ON $B(M, X)$-CC-PROJECTIVE MODULES

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Abstract: In this paper, $B(M, X)$-cc-projective modules are defined as generalization of $M$-cc-projective modules. Let $M$ be an $X$-amply supplemented module such that $B(M, X)$ is closed under supplement submodules. Then $M$ is $X$-lifting if and only if every module is $B(M, X)$-cc-projective. Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively projective modules $M_i$. Assume $M$ is an amply supplemented module. Then $M$ is $B(M, X)$-cc-projective if and only if $M_i$ is $B(M_i, X)$-cc-projective, for every $i \in \{1, \cdots, n\}$.

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1. Introduction

Throughout this paper all rings are associative with unity and all modules will be unital right $R$-modules. Let $M$ be a module, we write $(A \ll M) A \leq M$ to indicate that $A$ is a (small) submodule of $M$.

Let $M$ be a module and $A \leq B \leq M$. If $B/A \ll M/A$, then $A$ is called a coessential submodule of $B$ in $M$. A submodule $K$ of $M$ is called coclosed (denoted by $K \ll_{cc} M$) if $K$ has no proper coessential submodule in $M$. Let $N \leq M$, a submodule $K$ of $M$ is called a supplement of $N$ in $M$ if $M = N + K$ and $N \cap K \ll K$. A module $M$ is called weakly supplemented if for any submodule $A$ of $M$, there exists a submodule $B$ of $M$ such that $M = A + B$ and $A \cap B \ll M$. A module $M$ is called amply supplemented if for any submodules

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Let $X$ and $M$ be $R$-modules. Define the set
\[ B(M, X) = \{ A \leq M \mid \exists Y \leq X, \exists f \in \text{Hom}(M, X/Y), \ker f / A \ll M/A \}. \]
Consider the property
\[ B(M, X)-(D_1): \text{For all } A \in B(M, X), \text{there exists a direct summand } A^* \text{ of } M \text{ such that } A/A^* \ll M/A^*. \]

Following [1], $M$ is called $X$-lifting if $M$ satisfies $B(M, X)-(D_1)$ and $M$ is called $X$-amply supplemented if for any submodules $A, B$ of $M$ such that $A \in B(M, X)$ and $M = A + B$, there exists a supplement $P$ of $A$ such that $P \leq B$.

Let $M_1$ and $M_2$ be modules. Following [2], the module $M_2$ is $M_1$-cc-projective if every homomorphism $\alpha : M_2 \to M_1/K$, where $K \leq cc M_1$, can be lifted to a homomorphism $\beta : M_2 \to M_1$.

**Definition 1.1.** (see [1]) Let $M$ and $X$ be modules. A module $N$ is called small $B(M, X)$-projective if for any submodule $K$ of $M$ with $K \in B(M, X)$, every homomorphism $\alpha : N \to M/K$ with $\text{Im } \alpha \ll M/K$ can be lifted to a homomorphism $\beta : N \to M$.

**Definition 1.2.** Let $M$ and $X$ be modules. A module $N$ is called $B(M, X)$-cc-projective if every homomorphism $\alpha : N \to M/K$, where $K \leq cc M$ and $K \in B(M, X)$, can be lifted to a homomorphism $\beta : N \to M$.

The Prüfer p-group $\mathbb{Z}(p^{\infty})$ is a lifting $\mathbb{Z}$-module and so a $X$-lifting $\mathbb{Z}$-module. By Proposition 2.2, $\mathbb{Z}(p^{\infty})$ is $B(\mathbb{Z}(p^{\infty}), X)$-cc-projective, but it is not self-projective.

### 2. $B(M, X)$-cc-Projectivity

**Lemma 2.1.** Let $M, X$ be modules and $K$ be a coclosed submodule of $M$ with $K \in B(M, X)$. If $M/K$ is $B(M, X)$-cc-projective, then $K$ is a direct summand of $M$.

**Proof.** By hypothesis, there exists a homomorphism $\alpha : M/K \to M$ that lifts the identity $1 : M/K \to M/K$. It is easy to see that $M = K \oplus \alpha(M/K)$. Hence $K$ is a direct summand of $M$. \[\square\]

**Proposition 2.2.** Let $M$ be an $X$-amply supplemented module $M$ such that $B(M, X)$ is closed under supplement submodules. Then the following
statements are equivalent:

1. M is X-lifting.
2. Every module is \(B(M, X)\)-cc-projective.
3. For every coclosed submodule \(K\) of \(M\) with \(K \in B(M, X)\), \(M/K\) is \(B(M, X)\)-cc-projective.

Proof. (1) \(\Rightarrow\) (2) Let \(N\) be any module. Let \(\alpha : N \to M/K\) be any homomorphism with \(K \leq cc M\) and \(K \in B(M, X)\). Note that every coclosed submodule \(K\) of \(M\) with \(K \in B(M, X)\) is a direct summand of \(M\), now the proof is clear.

(2) \(\Rightarrow\) (3) Clearly: (2) implies (3), and by Lemma 2.1, (3) implies (1).

(3) \(\Rightarrow\) (1) By Lemma 2.1 and [1], Proposition 3.4. \(\square\)

Lemma 2.3. Let \(M_1, M_2\) and \(X\) be modules. If \(M_1\) is \(B(M_2, X)\)-cc-projective and \(M_2\) is weakly supplemented, then for every coclosed submodule \(N\) of \(M_2\), \(M_1\) is \(B(M_2/N, X)\)-cc-projective.

