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CONTINUOUS FIELDS OF C^*-ALGEBRAS BY ALMOST COMMUTING ISOMETRIES

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Abstract: We study the C^*-algebras involving almost commuting isometries and their structure, K-theory, and continuous fields.

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1. Introduction

Continuous fields of C^*-algebras have been studied of great interest (see Dixmier [4]). Especially, continuous fields of C^*-algebras with continuous trace as fibers are classified by the third cohomology of the base spaces, called the Dixmier-Douady invariant (or classification) (see Raeburn and Williams [8]).

A well known fact is that the group C^*-algebra of the discrete Heisenberg group can be viewed as a continuous field of rotation C^*-algebras (or noncommutative 2-tori) on the torus $\mathbb{T}$ (for instance, see Anderson and Paschke [1]). On the other hand, Exel [6] has shown that there exists a continuous field of the C^*-algebras defined by almost commuting unitaries, called the soft tori, on the interval $[0, 2]$, whose fiber at 0 is $C(\mathbb{T}^2)$ the C^*-algebra of all continuous functions on the 2-torus $\mathbb{T}^2$ and the fiber at 2 is $C^*(F_2)$ the full group C^*-algebra of the free group F_2 of two generators, and the K-theory (or K-groups) of the soft tori is computed in [5] by Exel. See also the author [10] for K-theory of continuous fields of quantum (or noncommutative) tori.

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Since the study on continuous fields of C^*-algebras has been focused on those of C^*-algebras generated by unitaries as mentioned above (among others), our first quesion was wheater or not there exists a non-trivial continuous field of C^*-algebras generated by isometries. In this paper we obtain a positive result to this quesion. In contrast with this, constructed by the author [11] is a discontinuous deformation from a C^*-algebra generated by isometries to a C^*-algebra generated by unitaries.

This paper is organized as follows. In Section 2 we consider the universal C^*-algebras generated by almost commuting isometries, which we call soft Toeplitz tensor products, and if we further impose some norm estimates from a technical requirement, we call them super-soft, and study their structure and K-theory. In Section 3 we show that there exists a continuous field with fibers given by the super-soft Toeplitz tensor products, so that there exists a continuous field between the C^*-tensor product of Toeplitz algebras and the full unital free product C^*-algebra of them. This interpolating result should be new and interesting. Some methods used in Sections 2 and 3 are taken analogously from both the methods of Exel [5] for the universal C^*-algebras generated by almost commuting unitaries and their structure and K-theory and those of Exel [6] for a continuous field between $C(\mathbb{T}^2)$ (that is isomorphic the C^*-tensor product $C(\mathbb{T}) \otimes C(\mathbb{T})$) and $C^*(F_2)$ (that is isomorphic to the full unital free product C^*-algebra $C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$). For the convenience to readers, the detailed proofs in our case are given. However, the main difference between our case and Exel's case is that we need to treat with the semigroup C^*-algebra of $\mathbb{N}$ of natural numbers and semigroup crossed products of C^*-algebras by endomorphisms of $\mathbb{N}$ in stead of the group C^*-algebra of $\mathbb{Z}$ of integers and ordinary crossed products of C^*-algebras by automorphisms of $\mathbb{Z}$ as in Exel's case. Fortunately, we can use techniques for semigroup crossed products by $\mathbb{N}$ such as their generators and conditional expectations and K-theory (see Rørdam [9]). More fortunately, it is found that we can use a reduction from our case to Exel's case by considering canonical quotients, which makes it possible to avoid considering technical difficulties using functional calculus (for isometries or unitaries) that is not useful for (proper) isometies. Also, the K-theory formulas such as those for unital free product C^*-algebras and tensor product C^*-algebras as well are available (see Blackadar [2, 10.11.11] and Wegge-Olsen [12]).

2. K-Theory

Let $\mathfrak{F}$ be the Toeplitz algebra, which is defined to the universal C^*-algebra generated by a proper (or non-unitary) isometry s, i.e., denoted by $\mathfrak{F} = C^*(s)$, where $s^*s = 1$ is the identity element (see Murphy [7] or [12]). The Toeplitz algebra is also regarded as the semi-group C^*-algebra of $\mathbb{N}$ the additive semi-group of natural numbers by endomorphisms by a non-unitary isometry u (in the sense that $\alpha_1(1_H) = u1_Hu^* = uu^*$ and $\alpha_n(1_H) = u^n(u^*)^n$ for $n \in \mathbb{N}$, where $1_H = u^*u$ is the identity map on a Hilbert space H), i.e., denoted by $\mathfrak{F} = C^*(\mathbb{N})$. Note also that the Toeplitz $\mathfrak{F}$ has the decomposition into the following short exact sequence (see [7]):

$$0 \to \mathbb{K} \to \mathfrak{F} \to C(\mathbb{T}) \to 0,$$

where $\mathbb{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space and $C(\mathbb{T})$ is the universal C^*-algebra generated by a unitary, and it is also the C^*-algebra of continuous functions on the torus $\mathbb{T}$.

For any $\varepsilon \geq 0$, the soft torus A_ε of Exel is defined to be the universal C^*-algebra generated by two unitaries $u_{\varepsilon,1}$, $u_{\varepsilon,2}$ such that $\|u_{\varepsilon,2}u_{\varepsilon,1} - u_{\varepsilon,1}u_{\varepsilon,2}\| \leq \varepsilon$. By definition, A_0 is isomorphic to the C^*-tensor product $C(\mathbb{T})\otimes C(\mathbb{T})$, that is the universal C^*-algebra generated by commuting two unitaries. If $\varepsilon \geq 2$, then A_ε is isomorphic to the unital full free product C^*-algebra $C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$, which can be viewed as the full group C^*-algebra $C^*(\mathbb{Z} * \mathbb{Z})$ of the free product $\mathbb{Z} * \mathbb{Z}$. This follows from universality of A_ε and that the inequality $\|u_{\varepsilon,2}u_{\varepsilon,1} - u_{\varepsilon,1}u_{\varepsilon,2}\| \leq 2$ always holds.

