International Journal of Pure and Applied Mathematics
Volume 44 No. 3 2008, 415-424

THE NORMAL LAPLACE APPROXIMATION TO COMPOUND DISTRIBUTIONS

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Abstract: The state of a Brownian motion after an exponentially distributed random time with normally distributed initial state generates a four-parameter Gaussian type distribution, called Normal-Laplace distribution, which exhibits two-sided fatter than normal tail behavior. Limiting cases include three-parameter Normal-Exponential distributions, which exhibit fatter than normal tail behavior in either the right-tail or the left-tail of the distribution. The application of these Gaussian type distributions to the analytical approximation of aggregate claims distributions is compared to earlier good approximations based on the translated gamma distribution, translated inverse Gaussian distribution and a mixture thereof. Quantitative improvement on the latter approximations is shown through numerical illustration.

AMS Subject Classification: 62P05, 91B30
Key Words: Brownian motion, Normal-Laplace, Normal-Exponential, gamma, translated gamma, translated inverse Gaussian, mixtures, aggregate claims

1. Introduction

The construction of analytical distributions, which can be used as analytical approximations to the aggregate claims distribution of an insurance portfolio, is an important subject of wide interest going beyond actuarial science. Well-known books from actuarial science, which include such analytical distributions, are Hogg and Klugman [5], Panjer and Willmot [9] and Klugman et al [7].

Received: January 31, 2008

The usefulness in actuarial science of some three-parameter tractable analytical distributions like the translated gamma, the translated inverse Gaussian and a four-parameter mixture thereof has been pointed out in Chaubey et al [3]. Another recent analytical approximation has been presented in Hürlimann [6].

This interesting and open subject is still not exhausted and a main purpose of the present study is to demonstrate that further simplification and improvement remain possible. In the present paper, further Gaussian type distributions are introduced in the actuarial context. In theory, several temporal stochastic phenomena can be modeled appropriately using a Brownian motion (BM), e.g. the logarithm of stock returns is often assumed to follow a normal distribution. In practice, the empirical data of such phenomena exhibits fatter than normal tail behavior, which contradicts the normal distribution behind BM. Recall the following recent reconciliation. A simple mechanism, which generates the fatter than normal tail behavior, consists to assume that the time of observation in a BM is itself a random variable, whose distribution is close to an exponential distribution. For example, the state of a BM after an exponentially distributed random time with fixed initial state generates the skew Laplace distribution as shown in Reed [10]. A natural generalization consists to look at the state of a BM after an exponentially distributed random time with normally distributed initial state. It generates a Normal-Laplace distribution, which has been considered in Reed [11] and Reed and Jorgensen [12]. A more detailed content of the paper follows.

Section 2 describes the steps necessary to derive the Normal-Laplace distribution from the Brownian motion. Closed-form analytical expressions for the density, distribution and first moments of the distribution are given. Limiting cases include Normal-Exponential type distributions, which exhibit fatter than normal tail behavior in either the right-tail or the left-tail of the distribution. Section 3 discusses the application of these Gaussian type analytical distributions to the numerical approximation of aggregate claims distributions and recalls the earlier approximations proposed in Chaubey et al [3]. A detailed numerical example is presented and discussed in Section 4.

2. From the Brownian Motion to the Normal-Laplace Distribution

It is often assumed that the time evolution of a stochastic phenomenon X_t can be modeled by a BM described by the stochastic differential equation

$$dX = \mu \cdot dt + \sigma \cdot dW, \tag{2.1}$$

where dW is the increment of a Wiener process. Since the increment of a BM in time dt has a systematic component $\mu \cdot dt$ and a random white noise component $\sigma \cdot dW$, BM can be viewed as a stochastic version of a simple linear growth model. Empirical studies on BM phenomena often exhibit behavior with fatter than normally distributed tails. However, the state of a BM after a fixed time T follows a normal distribution, which does not exhibit this behavior.

Why does one observe fatter than normal tail behavior for phenomena apparently evolving like a BM? A simple mechanism, which generates fatter than normal tail behavior, consists to assume that the time of observation T itself is a random variable, whose distribution is close to an exponential distribution. The idea for this generating mechanism is borrowed from Reed [10], which applies it to a geometric Brownian motion. The distribution of X_T with fixed initial state X_0 is described by the asymmetric *Laplace distribution* centered on X_0 (Kotz et al [8]), written $X_T \sim L(X_0, \alpha, \beta)$, with has the probability density function

$$f_{X_T}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} \exp\left\{\beta\left(x - X_0\right)\right\}, & x < X_0, \\ \frac{\alpha\beta}{\alpha+\beta} \exp\left\{-\alpha\left(x - X_0\right)\right\}, & x \geq X_0, \end{cases} \tag{2.2}$$

where $\alpha,\ \beta > 0$, and $\alpha,\ -\beta$ are the positive roots of the characteristic equation

$$\mu z + \frac{1}{2}\sigma^2 z^2 = \lambda, \tag{2.3}$$

where λ is the parameter of the exponentially distributed random variable T.

