, Berlin, New York (1972).

[4] R.A. De Vore, G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, New York (1993).

[5] Z. Ditzian, V. Totik, *Moduli of Smoothness*, Springer-Verlag, New York (1987).

[6] G.H. Kirov, A generalization of the Bernstein polynomials, *Math. Balcanica*, **2**, No. 2 (1992), 147-153.

[7] G.H. Kirov, L. Popova, A generalization of the linear positive operators, *Math. Balcanica*, **7**, No. 2 (1993), 149-162.

[8] O.T. Pop, About some linear and positive operators defined by infinite sum, *Demonstratio Math.*, **39**, No. 2 (2006), 377-388.

[9] L. Rempulska, M. Skorupka, On strong approximation of functions by certain linear operators, *Math. J. Okayama Univ.*, **46** (2004), 153-161.

[10] L. Rempulska, Z. Walczak, Modified Szász-Mirakyan operators, *Math. Balcanica*, **18** (2004), 53-63.