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# A REPEATED LEIBNIZ INTEGRAL RULE

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**Abstract:** We obtain a generalization of Leibniz' rule for differentiation under the integral sign.

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## 1. Introduction and Statement of the Result

One of the most classical formula of calculus is Leibniz' rule for differentiation under the integral sign: if $\phi$ is the function defined by

$$\phi(z) = \int_a^{b(z)} f(u,z)\,\mathrm{du} \tag{1}$$

then, under suitable conditions, we have

$$\phi'(z) = f(b(z),z)b'(z) + \int_a^{b(z)} \frac{\partial f(u,z)}{\partial z}\,\mathrm{du}. \tag{2}$$

When the bound $a$ depends on $z$, $a = a(z)$, we can use the additivity of the integral and apply (2) two times. A formula for the second derivative of $\phi$ is deduced from (2) and the chain rule; we have

$$\phi''(z) = f(b(z),z)b''(z) + \left.\frac{\partial f(u,v)}{\partial u}\right|_{\substack{u=b(z)\\ v=z}} \big(b'(z)\big)^2 + 2\left.\frac{\partial f(u,v)}{\partial v}\right|_{\substack{u=b(z)\\ v=z}} b'(z) + \int_a^{b(z)} \frac{\partial^2 f(u,z)}{\partial z^2}\,\mathrm{du}. \tag{3}$$

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A natural question then arises. Given a positive integer $n$, what is the formula for $\phi^{(n)}(z)$?

If the function $f$ does not depend on $z$, $f(u,v) = f(u)$, then $\phi(z) = F\big(b(z)\big)$, where $F$ is an antiderivative of $f$, and the answer is given by Faa Di Bruno's formula (see Riordan [1], p. 177):

$$\Big(F\big(G(z)\big)\Big)^{(n)} = \sum_{r=1}^{n} \sum_{\pi(n,r)} c(k_1,\ldots,k_n) \prod_{\mu=1}^{n} \big(G^{(\mu)}(z)\big)^{k_\mu} F^{(r)}\big(G(z)\big). \tag{4}$$

In (4) the notation $\pi(n,r)$ means that the summation is extended over the nonnegative integers $k_1, k_2, \ldots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$ and $k_1 + k_2 + \cdots + k_n = r$. The coefficients $c(k_1,\ldots,k_n)$ are given by

$$c(k_1,\ldots,k_n) = \frac{n!}{k_1!k_2!\cdots k_n!(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}}. \tag{5}$$

When the function $f$ has the form $f(u,v) = f(u)g(v)$, the answer to the aforementioned question is given by another classical formula of Leibniz, namely

$$\big(F(z)G(z)\big)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} F^{(r)}(z) G^{(n-r)}(z). \tag{6}$$

We will prove the following result.

**Theorem 1.** *If $\phi$ is the function defined by* (1) *then, for $n = 1, 2, 3, \ldots,$ we have*

$$\phi^{(n)}(z) = \sum_{r=1}^{n}\sum_{j=1}^{r} \binom{n}{r} \frac{\partial^{n-r+j-1} f(u,v)}{\partial u^{j-1}\partial v^{n-r}}\bigg|_{\substack{u=b(z)\\ v=z}} \sum_{\pi(r,j)} c(k_1,\ldots,k_r) \prod_{\mu=1}^{r} \big(b^{(\mu)}(z)\big)^{k_\mu} + \int_a^{b(z)} \frac{\partial^n f(u,z)}{\partial z^n}\,\mathrm{du}. \tag{7}$$

## 2. Proof of the Theorem

We prove (7) by mathematical induction, using formula (4) and the chain rule.

For $n = 1$, the result is the ordinary Leibniz' rule (2). Let us assume that (7) is true for a given $n > 1$. In view of (4), this hypothesis can be written as

$$\phi^{(n)}(z) = \sum_{r=1}^{n} \binom{n}{r} \frac{\partial^{n-r}}{\partial v^{n-r}} \left(\frac{\partial^r F(z,v)}{\partial z^r}\right)\bigg|_{v=z} + \int_a^{b(z)} \frac{\partial^n f(u,z)}{\partial z^n}\,\mathrm{du}, \tag{8}$$

where

$$F(u,v) := \int_a^{b(u)} f(t,v)\,\mathrm{dt}. \tag{9}$$

We then have, using the chain rule and (2),

$$\begin{aligned}\phi^{(n+1)}(z) = \sum_{r=1}^{n} \binom{n}{r} \frac{\partial^{n+1} F(u,v)}{\partial u^{r+1} \partial v^{n-r}}\bigg|_{\substack{u=z\\v=z}} + \sum_{r=1}^{n} \binom{n}{r} \frac{\partial^{n+1} F(u,v)}{\partial u^{r} \partial v^{n-r+1}}\bigg|_{\substack{u=z\\v=z}} \\ + \frac{\partial^n f(u,v)}{\partial v^n}\bigg|_{\substack{u=b(z)\\v=z}} b'(z) + \int_a^{b(z)} \frac{\partial^{n+1} f(u,z)}{\partial z^{n+1}}\,\mathrm{du}.\end{aligned} \tag{10}$$