Proof. Let \(L/N\) be a coclosed submodule of \(M_2/N\) with \(L/N \in B(M_2/N, X)\). By [1], Lemma 2.2, \(L \in B(M_2, X)\) and \(L\) is coclosed in \(M_2\) by [3], Lemma 1.4. Let \(f : M_1 \to (M_2/N)/(L/N) \cong M_2/L\). Since \(M_1\) is \(B(M_2, X)\)-cc-projective, it is easy to see that \(M_1\) is \(B(M_2/N, X)\)-cc-projective. \(\square\)

Lemma 2.4. Let \(M, X\) and \(\{N_i \mid i \in I\}\) be modules. Then \(\oplus_{i \in I} N_i\) is \(B(M, X)\)-cc-projective if and only if \(N_i\) is \(B(M, X)\)-cc-projective, for every \(i \in I\).

Proof. The proof follows as for projectivity (see for example [4], Proposition 16.10). \(\square\)

Corollary 2.5. Let \(M_1, M_2\) and \(X\) be modules and \(M = M_1 \oplus M_2\) a weakly supplemented module. If \(M\) is \(B(M, X)\)-cc-projective, then \(M_i\) is \(B(M_j, X)\)-cc-projective, for every \(i, j \in \{1, 2\}\). In particular, if \(M\) is \(B(M, X)\)-cc-projective and \(N\) is a direct summand of \(M\), then \(N\) is \(B(N, X)\)-cc-projective.

Proof. By Lemmas 2.3 and 2.4. \(\square\)

Lemma 2.6. Let \(M_1, M_2\) and \(X\) be modules such that \(M = M_1 \oplus M_2\) is amply supplemented and \(M_1\) is small \(B(M_2, X)\)-projective. If any module \(N\) is \(B(M_2, X)\)-cc-projective and \(M_1\)-projective, then it is \(B(M, X)\)-cc-projective.

Proof. Let \(K \leq cc M\) and \(K \in B(M, X)\). Then \((K + M_1)/K \in B(M/K, X)\) by [1], Lemma 3.5. Consider the homomorphism \(\alpha : N \to M/K\) and the natural
epimorphism $\pi : M \to M/K$. Since $M/K$ is amply supplemented, there exists a submodule $H/K$ of $M/K$ such that $H/K \leq (K + M_1)/K, (K + M_1)/H \ll M/H$ and $H/K \leq cc M$. Since $K + M_1 = H + M_1, (H + M_1)/H \ll M/H$ and $(H + M_1)/K \in B(M/K, X)$. Hence $H + M_1 \in B(M, X)$ and so $H \in B(M, X)$ by [1], Lemma 2.2. Since $M_1$ is small $B(M_2, X)$-projective, there exists a submodule $H'$ of $H$ such that $M = H' \oplus M_2$ by [1], Proposition 2.4. Since $M_2 \cong M/H'$, $N$ is $B(M/H', X)$-cc-projective. Let $\beta$ be the epimorphism from $M/K$ to $M/H$ defined by $\beta(m + K) = m + H$ for all $m + K \in M/K$ and $\pi_1$ the epimorphism from $M/H'$ to $M/H \cong (M/H')/(H'/H')$ defined by $\pi_1(m + H') = m + H$ for all $m + H' \in M/H'$. Since $N$ is $B(M/H', X)$-cc-projective, there exists a homomorphism $g : N \to M/H'$ such that $\pi_1g = \beta\alpha$. Now, consider the following homomorphisms:

$$N \xrightarrow{g} M/H' \xrightarrow{f} M_2 \xrightarrow{i_1} M_1 \xrightarrow{\pi} M/K,$$

where $i_1$ is the inclusion map and $f$ is the isomorphism from $M/H'$ to $M_2$. Then we have the homomorphism $\pi i_1fg : N \to M/K$. Take any element $n$ in $N$, and suppose $\alpha(n) = m' + K$ and $g(n) = m + H'$ with $m, m' \in M$. Therefore, $\pi_1g(n) = \beta\alpha(n)$ implies that $m - m' \in H$. Write $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. Now, $(\pi i_1fg - \alpha)(n) = \pi i_1fg(n) - \alpha(n) = \pi i_1f(m + H') - (m' + K) = \pi i_1f(m_1 + m_2 + H') - (m' + K) = \pi i_1f(m_2) - (m' + K) = m_2 + K - (m' + K) = m_1 + m_2 - m' - m_1 + K$ implies that $\text{Im}(\pi i_1fg - \alpha) \subseteq (H + M_1)/K = (K + M_1)/K$. Consider the inclusion map $i_2 : (K + M_1)/K \to M/K$. Since $\text{Im}(\pi i_1fg - \alpha) \subseteq \text{Im}(i_2) = (K + M_1)/K$, there exists a homomorphism $\gamma : N \to (K + M_1)/K$ such that $i_2\gamma = \pi i_1fg - \alpha$. Let $\pi_2 : M_1 \to (K + M_1)/K$ be the natural epimorphism. Since $N$ is $M_1$-projective, $\gamma$ can be lifted to a homomorphism $\phi : N \to M_1$. Consider, finally, the homomorphism $\theta = i_1fg - \phi : N \to M$. Let $n \in N$, then $\pi\theta(n) = \pi(i_1fg - \phi)(n) = \pi i_1fg(n) - \pi\phi(n) = \pi i_1fg(n) - \alpha(n) + \alpha(n) - \pi_2\phi(n) = (\pi i_1fg - \alpha)(n) + \alpha(n) - i_2\gamma(n) = \alpha(n)$. Therefore, $\alpha$ can be lifted to the homomorphism $\theta$ and $N$ is $B(M, X)$-cc-projective.

**Theorem 2.7.** Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively projective modules $M_i$. Assume $M$ is amply supplemented. Then $M$ is $B(M, X)$-cc-projective if and only if $M_i$ is $B(M_i, X)$-cc-projective, for every $i \in \{1, \cdots, n\}$.

Proof. By Lemmas 2.4 and 2.6, using induction. 

□
References


