International Journal of Pure and Applied Mathematics
Volume 44 No. 3 2008, 439-446

MULTIPLIERS FOR A QUOTIENT BANACH SPACE AND THE NEVANLINNA-PICK THEOREM

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Abstract: Let E be a Banach space on a set X and $M(E)$ the space of multipliers of E. In this paper, we study the space of multipliers of the quotient space E/K, where K is a closed $M(E)$ - invariant subspace in E. When E is the classical Hilbert-Hardy space, the Nevanlinna-Pick Theorem shows $M(E/K)$ is a quotient algebra of $M(E)$.

AMS Subject Classification: 47A20, 46J15
Key Words: quotient Bahach space, multiplier, Nevanlinna-Pick Theorem, Hardy space

1. Introduction

A Banach space E of functions on a set X is a Banach space whose elements are complex-valued functions defined on X with the usual pointwise addition and scalar multiplication. If ϕ is a complex-valued function on X and ϕf belongs to E for all f in E, then we write that ϕ is an element of $M(E)$, the space of multipliers of E. We assume that the point evaluations are continuous on E, that is, X is embeded in the dual space E^* and that there is no point in X where all the members of E vanish. It is known that $T_\phi : f \to \phi f$ is a bounded operator on E for each ϕ in $M(E)$, since by continuity of point evaluation, each such map has closed graph. $M(E)$ is a closed subalgebra of $\mathcal{B}(E)$, the set of

Received: February 24, 2008

bounded operators on E, indeed $M(E)$ is closed in the weak operator topology. Thus we assume that the space $M(E)$ of multipliers of E is an operator algebla on E.

If K is a closed subspace of E then E/K is also a Banach space. We want to define the space of multipliers $M(E/K)$ of E/K. For ϕ in $M(E)$ put

$$S_\phi(f+K) = \phi f + K \quad (f \in E).$$

In general, S_ϕ is not well defined on E/K. We need assume that $M(E)K \subset K$. Suppose $M(E/K) = \{S_\phi; \phi \in M(E)\}$. Then $M(E/K)$ is also an operator algebra on E/K. Put

$$\mathcal{K} = \{\phi \in M(E); \phi E \subset K\},$$

then $\mathcal{K}$ is a closed ideal in $M(E)$. By the definition, $S_\phi = 0$ if and only if $\phi E \subset K$. Hence $S_\phi = 0$ if and only if ϕ belongs to $\mathcal{K}$. Therefore $S_{\phi+\psi} = S_\phi$ for any ψ in $\mathcal{K}$ and so there exists a one-to-one map from $M(E/K)$ onto $M(E)/\mathcal{K}$. Moreover this map is contractive. In fact, for any g in K,

$$S_\phi(f+K) = \phi f + K = \phi(f+g) + K$$

and so

$$\|S_\phi(f+K)\| \le \|\phi(f+g)\| \le \|\phi\|\|f+g\|.$$

This implies that $\|S_\phi\| \le \|\phi\|$. Since $S_{\phi+\psi} = S_\phi$ for any ψ in $\mathcal{K}$, $\|S_\phi\| \le \|\phi + \mathcal{K}\|$. Now the following problem is natural.

Problem 1. Is $M(E/K)$ isometrically isomorphic onto $M(E)/\mathcal{K}$?

If Probrem 1 can be solved positively, then it shows that $M(E)/\mathcal{K}$ is an operator algebra on E/K. Suppose

$$M(E/K)' = \{A \in \mathcal{B}(E/K); S_\phi A = AS_\phi \text{ for any } \phi \in M(E)\}.$$

Then $M(E/K)'$ is a commutative algebra in $\mathcal{B}(E/K)$ which contains $M(E/K)$. We are interested in the following problem.

Problem 2. Is $M(E/K)'$ equal to $M(E/K)$?

Problem 2 is related to a problem of commuting dilation, that is, if $A \in \mathcal{B}(E/K)$ such that $AS_\phi = S_\phi A \quad (\phi \in M(E))$ then does exist $\tilde{A} \in \mathcal{B}(E)$ such that $\tilde{A}T_\phi = T_\phi \tilde{A} \quad (\phi \in M(E))$ and $A(f+K) = \tilde{A}f + K \quad (f \in E)$? If $M(E)' = M(E)$ then $\tilde{A} = T_\psi$ for some $\psi \in M(E)$ and so $A = S_\psi$.