A natural generalization of the Laplace law (2.2) has been considered in Reed [11] and Reed and Jorgensen [12]. They consider the mixture of normal distributions arising from a BM with normally distributed initial state $X_0 \sim N\left(\nu, \tau^2\right)$ (instead of a fixed initial state as above), which is killed or stopped with the constant killing rate λ. Equivalently, this is the distribution of the state of a BM after an exponentially distributed time of evolution. For completeness, let us derive shortly the form of this distribution. At fixed time t, this BM is such that $X_t \sim N\left(\nu + \mu t, \tau^2 + \sigma^2 t\right)$, which has the moment generating function (mgf)

$$M_{X_t}(s) = \exp\left\{\nu s + \tfrac{1}{2}\tau^2 s^2 + [\mu s + \tfrac{1}{2}\sigma^2 s^2]t\right\}. \tag{2.4}$$

Let now T be exponentially distributed with mgf

$$M_T(s) = \frac{\lambda}{\lambda - s}. \tag{2.5}$$

Then the state X_T of a Brownian motion after the random time T has mgf

$$M_{X_T}(s) = E_T[M_{X_t}(s)\,|t = T] = e^{\nu s + \frac{1}{2}\tau^2 s^2} \cdot \frac{\alpha\beta}{(\alpha - s)(\beta + s)}, \tag{2.6}$$

where $\alpha,\ -\beta$ are the positive roots of (2.3). This shows that $X_T = Z + W$ is the independent sum of a normally distributed random variable $Z \sim N\left(\nu, \tau^2\right)$ and a Laplace distribution $W \sim L(0, \alpha, \beta)$ with probability density function (pdf)

$$f_W(w) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta} e^{\beta w}, & w \leq 0, \\ \dfrac{\alpha\beta}{\alpha+\beta} e^{-\alpha w}, & w > 0. \end{cases} \tag{2.7}$$

Through convolution one obtains the pdf

$$f(x) = f_{X_T}(y) = \frac{\alpha\beta}{\alpha+\beta}\varphi\left(\frac{x-\nu}{\tau}\right) \times \left[R\left(\alpha\tau - \frac{x-\nu}{\tau}\right) + R\left(\beta\tau + \frac{x-\nu}{\tau}\right)\right], \tag{2.8}$$

where $R(\cdot)$ denotes Mill's ratio of the complementary cumulative distribution function (cdf) to the pdf of a standard normal distribution given by

$$R(z) = \frac{\bar{\Phi}(z)}{\varphi(z)}, \quad \bar{\Phi}(z) = 1 - \Phi(z). \tag{2.9}$$

By construction it is natural to call this a *Normal-Laplace* distribution and write $X_T \sim NL\left(\alpha, \beta, \tau, \nu^2\right)$. The cdf of the Normal-Laplace satisfies the analytical expression

$$F(x) = \Phi\left(\frac{x-\nu}{\tau}\right) - \varphi\left(\frac{x-\nu}{\tau}\right)\frac{\beta \cdot R\left(\alpha\tau - \frac{x-\nu}{\tau}\right) + \alpha \cdot R\left(\beta\tau + \frac{x-\nu}{\tau}\right)}{\alpha+\beta}. \tag{2.10}$$

Since a Laplace random variable is an independent difference of two exponential random variables, one has the alternative representation

$$X_T \overset{d}{=} \nu + \tau Z + \frac{1}{\alpha}E_1 - \frac{1}{\beta}E_2, \tag{2.11}$$

where $E_1,\ E_2$ are independent standard exponential random variables and Z is standard normal random variable, which is independent of $E_1,\ E_2$.

Some more comments on the Normal-Laplace distribution are in order. Since this distribution is infinitely divisible, it is possible to construct a Lévy process with increments following a Normal-Laplace distribution. The rich class of Lévy processes, extensively studied in Barndorff-Nielsen [1], can be used to model the logarithmic return of financial instruments. The attractive Normal-Laplace can thus be used to reflect the empirical fact that observed logarithmic returns for high frequency data have fatter tails than those of a normal distribution. In this setting, option pricing formulas can be derived applying the characteristic function approach (e.g. Schoutens [13], p. 20).