Definition 2.1. For any $\varepsilon \geq 0$, we define the C^*-algebra denoted by $\mathfrak{D}_\varepsilon$ to be the universal C^*-algebra generated by two isometries $s_{\varepsilon,1}, s_{\varepsilon,2}$ such that $\|s_{\varepsilon,2}s_{\varepsilon,1} - s_{\varepsilon,1}s_{\varepsilon,2}\| \leq \varepsilon$. We call it soft Toeplitz tensor product. In addition, if we assume the norm estimates:

$$\|\pi(s_{\varepsilon,2}^{n+1}s_{\varepsilon,1}(s_{\varepsilon,2}^*)^{n+1}) - \pi(s_{\varepsilon,2}^{n}s_{\varepsilon,1}(s_{\varepsilon,2}^*)^{n})\| \leq \varepsilon$$

for $n \in \mathbb{N}$ and $n = 0$, then we call $\mathfrak{D}_\varepsilon$ super-soft Toeplitz tensor product, where π is the canonical quotient map from $\mathfrak{D}_\varepsilon$ to A_ε. If necessary to mention, we call $\mathfrak{D}_\varepsilon$ soft or super-soft in those cases respectively, in what follows. Otherwise we always assume that $\mathfrak{D}_\varepsilon$ is super-soft.

Remark. There exists a canonical quotient map π from $\mathfrak{D}_\varepsilon$ to A_ε by sending $s_{\varepsilon,j}$ to $u_{\varepsilon,j}$ for $j = 1, 2$. Note that unitaries are isometries. The second norm estimate is required by a technical reason in what follows. Indeed, the first norm estimate in the soft torus case implies the second estimate. Therefore, it might be possible to remove the super-softness from the definition.

By definition, $\mathfrak{D}_0$ is isomorphic to the C^*-tensor product $\mathfrak{F} \otimes \mathfrak{F}$, that is the universal C^*-algebra generated by commuting two isometries. If $\varepsilon \geq 2$, then $\mathfrak{D}_\varepsilon$ is isomorphic to the full unital free product C^*-algebra $\mathfrak{F} *_{\mathbb{C}} \mathfrak{F}$, which can be viewed as the semi-group C^*-algebra of the free product semi-group $\mathbb{N} * \mathbb{N}$, denoted by $C^*(\mathbb{N} * \mathbb{N})$. This follows from the universality of $\mathfrak{F} *_{\mathbb{C}} \mathfrak{F}$ (or $C^*(\mathbb{N} * \mathbb{N})$) and that the inequality: $\|s_{\varepsilon,2}s_{\varepsilon,1} - s_{\varepsilon,1}s_{\varepsilon,2}\| \leq 2$ always holds. It is well known that the K-groups of the Toeplitz algebra $\mathfrak{F}$ are given by $K_0(\mathfrak{F}) \cong \mathbb{Z}$ and $K_1(\mathfrak{F}) \cong 0$ (see [12, 9.L]). The Künneth formula implies

$$K_0(\mathfrak{D}_0) = K_0(\mathfrak{F} \otimes \mathfrak{F}) \cong [K_0(\mathfrak{F}) \otimes K_0(\mathfrak{F})] \oplus [K_1(\mathfrak{F}) \otimes K_1(\mathfrak{F})] \cong \mathbb{Z},$$
$$K_1(\mathfrak{D}_0) = K_1(\mathfrak{F} \otimes \mathfrak{F}) \cong [K_0(\mathfrak{F}) \otimes K_1(\mathfrak{F})] \oplus [K_1(\mathfrak{F}) \otimes K_0(\mathfrak{F})] \cong 0$$

(see [12, 9.3.3]). On the other hand, the K-groups of the full unital free product C^*-algebra $\mathfrak{F} *_{\mathbb{C}} \mathfrak{F}$ are computed as

$$K_0(\mathfrak{F} *_{\mathbb{C}} \mathfrak{F}) \cong (K_0(\mathfrak{F}) \oplus K_0(\mathfrak{F}))/\mathbb{Z} \cong \mathbb{Z},$$
$$K_1(\mathfrak{F} *_{\mathbb{C}} \mathfrak{F}) \cong K_1(\mathfrak{F}) \oplus K_1(\mathfrak{F}) \cong 0$$

(see [2, 10.11.11]).

For any $\varepsilon \geq 0$, Exel considered the C^*-subalgebra B_ε of A_ε generated by unitaries $u_{\varepsilon,2}^n u_{\varepsilon,1}(u_{\varepsilon,2}^*)^n$ $(n \in \mathbb{Z})$ satisfying

$$\|u_{\varepsilon,2}^{n+1} u_{\varepsilon,1}(u_{\varepsilon,2}^*)^{n+1} - u_{\varepsilon,2}^n u_{\varepsilon,1}(u_{\varepsilon,2}^*)^n\| \leq \varepsilon.$$

We define B'_ε to be the C^*-algebra generated by the unitaries $u_{\varepsilon,2}^n u_{\varepsilon,1}(u_{\varepsilon,2}^*)^n$ for $n \in \mathbb{N}$ and $n = 0$.

Definition 2.2. We define E_ε to be the universal C^*-algebra generated by an isometry t_1 and the elements $t_{n+1} = u^n t_1 (u^*)^n$ $(n \in \mathbb{N})$ for an isometry u such that $\|ut_1 - t_1 u\| \leq \varepsilon$ (or E_ε may be defined as the C^*-subalgebra of $\mathfrak{D}_\varepsilon$ generated by the corresponding elements $s_{\varepsilon,1}$ and $s_{\varepsilon,2}^n s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n$ for $n \in \mathbb{N}$). Let α_ε be the endomorphism of E_ε defined by $\alpha_\varepsilon(t_n) = t_{n+1} = ut_n u^*$ for $n \in \mathbb{N}$. Let $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ be the semigroup crossed product C^*-algebra corresponding to the C^*-dynamical system $(E_\varepsilon, \alpha_\varepsilon, \mathbb{N})$. In addition, if necessary in what follows, we further assume the norm estimates

$$\|\pi(t_{n+1}) - \pi(t_n)\| \leq \varepsilon$$

for $n \in \mathbb{N}$, where π is the canonical quotient map from E_ε to B'_ε. If necessary to mention, we call E_ε soft or super-soft in those cases respectively, in what follows. Otherwise we assume that E_ε is super-soft.

Remark. There exists a canonical quotient map π from E_ε to B'_ε by sending t_1 and u to $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$ respectively.