Let H^p $(1 \le p \le \infty)$ be the usual Hardy space of analytic functions on the open unit dise D. When $E = H^2$, Sarason [4] solved Problems 1 and 2 positively. Then a theorem of Nevanlinna-Pick and a theorem of Carathéodory follow. When $E = H^p \quad (1 \le p \le \infty, p \ne 2)$ and $K = BH^p$ for a Blaschke product with simple zeros, Snyder [5] solved Problems 1 and 2. In this paper, we solve them when $E = H^p \quad (1 \le p \le \infty)$ and K is arbitrary.

In general, $M(E)$ may not be a supnorm algebra (see [6]). Even if E is a Hilbert space, $M(E)$ is a supnorm algebra on X and dim $M(E)/\mathcal{K} = 2$, it is known that we can solve negatively Problem 1 for some E and $\mathcal{K}$ (see [1]).

In this paper, for a subset S, $[S]$ denotes the closed linear span of S.

2. General Case

For each x in X, put $\tau_x(f) = f(x)$ for a function f on X. We assume that τ_x is bounded on E and $\|\tau_x\|$ denotes the norm of τ_x on E. τ_x is also bounded on $M(E)$ and the norm is just one. For

$$|\tau_x(\phi) \cdot \tau_x(f)| = |\tau_x(\phi f)| \le \|\tau_x\| \cdot \|\phi\| \cdot \|f\| \quad (\phi \in M(E), f \in E)$$

and so $|\tau_x(\phi)| \cdot \|\tau_x\| \le \|\tau_x\| \cdot \|\phi\|$. Put $E_x = \ker\, \tau_x \cap E$ and $M(E)_x = \ker\, \tau_x \cap M(E)$. If $K = \{0\}$ then $\mathcal{K} = \{0\}$ and so Problem 1 can be solved trivially. Moreover the following Proposition 1 solves Problem 2.

Proposition 1. *If $M(E)_x E$ is dense in E_x for any x in X then $M(E)' = M(E)$.*

Proof. It is clear that $M(E) \subset M(E)'$. Suppose $A \in M(E)'$. If $T_\phi \in M(E)$ then $T_\phi^* \tau_x = \overline{\phi(x)} \tau_x$ for any $x \in X$ because $\tau_x \in E^*$. Hence $T^*_{\phi - \phi(x)}(A^* \tau_x) = A^*(T^*_{\phi-\phi(x)} \tau_x) = 0$ because $A T_{\phi-\phi(x)} = T_{\phi - \phi(x)} A$. Therefore for any $f \in E$, $\langle T_{\phi - \phi(x)} f, A^* \tau_x \rangle = 0$ and so $A^* \tau_x = 0$ on E_x because $M(E)_x E$ is deuse in E_x. Thus $A^* \tau_x = \overline{\psi(x)} \tau_x \quad (x \in X)$ and so for any $f \in E$, $\psi(x) f(x) = \langle f, A^* \tau_x \rangle = \langle Af, \tau_x \rangle = (Af)(x)$. Hence $Af = \psi f = T_\psi f \quad (f \in E)$. This implies A belongs to $M(E)$. □

Proposition 2. *If $M(E) + K = E$, then $M(E/K)' = M(E/K)$.*

Proof. For any $f \in E$, put $\tilde{f} = f + K$. Then we may assume that $f \in M(E)$ by hypothesis $M(E) + K = E$. For any $g \in M(E)$, if $A \in M(E/K)'$, then for any x in $(E/K)^*$

$$\begin{aligned} \langle A\tilde{g}, x \rangle = \langle \tilde{g}, A^* x \rangle = \langle \tilde{g} \cdot \tilde{1}, A^* x \rangle \\ = \langle A S_g \tilde{1}, x \rangle = \langle S_g A \tilde{1}, x \rangle = \langle S_g \tilde{\phi}, x \rangle = \langle \tilde{\phi} \tilde{g}, x \rangle\,, \end{aligned}$$

where $\tilde{\phi} = A\tilde{1}$. Since $M(E) + K = E$, we may assume that $\phi \in M(E)$ and so $A = S_\phi$. □