In the following, some useful properties of the Normal-Laplace distribution are listed. The two special cases, obtained from the limiting cases $\alpha \to \infty$ and $\beta \to \infty$, are of main interest. They define two Normal-Exponential distributions, the *right-tailed Normal-Exponential* distribution $X_1 \sim NEr\left(\alpha, \nu, \tau^2\right)$ and the *left-tailed Normal-Exponential* distribution $X_2 \sim NEl\left(\beta, \nu, \tau^2\right)$ with pdf's

$$\begin{aligned} f_1(x) &= \alpha\varphi\left(\frac{x-\nu}{\tau}\right) R\left(\alpha\tau - \frac{x-\nu}{\tau}\right), \\ f_2(x) &= \beta\varphi\left(\frac{x-\nu}{\tau}\right) R\left(\beta\tau + \frac{x-\nu}{\tau}\right). \end{aligned} \tag{2.12}$$

With (2.8) these formulas show that the Normal-Laplace satisfies the mixture representation

$$f(x) = \frac{\beta}{\alpha+\beta} f_1(x) + \frac{\alpha}{\alpha+\beta} f_2(x). \tag{2.13}$$

The mean, the variance and the third and fourth order cumulants of the Normal-Laplace are given by

$$\begin{aligned} &E[X] = \nu + \alpha^{-1} - \beta^{-1}, \quad \operatorname{Var}[X] = \tau^2 + \alpha^{-2} + \beta^{-2}, \\ &\kappa_3 = 2\alpha^{-3} - 2\beta^{-3}, \quad \kappa_4 = 6\alpha^{-4} + 6\beta^{-4}. \end{aligned} \tag{2.14}$$

Unfortunately, solving the third and fourth order cumulants equations in (2.14) does not always lead to real solutions, a fact which must be taken into account in applications.

3. Analytical Approximations to Aggregate Claims Distributions

From Chaubey et al [3] one knows that the analytical approximations based on the gamma, translated gamma, translated inverse Gaussian and a mixture of the last two distributions yield usually good approximations of aggregate claims distributions or compound distributions. Even more, Chaubey et al [3] claim that the latter mixture seems to be a superb approximation to the aggregate claims distribution. These approximations are based on the method of moments. It is interesting and instructive to analyze whether the 3-parameter right-tailed Normal-Exponential distribution, as well as 4-parameter mixtures thereof with the translated gamma and the translated inverse Gaussian distributions, and the 4-parameter Normal-Laplace distribution are able to yield similar or even better approximations to the aggregate claims distribution as those provided by the above ones.

The most often used compound Poisson and compound negative binomial approximations use at most 8 parameters. For the number of claims random variable N, these are the mean $\mu N = E[N]$, the coefficient of variation $kN = \sqrt{\mathrm{Var}[N]}\Big/E[N]$, the 3-rd order cumulant $\kappa 3N = E[(N-\mu N)^3]$ and the 4-th order cumulant $\kappa 4N = E[(N-\mu N)^4]$. For the claim size random variable Y, these are the mean $\mu Y = E[Y]$, the standard deviation $\sigma Y = \sqrt{\mathrm{Var}[Y]}$, and the 3-rd and 4-th order moments $m3Y = E\left[Y^3\right]$, $m4Y = E\left[Y^4\right]$. From the latter quantities, we will also use the following derived quantities: the coefficient of variation $kY = \sqrt{\mathrm{Var}[Y]}\Big/E[Y]$, the 3-rd order cumulant $\kappa 3Y = m3Y - 3\mu Y\cdot\sigma Y^2 - \mu Y^3$ and the 4-th order cumulant $\kappa 4Y = m4Y - 4\mu Y\cdot m3Y + 6\mu Y^2\cdot\sigma Y^2 + 3\cdot\mu Y^4 - 3\cdot\sigma Y^4$. The relevant characteristics for the aggregate claims random variable X are the mean $\mu X = \mu N\cdot\mu Y$, the coefficient of variation $kX = \sqrt{kY^2/\mu N + kN^2}$, the standard deviation $\sigma X = kX\cdot\mu X$, the 3-rd order cumulant $\kappa 3X = \mu N\cdot\kappa 3Y + 3\sigma N^2\cdot\mu Y\cdot\sigma Y^2 + \kappa 3N\cdot\mu Y^3$, the 4-th order cumulant $\kappa 4Y = m4Y - 4\mu Y\cdot m3Y + 6\mu Y^2\cdot\sigma Y^2 + 3\cdot\mu Y^4 - 3\cdot\sigma Y^4$, the skewness $\gamma X = \kappa 3X\big/\sigma X^3$ and the kurtosis $\kappa X = \kappa 4X\big/\sigma X^4$. Recall the formulas required for a numerical evaluation of various analytical approximations to the aggregate claims X.