Recall from [9] that an endomorphism ρ on a unital C^*-algebra $\mathfrak{B}$ by an isometry s is a corner endomorphism if $\rho(\mathfrak{B}) = s\mathfrak{B}s^*$ is equal to the corner $\rho(1)\mathfrak{B}\rho(1) = ss^*\mathfrak{B}ss^*$. Then it follows $s^*\mathfrak{B}s = \mathfrak{B}$ that we use later. In this case we can also use the Pimsner-Voiculesce exact sequence and conditional expectations as given below. The above endomorphism α_ε of E_ε is a corner endomorphism. Indeed,

$$\alpha_\varepsilon(1) = \alpha_\varepsilon(t_1^* t_1) = (ut_1^* u^*)(ut_1 u^*) = uu^*.$$

Note that $E_0 \rtimes_{\alpha_0} \mathbb{N} \cong \mathfrak{F} \otimes \mathfrak{F}$.

Proposition 2.3. *For $\varepsilon \in [0,2]$, we have $\mathfrak{D}_\varepsilon \cong E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ for $\mathfrak{D}_\varepsilon$ soft or super-soft.*

Proof. Let u be the canonical implementing isometry of $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ such that $ut_n u^* = t_{n+1}$ and $\|ut_1 - t_1 u\| \le \varepsilon$. By universality, it follows that there exists a unique $*$-homomorphism φ from $\mathfrak{D}_\varepsilon$ to $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ such that $\varphi(s_{\varepsilon,1}) = t_1$ and $\varphi(s_{\varepsilon,2}) = u$.

On the other hand, note that

$$s_{\varepsilon,2}^n s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n = (s_{\varepsilon,2}s_{\varepsilon,2}^*)(s_{\varepsilon,2}^n s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n)(s_{\varepsilon,2}s_{\varepsilon,2}^*),$$

from which an endmorphism β_ε on $\mathfrak{D}_\varepsilon$ defined by

$$\beta_\varepsilon(s_{\varepsilon,2}^n s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n) = s_{\varepsilon,2}^{n+1} s_{\varepsilon,1}(s_{\varepsilon,2}^*)^{n+1}$$

is a corner endomorphism. Thus, by universality there exists a $*$-homomorphism ψ from E_ε to $\mathfrak{D}_\varepsilon$ such that $\psi(t_{n+1}) = s_{\varepsilon,2}^n s_{\varepsilon,1}(s_{\varepsilon,2}^*)^n$ for $n \in \mathbb{N}$ and $n = 0$. Since $s_{\varepsilon,2}\psi(t_n)s_{\varepsilon,2}^* = \psi(t_{n+1})$, we can extend ψ to $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ by $\psi(u) = s_{\varepsilon,2}$. By construction, ψ is the inverse of φ, and vise versa. □

Let $u_1 = s \otimes 1$, $u_2 = 1 \otimes s$ be the generating isometries of $\mathfrak{F} \otimes \mathfrak{F}$, where $\mathfrak{F} = C^*(s)$. Since u_1 and u_2 commute, for any $\varepsilon > 0$, there exists a unique $*$-homomorphism $\varphi_\varepsilon : \mathfrak{D}_\varepsilon \to \mathfrak{F} \otimes \mathfrak{F}$ such that $\varphi_\varepsilon(s_{\varepsilon,1}) = u_1$ and $\varphi_\varepsilon(s_{\varepsilon,2}) = u_2$. Also, there exists a unique $*$-homomorphism $\psi_\varepsilon : E_\varepsilon \to \mathfrak{F}$ such that $\psi_\varepsilon(t_n) = s$ for $n \in \mathbb{N}$. This map extends to a $*$-homomorphism from $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ to $\mathfrak{F} \otimes \mathfrak{F}$.

Theorem 2.4. *For any $0 < \varepsilon < 2$ and super-soft E_ε, we have*

$$K_0(E_\varepsilon) \cong \mathbb{Z}, \quad K_1(E_\varepsilon) \cong 0.$$

Consequently, $K_j(E_\varepsilon) \cong K_j(\mathfrak{F})$ for $j = 0, 1$.

Proof. Define $\sigma : \mathfrak{F} \to E_\varepsilon$ by $\sigma(s) = t_1$. Then the composition $\psi_\varepsilon \circ \sigma$ is the identity map of $\mathfrak{F}$. Indeed, $\psi_\varepsilon \circ \sigma(s) = \psi_\varepsilon(t_1) = s$.

Let $\pi : E_\varepsilon \to \pi(E_\varepsilon) = B_\varepsilon'$ be the canonical quotient map. Then $\pi(t_n)$ for $n \in \mathbb{N}$ are unitaries, and $\pi(E_\varepsilon)$ is the C^*-algebra generated by unitaries $\pi(t_n)$ such that $\|\pi(t_{n+1}) - \pi(t_n)\| \le \varepsilon$ for $n \in \mathbb{N}$. Note that $\pi(E_0) = \pi(\mathfrak{F}) \cong C(\mathbb{T})$. By using the method for [5, Theorem 2.2] there exists a homotopy between $\pi(E_\varepsilon)$ and $C(\mathbb{T})$ such that for the map $[\sigma] : C(\mathbb{T}) \to \pi(E_\varepsilon)$ defined by $[\sigma](\pi(s)) = \pi(t_1)$ and $[\psi]_\varepsilon : \pi(E_\varepsilon) \to C(\mathbb{T})$ defined as in [5, Theorem 2.2], the composition $[\psi]_\varepsilon \circ [\sigma]$ is the identity map of $C(\mathbb{T})$, and the composition $[\sigma] \circ [\psi]_\varepsilon$ is homotopic to the identity map $\mathrm{id}_{\pi(E_\varepsilon)}$ of $\pi(E_\varepsilon)$.