For a subset S of X, let $E|S$ be the restriction of E to S and put $K = \{f \in E; f = 0 \text{ on } S\}$. Then, $E|S$ becomes a Banach space of functions on S

under the quotient norm of E/K. We may assume that $E|S \cong E/K$. Put $\mathcal{K} = \{\phi \in M(E); \phi = 0 \text{ on } S\}$, then $M(E)|S \cong M(E)/\mathcal{K}$. Even if K is such a special case, Probrems 1 and 2 cannot be solved in general. Snyder [5] studied Problem 1, that is, whether $M(E)|S = M(E|S)$. In this special case, Problem 1 is just an interpolation problem. That is, if f is a function on $S \subset X$ and $f(E|S) \subset E|S$, then does there exist a function F on X such that $FE \subset E$ and $F|S = f$ and $\|F\| = \|f\|$? Therefore the research of Snyder [5] is contained in our one.

Corollary 1. *If E is a commutative Banach algebra with unit, then $M(E/K)' = M(E/K)$.*

Proof. If E is a commutative Banach algebra with unit, then $M(E) = E$ and $\mathcal{K} = K$. Hence $M(E) + K = E$. Proposition 2 implies that $M(E/K)' = M(E/K)$. □

Proposition 3. *If E is a commutative Banach algebra with unit, then $M(E)/\mathcal{K} = M(E/K)$, where $K = \mathcal{K}$.*

Proof. By the proof of Corollary 1, $M(E) = E$ and it is easy to see that $M(E)$ is isometrically isomorphic to E. Similarly $M(E/K)$ is isometrically isomorphic to E/K. This implies the proposition. □

3. Two Dimensional Case

In this section we assume that $M(E) \subset E$. (1) of Theorem 1 is due to Snyder [5] and (2) of Theorem 1 is new.

d_x is called the derivation at x if $d_x(fg) = d_x(f)\tau_x(g) + \tau_x(f)d_x(g) \quad (f, g \in M(E))$.

Proposition 4. *Suppose E/K and $M(E/K)$ are of finite dimension 2. Then $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$ or $(E/K)^* = [\tau_x, d_x]$ for $x \in X$, where d_x is a point derivation at x.*

Proof. By hypothesis, $M(E/K) = E/K$ as a set. Since $M(E/K)$ is a commutative Banach algebra and dim $M(E/K) = 2$, by [2, Proposition 1] it is easy to see that $M(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$ or $M(E/K)^* = [\tau_x, d_x]$ for $x \in X$. □

Lemma 1. *Suppose $M(E) + K$ is dense in E. If $\phi \in M(E)$, then $S_\phi^* d_x = \overline{d_x(\phi)}\tau_x + \overline{\tau_x(\phi)}d_x$.*

Proof. For $f \in M(E)$

$$\langle f + K, S_\phi^* d_x\rangle = \langle \phi f + K, d_x\rangle = \langle \phi f, d_x\rangle$$
$$= d_x(\phi)\tau_x(f) + \tau_x(\phi)d_x(f) = \langle f + K, \overline{d_x(\phi)}\tau_x + \overline{\tau_x(\phi)}d_x\rangle\,. \quad \square$$

Theorem 1. *Suppose that $M(E) + K = E$, E/K and $M(E/K)$ are of two dimension. If $M(E/K)$ is isometrically isomorphic to $M(E)/\mathcal{K}$, then the following (1) and (2) are valid.*

(1) When $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u$, $\tau_y(\phi) = v$ and $\|\phi + \mathcal{K}\| \leq 1$ if and only if

$$\|\alpha\bar{u}\tau_x + \beta\bar{v}\tau_y\|_* \leq \|\alpha\tau_x + \beta\tau_y\|_* \quad (\alpha, \beta \in \mathbb{C}).$$

(2) When $(E/K)^* = [\tau_x, d_x]$ for $x \in X$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, d_x(\phi) = v$ and $\|\phi + \mathcal{K}\| \leq 1$ if and only if

$$\|(\alpha\bar{u} + \beta\bar{v})\tau_x + \beta\bar{u}d_x\|_* \leq \|\alpha\tau_x + \beta d_x\|_* \quad (\alpha, \beta \in \mathbb{C}).$$