Gamma Approximation. The distribution of X is approximated by a gamma distribution $\Gamma(\beta,\alpha)$ with parameters $\alpha = k_X^{-2}$, $\quad\beta = \left(k_X^2\mu_X\right)^{-1}$. This approximation matches the mean and variance of X.

Translated Gamma Approximation (TG). According to Dickson and Waters [4], it is natural to approximate X by $X_{TG} = X_G + \gamma$, where $X_G \sim \Gamma(\beta,\alpha)$, with the parameters defined by

$$\alpha = \frac{4}{\gamma X^2}, \qquad \beta = \frac{\sigma X\cdot\gamma X}{2}, \qquad \gamma = \mu X - 2\cdot\frac{\sigma X}{\gamma X}. \tag{3.1}$$

This approximation matches the mean, variance and skewness of X.

Translated Inverse Gaussian Approximation (TIG). Chaubey et al [3] propose to approximate X by $X_{TIG} = X_{IG} + \delta$, where $X_{IG} \sim IG(a,b)$ has an inverse Gaussian distribution, with the parameters defined by

$$a = 3\cdot\frac{\sigma X}{\gamma X}, \qquad b = \frac{1}{3}\cdot\sigma X\cdot\gamma X, \qquad \delta = \mu X - a. \tag{3.2}$$

This approximation matches the mean, variance and skewness of X.

Mixture of Translated Gamma and Translated Inverse Gaussian. Let $F_{TG}(x) = F_G(x-\gamma)$ and $F_{TIG}(x) = F_{IG}(x-\delta)$ be the distribution functions of the translated gamma and translated inverse Gaussian approximations. Following Chaubey [2], the mixture defined by

$$F_{mix}(x) = w \cdot F_{TG}(x) + (1 - w) \cdot F_{TIG}(x) \tag{3.3}$$

matches the mean, variance, skewness and kurtosis of X provided

$$w = \frac{\kappa_X - \kappa_{TIG}}{\kappa_{TG} - \kappa_{TIG}}, \tag{3.4}$$

where the kurtosis parameters of the TG and the TIG are defined by

$$\kappa_{TG} = \frac{6}{\alpha}, \qquad \kappa_{TIG} = 15 \cdot \frac{b}{a}. \tag{3.5}$$

Normal-Exponential Approximation (NE). According to the limiting case of (2.14) for $\beta \to \infty$, the parameters of the Normal-Exponential $NE\left(\alpha, \nu, \tau^2\right)$ are set equal to

$$\alpha = \left(\frac{2}{\kappa 3X}\right)^{\frac{1}{3}}, \quad \nu = \mu X - \alpha^{-1}, \quad \tau = \sqrt{\sigma X^2 - \alpha^{-2}}. \tag{3.6}$$

This approximation matches the mean, variance and skewness of X. In case $\alpha > \sigma X$ the parameter τ is not defined and the method of moments cannot be applied.

Mixture of Translated Gamma and Normal-Exponential. Let $F_{TG}(x) = F_G(x - \gamma)$ and $F_{NE}(x)$ be the distribution functions of the translated gamma and Normal-Exponential approximations. The mixture defined by

$$F_{mix}(x) = w \cdot F_{TG}(x) + (1 - w) \cdot F_{NE}(x) \tag{3.7}$$

matches the mean, variance, skewness and kurtosis of X provided

$$w = \frac{\kappa_X - \kappa_{NE}}{\kappa_{TG} - \kappa_{NE}}, \tag{3.8}$$

where the kurtosis parameters of the TG and the NE are defined by

$$\kappa_{TG} = \frac{6}{\alpha}, \qquad \kappa_{NE} = \frac{6}{\left(1 + \alpha^2 \tau^2\right)^2}. \tag{3.9}$$