Consider the following exact sequence:

$$0 \to \mathfrak{I}_\varepsilon \to E_\varepsilon \to \pi(E_\varepsilon) \to 0,$$

where $\mathfrak{I}_\varepsilon$ is the kernel of π. By definition, $\mathfrak{I}_\varepsilon$ consists of elements of E_ε in $\mathbb{K} + u\mathbb{K}u^* + \cdots + u^n\mathbb{K}(u^*)^n + \cdots$, where this $\mathbb{K}$ is the closed ideal of compact operators in the C^*-algebra $C^*(t_1)$ generated by t_1 such that $C^*(t_1)/\mathbb{K} \cong C(\mathbb{T})$. Since $u^n\mathbb{K}(u^*)^n \cong \mathbb{K}$, it is not hard to see that $\mathfrak{I}_\varepsilon$ has the same K-theory as $\mathbb{K}$. Indeed, this follows from homotopy invariance and continuity of K-theory. Thus, we have the six term exact sequence:

$$\begin{array}{ccccc} K_0(\mathbb{K}) & \longrightarrow & K_0(E_\varepsilon) & \longrightarrow & K_0(\pi(E_\varepsilon)) \\ \uparrow & & & & \downarrow \\ K_1(\pi(E_\varepsilon)) & \longleftarrow & K_1(E_\varepsilon) & \longleftarrow & K_1(\mathbb{K}). \end{array}$$

Note that $K_0(\mathbb{K}) \cong \mathbb{Z}$ and $K_1(\mathbb{K}) \cong 0$. It follows from the homotopy equivalence between $\pi(E_\varepsilon)$ and $C(\mathbb{T})$ shown above that $K_j(\pi(E_\varepsilon)) \cong K_j(C(\mathbb{T}))$ for $j = 0, 1$. Note that $K_j(C(\mathbb{T})) \cong \mathbb{Z}$ for $j = 0, 1$. Furthermore, we have $K_0(\mathbb{K}) \cong \mathbb{Z} \cong K_1(\pi(E_\varepsilon))$ by the index map. Therefore, we obtain $K_0(E_\varepsilon) \cong \mathbb{Z}$ and $K_1(E_\varepsilon) \cong 0$. □

Theorem 2.5. *For any $0 < \varepsilon < 2$, the induced map $(\varphi_\varepsilon)_*$ (or $(\psi_\varepsilon)_*$) from $K_j(\mathfrak{D}_\varepsilon)$ (or $K_j(E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N})$) to $K_j(\mathfrak{F} \otimes \mathfrak{F})$ (or $K_j(\mathfrak{F} \rtimes_{\mathrm{id}} \mathbb{N})$) is an isomorphism for $j = 0, 1$.*

Proof. The Pimsner-Voiculescu six-term exact sequence for semigroup crossed products of C^*-algebras by $\mathbb{N}$ (see Rørdam [9, Corollary 2.2]) implies

$$\begin{array}{ccccc} K_0(E_\varepsilon) & \xrightarrow{(\mathrm{id}-\alpha_\varepsilon)_*} & K_0(E_\varepsilon) & \xrightarrow{i_*} & K_0(E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) & \xleftarrow[i_*]{} & K_1(E_\varepsilon) & \xleftarrow[(\mathrm{id}-\alpha_\varepsilon)_*]{} & K_1(E_\varepsilon)\,, \end{array}$$

where i is the canonical inclusion from E_ε to $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$. In fact, this six term diagram is induced from the following commutative diagram:

$$\begin{array}{ccccc} E_\varepsilon & \xrightarrow{\alpha_\varepsilon} & E_\varepsilon & \xrightarrow{i} & E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N} \\ \downarrow{\lambda} & & \downarrow{\lambda} & & \downarrow{\lambda} \\ E_\varepsilon \otimes \mathbb{K} & \xrightarrow{\beta_\varepsilon} & E_\varepsilon \otimes \mathbb{K} & \xrightarrow{i} & (E_\varepsilon \otimes \mathbb{K}) \rtimes_{\beta_\varepsilon} \mathbb{Z} \end{array}$$

and its usual Pimsner-Voiculescu six-term exact sequence for crossed products of C^*-algebras by $\mathbb{Z}$:

$$\begin{array}{ccccc} K_0(E_\varepsilon \otimes \mathbb{K}) & \xrightarrow{(\mathrm{id}-\beta_\varepsilon)_*} & K_0(E_\varepsilon \otimes \mathbb{K}) & \xrightarrow{i_*} & K_0((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\beta_\varepsilon} \mathbb{Z}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\beta_\varepsilon} \mathbb{Z}) & \xleftarrow[i_*]{} & K_1(E_\varepsilon \otimes \mathbb{K}) & \xleftarrow[(\mathrm{id}-\beta_\varepsilon)_*]{} & K_1(E_\varepsilon \otimes \mathbb{K})\,, \end{array}$$

where $\mathbb{K}$ is the C^*-algebra of compact operators, and λ is an embedding from E_ε onto a corner of $E_\varepsilon \otimes \mathbb{K}$ and β_ε is an automorphism on E_ε so that λ extends to an embedding from $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ onto a corner of $(E_\varepsilon \otimes \mathbb{K}) \rtimes_{\beta_\varepsilon} \mathbb{Z}$. Moreover, the induced maps λ_* on K_0, K_1-groups are isomorphisms, and the maps $(\mathrm{id} - \beta_\varepsilon)_*$ are zero maps, which implies that the maps $(\mathrm{id}-\alpha_\varepsilon)_*$ are also trivial. Therefore, we obtain the following short exact sequences:

$$0 \to K_j(E_\varepsilon) \to K_j(E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \to K_{j+1}(E_\varepsilon) \to 0$$

for $j = 0, 1$, where $j + 1$ (mod 2).

On the other hand, the Pimsner-Voiculescu six-term exact sequence implies the following:

$$\begin{array}{ccccc} K_0(\mathfrak{F}) & \xrightarrow{(\mathrm{id}-\mathrm{id})_*} & K_0(\mathfrak{F}) & \xrightarrow{i_*} & K_0(\mathfrak{F} \rtimes_{\mathrm{id}} \mathbb{N}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(\mathfrak{F} \rtimes_{\mathrm{id}} \mathbb{N}) & \xleftarrow[i_*]{} & K_1(\mathfrak{F}) & \xleftarrow[(\mathrm{id}-\mathrm{id})_*]{} & K_1(\mathfrak{F}), \end{array}$$

which splits into the following two short exact sequences:

$$0 \to K_j(\mathfrak{F}) \to K_j(\mathfrak{F} \rtimes_{\mathrm{id}} \mathbb{N}) \to K_{j+1}(\mathfrak{F}) \to 0$$

for $j = 0, 1$, where $j + 1$ (mod 2).