Replacing $r$ by $r-1$ in the first summation, and using the relation $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$, we readily obtain

$$\begin{aligned}\phi^{(n+1)}(z) = \sum_{r=2}^{n+1} \binom{n+1}{r} \frac{\partial^{n+1} F(u,v)}{\partial u^{r} \partial v^{n-r+1}}\bigg|_{\substack{u=z\\v=z}} + n\frac{\partial^{n+1} F(u,v)}{\partial u\, \partial v^{n}}\bigg|_{\substack{u=z\\v=z}} \\ + \frac{\partial^n f(u,v)}{\partial v^n}\bigg|_{\substack{u=b(z)\\v=z}} b'(z) + \int_a^{b(z)} \frac{\partial^{n+1} f(u,z)}{\partial z^{n+1}}\,\mathrm{du}.\end{aligned} \tag{11}$$

We now observe that

$$\begin{aligned}\frac{\partial^{n+1} F(u,v)}{\partial u\, \partial v^{n}}\bigg|_{\substack{u=z\\v=z}} &= \frac{\partial^n}{\partial v^n}\left(\frac{\partial}{\partial u}\int_a^{b(u)} f(t,v)\,\mathrm{dt}\right)\bigg|_{\substack{u=z\\v=z}} \\ &= \frac{\partial^n f(b(u),v)}{\partial v^n} b'(u)\bigg|_{\substack{u=z\\v=z}} \\ &= \frac{\partial^n f(u,v)}{\partial v^n}\bigg|_{\substack{u=b(z)\\v=z}} b'(z).\end{aligned}$$

We can thus write, from (11),

$$\phi^{(n+1)}(z) = \sum_{r=1}^{n+1} \binom{n+1}{r} \frac{\partial^{n+1} F(u,v)}{\partial u^{r} \partial v^{n+1-r}}\bigg|_{\substack{u=z\\v=z}} + \int_a^{b(z)} \frac{\partial^{n+1} f(u,z)}{\partial z^{n+1}}\,\mathrm{du}, \tag{12}$$

which completes the proof. □

## 3. Examples

Let $b(z) = z$ in (7), so that $b^{(\mu)}(z) = 0$ for $\mu > 1$; we obtain the

**Corollary 1.** *If* $\phi(z) := \int_a^z f(u,z)\,\mathrm{du}$ *then we have*

$$\phi^{(n)}(z) = \sum_{j=1}^{n} \binom{n}{j} \frac{\partial^{n-1} f(u,v)}{\partial u^{j-1} \partial v^{n-j}}\bigg|_{\substack{u=z\\ v=z}} + \int_a^z \frac{\partial^{n+1} f(u,z)}{\partial z^{n+1}}\,\mathrm{du}. \tag{13}$$

If we take $b(z) = z^2$ in (7) then the summation is extended over the non-negative integers $k_1$, $k_2$ such that $k_1 + 2k_2 = r$, $k_1 + k_2 = j$, whence $k_1 = 2j - r$ and $k_2 = r - j$. We obtain the

**Corollary 2.** *If* $\phi(z) := \int_a^{z^2} f(u,z)\,\mathrm{du}$ *then we have*

$$\phi^{(n)}(z) = \sum_{r=1}^{n} \sum_{j=1}^{r} \frac{r!\binom{n}{r}}{(r-j)!(2j-r)!} \frac{\partial^{n-r+j-1} f(u,v)}{\partial u^{j-1} \partial v^{n-r}}\bigg|_{\substack{u=z^2\\ v=z}} (2z)^{2j-r} + \int_a^{z^2} \frac{\partial^n f(u,z)}{\partial z^n}\,\mathrm{du}. \tag{14}$$

When $f(u,z) = f(u)$ does not depend on $z$, formula (14) gives

$$\big(F(z^2)\big)^{(n)} = \sum_{\frac{n}{2} \le j \le n} \frac{n!}{(n-j)!(2j-n)!} (2z)^{2j-n} F^{(j)}(z^2). \tag{15}$$

Of course, the last formula can be obtained directly by (4).

## References

[1] J. Riordan, *Combinatorial Identities*, Wiley and Sons, New York (1968).