Proof. (1) If there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, \tau_y(\phi) = v$ with $\|\phi + \mathcal{K}\| \leq 1$ then $\|S_\phi^*\| \leq 1$ by hypothesis. This implies that

$$\|\alpha\bar{u}\tau_x + \beta\bar{v}\tau_y\|_* \leq \|\alpha\tau_x + \beta\tau_y\|_* \quad (\alpha, \beta \in \mathbb{C})$$

because $S_\phi^*\tau_x = \overline{\tau_x(\phi)}\tau_x = \bar{u}\tau_x$ and $S_\phi^*\tau_y = \bar{v}\tau_y$. For the converse, put $A \in \mathcal{B}(H/K)$, $A^*\tau_x = \bar{u}\tau_x$ and $A^*\tau_y = \bar{v}\tau_y$, then $\|A^*\| \leq 1$ and A belongs to $M(E/K)'$. Since $M(E)+K = E$, by Proposition 2 $A = S_\phi$ for some $\phi \in M(E)$. By hypothesis, $\|\phi + \mathcal{K}\| \leq 1$ and $\tau_x(\phi) = u$ and $\tau_y(\phi) = v$.

(2) If there exists $\phi \in M(E)$ with $\tau_x(\phi) = u, d_x(\phi) = v$ with $\|\phi + \mathcal{K}\| \leq 1$ then $\|S_\phi^*\| \leq 1$ by hypothesis. This and Lemma 1 imply

$$\|(\alpha\bar{u} + \beta\bar{v})\tau_x + \beta\bar{u}d_x\|_* \leq \|\alpha\tau_x + \beta d_x\|_* \quad (\alpha, \beta \in \mathbb{C}).$$

For the converse, put $A \in \mathcal{B}(H/K)$, $A^*\tau_x = \bar{u}\tau_x$ and $A^*d_x = \bar{v}\tau_x + \bar{u}d_x$, then $\|A^*\| \leq 1$ and A belongs to $M(E/K)'$ by Lemma 1. By Proposition 2 $A = S_\phi$ for some $\phi \in M(E)$. By hypothesis, $\|\phi+\mathcal{K}\| \leq 1$ and $\tau_x(\phi) = u$ and $\tau_y(\phi) = v$. $\square$

In Theorem 1, if (1) or (2) is valid, then $M(E/K)$ is isometrially isomorphic to $M(E)/\mathcal{K}$.

Corollary 2. *In Theorem 1, if E is a Hilbert space, then there exist k_x and h_x in E such that*

$$\tau_x(f) = (f, k_x) \quad (f \in E)$$

and

$$d_x(f) = (f, h_x) \quad (f \in E)$$

and the following (1) and (2) are valid.

(1) When $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, \tau_y(\phi) = v$ and $\|\phi + \mathcal{K}\| \leq 1$ if and only if

$$|\alpha|^2(1-|u|^2)(k_x,k_x) + \alpha\bar{\beta}(1-\bar{u}v)(k_x,k_y) + \bar{\alpha}\beta(1-u\bar{v})(k_y,k_x) + |\beta|^2(1-|v|^2)(k_y,k_y) \geq 0\,,$$

for any $\alpha, \beta \in \mathbb{C}$.

(2) When $(E/K)^* = [\tau_x, d_x]$ for $x \in X$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, d_x(\phi) = v$ and $\|\phi + \mathcal{K}\| \leq 1$ if and only if

$$(|\alpha|^2 - |\alpha\bar{u} + \beta\bar{v}|^2)(k_x,k_x) + (\alpha\bar{\beta} - (\alpha\bar{\beta}|u|^2 + |\beta|^2 u\bar{v}))(k_x,h_x) + (\bar{\alpha}\beta - (\bar{\alpha}\beta|u|^2 + |\beta|^2\bar{u}v))(h_x,k_x) + |\beta|^2(1-|u|^2)(h_x,h_x) \geq 0\,,$$

for any $\alpha, \beta \in \mathbb{C}$.

The condition of (1) in Corollary 2 shows that the 2×2 matrix $\{(1 - |u|^2)(k_x,k_x), (1-\bar{u}v)(k_x,k_y), (1-u\bar{v})(k_y,k_x), (1-|v|^2)(k_y,k_y)\}$ is nonnegative. When $(k_x, h_x) = 0$, the condition of (2) in Corollary 2 shows that the 2×2 matrix $\{(1-|u|^2)(k_x,k_x), \bar{u}v(h_x,k_x), u\bar{v}(k_x,h_x), (1-|u|^2-|v|^2)(h_x,h_x)\}$ is nonnegative.