Since $\mathfrak{D}_\varepsilon \cong E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ by Proposition 2.3 and $\mathfrak{F} \rtimes_{\mathrm{id}} \mathbb{N} \cong \mathfrak{F} \otimes \mathfrak{F}$, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_j(E_\varepsilon) & \longrightarrow & K_j(\mathfrak{D}_\varepsilon) & \longrightarrow & K_{j+1}(E_\varepsilon) & \longrightarrow & 0 \\ \downarrow & & \downarrow{(\psi_\varepsilon)_*} & & \downarrow{(\varphi_\varepsilon)_*} & & \downarrow{(\psi_\varepsilon)_*} & & \downarrow \\ 0 & \longrightarrow & K_j(\mathfrak{F}) & \longrightarrow & K_j(\mathfrak{F} \otimes \mathfrak{F}) & \longrightarrow & K_{j+1}(\mathfrak{F}) & \longrightarrow & 0\,, \end{array}$$

where the maps $(\psi_\varepsilon)_*$ are isomorphisms by Theorem 2.4. The Lemma 5 implies that $(\varphi_\varepsilon)_*$ is an isomorphism as desired. □

Corollary 2.6. *For any* $\varepsilon \in [0,2]$,

$$K_0(\mathfrak{D}_\varepsilon) \cong \mathbb{Z}, \quad K_1(\mathfrak{D}_\varepsilon) \cong 0.$$

3. Continuity

For any $0 \le \varepsilon_1 \le \varepsilon_2 \le 2$, by universality there exists a unique $*$-homomorphism from $\mathfrak{D}_{\varepsilon_2}$ to $\mathfrak{D}_{\varepsilon_1}$ sending $s_{\varepsilon_2,j}$ to $s_{\varepsilon_1,j}$ for $j = 1,2$. In particular, for $\varepsilon_2 = 2$ and $\varepsilon_1 = \varepsilon$, let q_ε be the unique $*$-homomorphism from $\mathfrak{D}_2$ to $\mathfrak{D}_\varepsilon$. As the main result of this section, it is obtained that

Theorem 3.1. *There exists a continuous field of* C^**-algebras on the interval* $[0,2]$ *such that* $\mathfrak{D}_\varepsilon$ *is the fiber at* $\varepsilon \in [0,2]$ *and the maps* f_a *defined by* $[0,2] \ni \varepsilon \to q_\varepsilon(a) = f_a(\varepsilon) \in \mathfrak{D}_\varepsilon$ *for* $a \in \mathfrak{D}_2$ *are continuous.*

The proof for this is given later. Let I_ε be the kernel of q_ε. Then the norm $\|q_\varepsilon(a)\|$ for $a \in \mathfrak{D}_2$ is equal to the distance $d(a, I_\varepsilon)$ between a and I_ε. Note that if $\varepsilon < \varepsilon'$, then $I_\varepsilon \supset I_{\varepsilon'}$ and $\|q_\varepsilon(a)\| \le \|q_{\varepsilon'}(a)\|$ since $\|q_\varepsilon(a)\| = \inf_{b \in I_\varepsilon} \|a - b\|$ by definition. Let I_ε^+ be the norm closure of the union $\cup_{\varepsilon < \varepsilon' \le 2} I_{\varepsilon'}$, and I_ε^- the intersection $\cap_{0 \le \varepsilon' < \varepsilon} I_{\varepsilon'}$.

Proposition 3.2. *For* $\varepsilon \in [0,2)$, *if* $I_\varepsilon = I_\varepsilon^+$, *then* f_a *is right continuous at* ε *for* $a \in \mathfrak{D}_2$. *For* $\varepsilon \in (0,2]$, *if* $I_\varepsilon = I_\varepsilon^-$, *then* f_a *is left continuous at* ε *for* $a \in \mathfrak{D}_2$.

Proof. For any $a \in \mathfrak{D}_2$ we have

$$d(a, I_\varepsilon^+) = \inf_{\varepsilon < \varepsilon' \le 2} d(a, I_{\varepsilon'}), \quad d(a, I_\varepsilon^-) = \sup_{0 \le \varepsilon' < \varepsilon} d(a, I_{\varepsilon'}).$$

Indeed, by definition $d(a, I_\varepsilon^+) = \inf_{b \in I_\varepsilon^+} \|a - b\|$. Since $I_\varepsilon^+ \supset I_{\varepsilon'}$ for $\varepsilon < \varepsilon'$,

$$d(a, I_\varepsilon^+) \le \inf_{\varepsilon < \varepsilon' \le 2} d(a, I_{\varepsilon'}).$$

There exsits a sequence of $b_n \in I_{\varepsilon'_n}$ such that $\varepsilon < \varepsilon'_{n+1} < \varepsilon'_n$ and $d(a, I_\varepsilon^+) = \lim_{n\to\infty} \|a - b_n\|$. Since $\|a - b_n\| \ge d(a, I_{\varepsilon'_n})$, it follows that

$$d(a, I_\varepsilon^+) \ge \inf_{\varepsilon < \varepsilon' \le 2} d(a, I_{\varepsilon'}).$$

Now consider the map q from $\mathfrak{D}_2$ to the direct product $\Pi_{0 \le \varepsilon' < \varepsilon} \mathfrak{D}_{\varepsilon'}$ defined by

$q(a) = (q_{\varepsilon'}(a))_{0\le\varepsilon'<\varepsilon}$. Then the kernel of q is I_ε^-. Therefore,

$$d(a, I_\varepsilon^-) = \|q(a)\| = \sup_{0\le\varepsilon'<\varepsilon} \|q_{\varepsilon'}(a)\| = \sup_{0\le\varepsilon'<\varepsilon} d(a, I_{\varepsilon'}).$$

If $I_\varepsilon = I_\varepsilon^+$, then for $\varepsilon \in [0, 2)$,

$$\|q_\varepsilon(a)\| = d(a, I_\varepsilon) = d(a, I_\varepsilon^+) = \inf_{\varepsilon<\varepsilon'\le 2} d(a, I_{\varepsilon'}) = \lim_{\varepsilon<\varepsilon'\le 2} \|q_{\varepsilon'}(a)\|,$$

and if $I_\varepsilon = I_\varepsilon^-$, then for $\varepsilon \in (0, 2]$,

$$\|q_\varepsilon(a)\| = d(a, I_\varepsilon) = d(a, I_\varepsilon^-) = \sup_{0\le\varepsilon'<\varepsilon} d(a, I_{\varepsilon'}) = \lim_{0\le\varepsilon'<\varepsilon} \|q_{\varepsilon'}(a)\|. \qquad \square$$