Mixture of Translated Inverse Gaussian and Normal-Exponential. Let $F_{TIG}(x) = F_{IG}(x - \delta)$ and $F_{NE}(x)$ be the distribution functions of the translated inverse Gaussian and Normal-Exponential approximations. The mixture defined by

$$F_{mix}(x) = w \cdot F_{TIG}(x) + (1 - w) \cdot F_{NE}(x) \tag{3.10}$$

matches the mean, variance, skewness and kurtosis of X provided

$$w = \frac{\kappa_X - \kappa_{NE}}{\kappa_{TIG} - \kappa_{NE}}, \tag{3.11}$$

Distribution	mean	st. dev.	coeff. of variation	skewness	kurtosis
claim number	10	4.472	0.447	0.671	0.650
claim size	100	250	2.500	0.016	8.788
aggregate claims	1000	908.295	0.908	0.584	1.220

Table 4.1: Characteristics of the aggregate claims distribution

where the kurtosis parameters of the TIG and the NE are defined by

$$\kappa_{TIG} = 15 \cdot \frac{b}{a}, \qquad \kappa_{NE} = \frac{6}{(1+\alpha^2\tau^2)^2}. \tag{3.12}$$

Normal-Laplace Approximation. The four parameters of the Normal-Laplace distribution $NL\left(\alpha,\beta,\tau,\nu^2\right)$ are implicitly defined as the solution of the moment equations (2.14), provided there is a solution. Necessarily, the parameter α will solve the equation

$$27 \cdot \left(\kappa 3X \cdot x^3 - 2\right)^4 = 2 \cdot \left(\kappa 4X \cdot x^4 - 6\right)^3. \tag{3.13}$$

The other parameter are then given by

$$\begin{aligned} &\beta = 3\alpha \cdot \frac{2-\kappa 3X \cdot \alpha^3}{\kappa 4X \cdot \alpha^4 - 6}, \quad \nu = \mu X - \alpha^{-1} + \beta^{-1}, \\ &\tau = \sqrt{\sigma X^2 - \alpha^{-2} - \beta^{-2}}. \end{aligned} \tag{3.14}$$

Unfortunately, this algorithm does not always lead to real solutions.

4. Numerical Illustration

It is important to analyze the numerical sensitivity of these different analytical approximations to the aggregate claims distributions. The following numerical example compares the different approximations and measures their tail deviation with respect to the Normal-Laplace distribution as analytical benchmark approximation. The reason for this choice is that the Normal-Laplace distribution is the only purely 4-parameter distribution and therefore it is expected to yield the closest approximation to the aggregate claims distribution. In this example, the claim number distribution is assumed to be a negative binomial distribution. The Tables 4.1 to 4.3 summarize this analysis.

The 2-parameter Gamma approximation has fatter tail area (from the 90% up to the 99% percentile) than the other multi-parameter approximations. The 3-parameter Normal-exponential approximation is closer in this tail area to the 4-parameter approximations than the other two 3-parameter TG and TIG

Model	80%	90%	95%	99%
gamma	1581.9	2195.8	2800.8	4186.6
TG	1727.5	2205.9	2628.9	3492.4
TIG	1724.0	2201.6	2625.8	3499.0
mixture TG with TIG	1685.4	2149.5	2586.4	3586.6
NE	1694.2	2152.9	2582.5	3555.6
mixture NE with TG	1691.3	2148.0	2578.0	3562.1
mixture NE with TIG	1691.4	2148.0	2577.9	3561.9
NL	1717.3	2159.4	2583.8	3562.7

Table 4.2: Percentiles of the aggregate claims distribution

Model	80%	90%	95%	99%	90% total of absolute value of relative deviations
gamma	-7.89%	1.69%	8.40%	17.51%	27.60%
TG	0.60%	2.16%	1.75%	-1.98%	5.88%
TIG	0.39%	1.95%	1.63%	-1.79%	5.37%
mixture TG / TIG	-1.86%	-0.46%	0.10%	0.67%	1.23%
NE	-1.35%	-0.30%	-0.05%	-0.20%	0.55%
mixture NE / TG	-1.51%	-0.53%	-0.22%	-0.02%	0.77%
mixture NE / TIG	-1.51%	-0.53%	-0.23%	-0.02%	0.78%
NL	0.00%	0.00%	0.00%	0.00%	0.00%

Table 4.3: Relative deviations of percentiles to Normal-Laplace approximation

approximations. The 4-parameter Normal-Laplace has a slightly fatter tail area than the 3-parameter approximations and the 4-parameter mixtures. This discussion suggests that the following analytical approximations of aggregate claims can be selected and recommended, depending on the number of desired model parameters:

2-parameter model: Gamma approximation;

3-parameter model: Normal-Exponential approximation;

4-parameter model: Normal-Laplace approximation.

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