Proposition 3.3. *For any $\varepsilon \in [0, 2)$, we have $I_\varepsilon^+ = I_\varepsilon$.*

Proof. First assume that $\mathfrak{D}_\varepsilon$ is soft. For the generating isometries $s_{2,1}, s_{2,2} \in \mathfrak{D}_2$, denote by $s_{\varepsilon,1}^+, s_{\varepsilon,2}^+$ their images in $\mathfrak{D}_2/I_\varepsilon^+$ respectively. Then for any $\varepsilon < \varepsilon' \le 2$,

$$\begin{aligned} &\|s_{\varepsilon,2}^+ s_{\varepsilon,1}^+ - s_{\varepsilon,1}^+ s_{\varepsilon,2}^+\| = \inf_{b\in I_\varepsilon^+} \|s_{2,2}s_{2,1} - s_{2,1}s_{2,2} + b\| \\ &\le \inf_{b\in I_{\varepsilon'}} \|s_{2,2}s_{2,1} - s_{2,1}s_{2,2} + b\| = \|q_{\varepsilon'}(s_{2,2}s_{2,1} - s_{2,1}s_{2,2})\| \\ &= \|q_{\varepsilon'}(s_{2,2})q_{\varepsilon'}(s_{2,1}) - q_{\varepsilon'}(s_{2,1})q_{\varepsilon'}(s_{2,2})\| = \|s_{\varepsilon',2}s_{\varepsilon',1} - s_{\varepsilon',1}s_{\varepsilon',2}\| \le \varepsilon'. \end{aligned}$$

Hence, it follows $\|s_{\varepsilon,2}^+ s_{\varepsilon,1}^+ - s_{\varepsilon,1}^+ s_{\varepsilon,2}^+\| \le \varepsilon$. By the universality of $\mathfrak{D}_\varepsilon$, there exists a $*$-homomorphism from $\mathfrak{D}_\varepsilon \cong \mathfrak{D}_2/I_\varepsilon$ to $\mathfrak{D}_2/I_\varepsilon^+$ sending $s_{\varepsilon,j}$ to $s_{\varepsilon,j}^+$ for $j = 1, 2$. Thus, I_ε must be contained in I_ε^+.

On the other hand, for any $a \in I_\varepsilon^+$, there exists a sequence of $b_n \in I_{\varepsilon_n}$ such that $\varepsilon < \varepsilon_n$ and (b_n) converges to a. Then $q_\varepsilon(a) = \lim_{n\to\infty} q_\varepsilon(b_n) = 0$ because the following diagram:

$$\begin{array}{ccc} \mathfrak{D}_2 & = & \mathfrak{D}_2 \\ {\scriptstyle q_{\varepsilon_n}}\downarrow & & \downarrow{\scriptstyle q_\varepsilon} \\ \mathfrak{D}_{\varepsilon_n} & \xrightarrow{q_{\varepsilon_n,\varepsilon}} & \mathfrak{D}_\varepsilon\,, \end{array}$$

commutes, where $q_{\varepsilon_n,\varepsilon}$ is a unique $*$-homomorphism by universality, and $q_\varepsilon(b_n) = q_{\varepsilon_n,\varepsilon}(q_{\varepsilon_n}(b_n)) = 0$. Therefore, I_ε^+ is contained in I_ε.

Now assume that $\mathfrak{D}_\varepsilon$ is super-soft. For this case, we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\varepsilon & \longrightarrow & \mathfrak{D}_2 & \xrightarrow{q_\varepsilon} & \mathfrak{D}_\varepsilon & \longrightarrow & 0 \\ & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & \\ 0 & \longrightarrow & J_\varepsilon & \longrightarrow & A_2 & \xrightarrow{p_\varepsilon} & A_\varepsilon & \longrightarrow & 0\,, \end{array}$$

where p_ε is the canonical onto $*$-homomorphism and J_ε is the kernel of p_ε. Indeed, note that if $x \in I_\varepsilon$ so that $q_\varepsilon(x) = 0$, then $p_\varepsilon(\pi(x)) = \pi(q_\varepsilon(x)) = 0$. Hence $\pi(x) \in J_\varepsilon$. This diagram induces the following:

$$\begin{array}{ccc} \mathfrak{D}_2 & \longrightarrow & \mathfrak{D}_2/I_\varepsilon^+ \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_2/J_\varepsilon^+ \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{D}_\varepsilon & \longrightarrow & \mathfrak{D}_2/I_\varepsilon^+ \\ \downarrow & & \downarrow \\ A_\varepsilon & \longrightarrow & A_2/J_\varepsilon^+ , \end{array}$$

where $J_\varepsilon^+ = J_\varepsilon$. By the same argument as for $\mathfrak{D}_\varepsilon$ soft, the conclusion follows. □

Recall the following fact:

Lemma 3.4. (see [9]) *The semigroup crossed product $\mathfrak{B} \rtimes_\alpha \mathbb{N}$ of a unital C^*-algebra $\mathfrak{B}$ by a corner endomorphism α defined by $\alpha(b) = sbs^*$ for a proper isometry s has a dense $*$-subalgebra generated by $\mathfrak{B}$ and s, whose elements are of the form*:

$$(s^*)^n b_{-n} + \cdots + s^* b_{-1} + b_0 + b_1 s + \cdots + b_n s^n \equiv f$$

for some $n \in \mathbb{N}$ and $b_j \in \mathfrak{B}$ for $j = -n, \cdots, n$.

Moreover, an action β of the torus $\mathbb{T}$ on $\mathfrak{B} \rtimes_\alpha \mathbb{N}$ is given by $\beta_z(b) = b$ and $\beta_z(s) = zs$ for $b \in \mathfrak{B}$, $z \in \mathbb{T}$. A conditional expectation $E : \mathfrak{B} \rtimes_\alpha \mathbb{N} \to \mathfrak{B}$ is defined by

$$E(f) = \int_{\mathbb{T}} \beta_z(f) d\mu(z),$$

where $d\mu = z^{-1}dz$ is the normalized Lebesque measure on $\mathbb{T}$, and E satisfies the equations $E(ba) = bE(a)$, $E(ab) = E(a)b$, and $E(b) = b$ for $a \in \mathfrak{B} \rtimes_\alpha \mathbb{N}$ and $b \in \mathfrak{B}$. For the element f of the form above, we have $E(f) = b_0$.

Furthermore, the above element $f = 0$ if and only if $E(f) = 0$, $E(fs^m) = 0$ and $E(f(s^*)^m) = 0$ for every $m \in \mathbb{N}$, from which f can be replaced with any element of $\mathfrak{B}$.

Proof. Since α is a corner endomorphism, we have $s^*\mathfrak{B}s = \mathfrak{B}$. Note that

$$sb = sbs^*ss = \alpha(b)s, \quad \text{and} \quad bs^* = s^*sbs^* = s^*\alpha(b)$$

from which the first part of the statement follows.

For the second part of the statement, we compute

$$E(b_k s^k) = \int_{\mathbb{T}} \beta_z(b_k s^k) d\mu(z) = b_k s^k \int_{\mathbb{T}} z^k d\mu(z),$$
$$E((s^*)^k b_{-k}) = \int_{\mathbb{T}} \beta_z((s^*)^k b_{-k}) d\mu(z) = (s^*)^k b_{-k} \int_{\mathbb{T}} z^{-k} d\mu(z)$$

for $0 \le k \le n$. Since we have

$$\int_{\mathbb{T}} z^k d\mu(z) = (2\pi i)^{-1} \int_0^{2\pi} e^{i(k-1)\theta} i e^{i\theta} d\theta = \begin{cases} 1 & k = 0, \\ 0 & 0 \ne k \in \mathbb{Z}, \end{cases}$$

where $z = e^{i\theta} \in \mathbb{T}$, it follows that

$$E(f) = \sum_{k=0}^{n} E(b_k s^k) + \sum_{k=1}^{n} E((s^*)^k b_{-k}) = b_0 \in \mathfrak{B}.$$

Therefore, we obtain $E(b) = b$ for any $b \in \mathfrak{B}$. Since

$$\begin{aligned} bf &= b(s^*)^n b_{-n} + \cdots + bs^* b_{-1} + bb_0 + bb_1 s + \cdots + bb_n s^n \\ &= (s^*)^n \alpha_n(b) b_{-n} + \cdots + s^* \alpha_1(b) b_{-1} + bb_0 + bb_1 s + \cdots + bb_n s^n, \\ fb &= (s^*)^n b_{-n} b + \cdots + s^* b_{-1} b + b_0 b + b_1 s b + \cdots + b_n s^n b \\ &= (s^*)^n b_{-n} b + \cdots + s^* b_{-1} b + b_0 b + b_1 \alpha_1(b) s + \cdots + b_n \alpha_n(b) s^n, \end{aligned}$$

where $\alpha_n = \alpha \circ \cdots \circ \alpha$ (the n-times composed map), we have $E(bf) = bE(f)$ and $E(fb) = E(f)b$. Therefore, $E(ba) = bE(a)$ and $E(ab) = E(a)b$ for any $a \in \mathfrak{B} \rtimes_\alpha \mathbb{N}$ and $b \in \mathfrak{B}$.

For the third part of the statement, note that

$$\begin{aligned} (b_k s^k)s &= b_k s^{k+1} \in \mathfrak{B} s^{k+1}, \\ (s^*)^k b_{-k} s &= (s^*)^{n-1}(s^* b_{-k} s) \in (s^*)^{n-1} \mathfrak{B}, \quad \text{and} \\ (b_k s^k)s^* &= b_k s^{k-1}(ss^*) = b_k s^{k-1}(ss^*)(s^*)^{k-1} s^{k-1} \\ &= b_k s^k (s^*)^k s^{k-1} \in \mathfrak{B} s^{k-1}, \\ (s^*)^k b_{-k} s^* &= (s^*)^k s^* s b_{-k} s^* = (s^*)^{k+1}(s b_{-k} s^*) \in (s^*)^{k+1} \mathfrak{B}. \end{aligned}$$

Hence, it follows that $E(fs^m) = (s^*)^m b_{-m} s^m$ and $E(f(s^*)^m) = b_m s^m (s^*)^m$ for $m \le n$, and if $m > n$, then $E(fs^m) = 0 = E(f(s^*)^m)$. Thus, $E(fs^m) = 0$ if and only if $b_{-m} = 0$, and $E(f(s^*)^m) = 0$ if and only if $b_m s^m = 0$. □

Proposition 3.5. *There exists a sequence $(\gamma_m)_{m\in\mathbb{N}}$ of endomorphisms of $\pi(E_\varepsilon)$ converging pointwise to the identify map on $\pi(E_\varepsilon)$ such that*

$$\sup_{n\in\mathbb{N}} \|\gamma_m(\pi(t_n)) - \gamma_m(\pi(t_{n+1}))\| < \varepsilon,$$

where $\pi : E_\varepsilon \to \pi(E_\varepsilon) = B'_\varepsilon$ is the canonical quotient map.

Proof. Use [6, Proposition 2.2]. □

Now define L_ε to be the kernel of the map $\rho_\varepsilon : E_2 \to E_\varepsilon$ by universality. Also define $[L]_\varepsilon$ to be the kernel of the map $[\rho]_\varepsilon : \pi(E_2) \to \pi(E_\varepsilon)$.

Proposition 3.6. *We have* $[L]_\varepsilon = \cap_{0\le\varepsilon'<\varepsilon}[L]_{\varepsilon'}$.

Proof. This is [6, Theorem 2.4] using the above proposition. □

Proposition 3.7. *We have* $L_\varepsilon = \cap_{0\le\varepsilon'<\varepsilon}L_{\varepsilon'}$.

Proof. Note that $L_\varepsilon \cap \mathfrak{I}_2 = \{0\}$ since ρ_ε maps $\mathfrak{I}_2$ to $\mathfrak{I}_\varepsilon$, Therefore,

$$[L]_\varepsilon = L_\varepsilon + \mathfrak{I}_2/\mathfrak{I}_2 \cong L_\varepsilon/(L_\varepsilon \cap \mathfrak{I}_2) \cong L_\varepsilon.$$

Hence the conclusion follows from the above proposition. □

Lemma 3.8. *Let $h : \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism of C^*-algebras $\mathfrak{A}, \mathfrak{B}$. Suppose that h is equivariant with respect to corner endomorphisms α, β of $\mathfrak{A}$, $\mathfrak{B}$ respectively, i.e., $h(\alpha(a)) = \beta(h(a))$ for $a \in \mathfrak{A}$. Let $h^\sim : \mathfrak{A} \rtimes_\alpha \mathbb{N} \to \mathfrak{B} \rtimes_\beta \mathbb{N}$ be the $*$-homomorphism induced by h. Then*

$$\ker(h^\sim) = \{f \in \mathfrak{A} \rtimes_\alpha \mathbb{N} \,|\, E_{\mathfrak{A}}(fs^n), E_{\mathfrak{A}}(f(s^*)^n) \in \ker(h) \quad \text{for } n \in \mathbb{N},\, n = 0\},$$

where $E_{\mathfrak{A}} : \mathfrak{A} \rtimes_\alpha \mathbb{N} \to \mathfrak{A}$ is the conditional expectation, $\ker(\cdot)$ means the kernel, and s is the isometry implementing α.

Proof. The following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A} \rtimes_\alpha \mathbb{N} & \xrightarrow{h^\sim} & \mathfrak{B} \rtimes_\beta \mathbb{N} \\ {\scriptstyle E_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle E_{\mathfrak{B}}} \\ \mathfrak{A} & \xrightarrow{h} & \mathfrak{B}. \end{array}$$

Indeed, for $f = (s^*)^n a_{-n} + \cdots + s^* a_{-1} + a_0 + a_1 s + \cdots + a_n s^n$, $a_j, a_{-j} \in \mathfrak{A}$,

$$h^\sim(f) = (t^*)^n h(a_{-n}) + \cdots + t^* h(a_{-1}) + h(a_0) + h(a_1)t + \cdots + h(a_n)t^n\,,$$

where $t = h^\sim(s)$ is the isometry implementing β. Therefore,

$$(h \circ E_{\mathfrak{A}})(f) = h(a_0) = (E_{\mathfrak{B}} \circ h^\sim)(f).$$

For any $g \in \mathfrak{A} \rtimes_\alpha \mathbb{N}$ with $g \in \ker(h^\sim)$, we have $h^\sim(g) = 0$, which is equivalent to that $E_{\mathfrak{B}}(h^\sim(g)t^n) = 0$ and $E_{\mathfrak{B}}(h^\sim(g)(t^*)^n) = 0$ for $n \in \mathbb{N}$ and $n = 0$. This says that $h(E_{\mathfrak{A}}(g)s^n) = 0$ and $h(E_{\mathfrak{A}}(g)(s^*)^n) = 0$ for $n \in \mathbb{N}$ and $n = 0$. □

Theorem 3.9. *For any $\varepsilon \in (0, 2)$, one has $I_\varepsilon^- = I_\varepsilon$.*

Proof. Let $E : \mathfrak{D}_2 \to E_2$ be the conditional expectation induced by the isomorphism $\mathfrak{D}_2 \cong E_2 \rtimes_{\alpha_2} \mathbb{N}$. For $a \in I_\varepsilon^-$, we have $E(as^n) \in L_{\varepsilon'}$ and $E(a(s^*)^n) \in$

$L_{\varepsilon'}$ for all $n \in \mathbb{N}$ and $n = 0$ and $\varepsilon' < \varepsilon$. Thus,

$$E(as^n), E(a(s^*)^n) \in \cap_{0 \leq \varepsilon' < \varepsilon} L_{\varepsilon'} = L_\varepsilon$$

for all $n \in \mathbb{N}$ and $n = 0$. This shows that $a \in I_\varepsilon$.

On the other hand, since $I_{\varepsilon'} \supset I_\varepsilon$ for $\varepsilon' < \varepsilon$, we have

$$I_\varepsilon^- = \cap_{0 \leq \varepsilon' < \varepsilon} I_{\varepsilon'} \supset I_\varepsilon. \qquad \square$$

Theorem 3.10. *We have $I_2^- = I_2$.*

Proof. Let $0 \neq a \in I_2^-$. Note that $\pi(\mathfrak{D}_2)$ is isomorphic to the full group C^*-algebra $C^*(F_2)$ of the free group F_2 of two generators. It is shown by Choi [3] that $C^*(F_2)$ has a separating family of finite dimensional representations. Since the map q_ε maps $\mathfrak{I}_2$ to $\mathfrak{I}_\varepsilon$, we have $I_\varepsilon = \ker(q_\varepsilon) \cong [I]_\varepsilon$, where $[I]_\varepsilon = I_\varepsilon + \mathfrak{I}_2/\mathfrak{I}_2$ (in particular $\varepsilon = 2$). Thus, we may assume that $\pi(a) \neq 0$. Therefore, there exists a finite dimensional representation χ of $\pi(\mathfrak{D}_2)$ to $M_n(\mathbb{C})$ such that $\chi(\pi(a)) \neq 0$. Let $u_1 = \chi(\pi(s_{2,1}))$ and $u_2 = \chi(\pi(s_{2,2}))$. Then using [6, Proposition 3.2] we have $u_1 = \lim_{j\to\infty} u'_j$ for the unitaries $u'_j \in M_n(\mathbb{C})$ such that $\|u'_j u_2 - u_2 u'_j\| < 2$. For each j, there exists a representation χ_j of $\pi(\mathfrak{D}_2)$ such that $\chi_j(\pi(s_{2,1})) = u'_j$ and $\chi_j(\pi(s_{2,2})) = u_2$. If $\|u'_j u_2 - u_2 u'_j\| = \varepsilon_j < 2$, then χ_j vanishes on J_{ε_j}, where J_{ε_j} is the kernel of the map: $\pi(\mathfrak{D}_2) \to \pi(\mathfrak{D}_{\varepsilon_j})$. Thus, χ_j vanishes on the intersection of J_{ε_j} for $1 \leq j < \infty$. Note that $\pi(I_\varepsilon) = J_\varepsilon$. Hence, $\chi_j(\pi(a)) = 0$. Furthermore, χ_j converge pointwise to χ so that $\chi(\pi(a)) = 0$, which is a contradiction. $\square$

Proof of Theorem 3.1. Summing up the above arguments we obtain the desired result. $\square$

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