When $\dim E/K \geq 3$, even if $\dim E/K$ is finite, it is difficult to describe $(E/K)^*$ except $K = \{f \in E : f(x_j) = 0 \ \ 1 \leq j \leq \dim E/K\}$ and $x_i \neq x_j (i \neq j)$. Therefore we could not generalize (2) of Theorem 1.

4. Hardy Space H^p $(1 \leq p \leq \infty)$

In this section, we solve Problems 1 and 2 when $E = H^p$ for $1 \leq p \leq \infty$. When $E = H^\infty$, we can solve trivially Problems 1 and 2 by Corollary 2 and Proposition 4. If $\dim H^p/K < \infty$ then $M(H^p) + K = H^p$ and so $M(H^p/K)' = M(H^p/K)$ by Proposition 2. However we have to work more in order to prove $M(H^p/K) = M(H^p)/\mathcal{K}$.

Let W be a nonnegative function in L^1 with $\log W$ in $L^1 = L^1(d\theta/2\pi)$. Then there exists an outer function h in H^1 with $W = |h|$. For $1 \leq p < \infty$, $H^p(W)$ denotes the closure of analytic polynomials in $L^p(W) = L^1(W d\theta/2\pi)$. Then $H^p(W) = h^{-1/p}H^p$ and so we may assume that $H^p(W)$ is a Banach space of analytic functions on D. It is known that the point evaluations of points in D are continuous on $H^p(W)$. It is well known that $M(H^p(W)) = H^\infty$.

Theorem 2. *For $1 \leq p \leq \infty$, let K be a closed subspace of $H^p(W)$ with $M(H^p(W))K \subseteq K$. Then $M(H^p(W)/K)' = M(H^p(W)/K)$ and $M(H^p(W)/K) = M(H^p(W))/\mathcal{K}$, where $\mathcal{K} = \{\phi \in M(H^p(W)) : \phi H^p(W) \subseteq K\}$.*

Proof. Since $M(H^p(W)) = H^\infty$, $\mathcal{K} = QH^\infty$ for some inner function and $K = QH^p(W)$. Since $M(H^p(W)/K) \subseteq M(H^p(W)/K)'$, we will show that $M(H^p(W)/K))' \subseteq M(H^p(W)/K)$. If $A \in M(H^p(W)/K)'$, then there exists ψ in $H^p(W)$ such that $A(1+K) = \psi + K$. For any polynomial $f = h^{-1/p}F$ in $H^p(W)$, $\|\psi f + K\|_W \leq \|A\| \|f + K\|_W$. Since $K = QH^p(W) = h^{-1/p}QH^p$, if $1/p + 1/q = 1$

$$\begin{aligned} &\|\psi f + QH^p(W)\|_W \\ &\quad = \sup\{|\langle \psi f, g\rangle_W| \ : \ g \in \{QH^p(W)\}^\perp \text{ and } \|g\|_W \leq 1\} \\ &\quad = \sup\left\{|\int \psi h^{-1/p} F \bar{Q} h^{-1/q} G |h| dm| \ : \ G \in H_0^q \text{ and } \|G\|_q \leq 1\right\} \\ &\quad = \sup\left\{|\int \psi \bar{Q} F G dm| \ : \ G \in H_0^q \text{ and } \|G\|_q \leq 1\right\} \\ &\quad \leq \|A\| \|f\|_W = \|A\| \|F\|_p \end{aligned}$$

because $\{QH^p(W)\}^\perp = Q\bar{h}^{1/p}|h|^{-1}\bar{H}_0^q$. Thus

$$\sup\left\{|\int \psi \bar{Q} F G d\theta/2\pi| \ : \ F \in H^p, G \in H_0^q, \|F\|_p \leq 1 \text{ and } \|G\|_q \leq 1\right\} \leq \|A\|.$$

By the factorization theorem of H^1,

$$\sup\{|\int \psi \bar{Q} K d\theta/2\pi| : K \in H_0^1 \text{ and } \|K\|_1 \leq 1\} \leq \|A\|.$$

Since $(\bar{Q}H_0^1)^* = L^\infty/QH^\infty$, $\|\psi + QH^\infty\| \leq \|A\|$. Hence there exists a function ϕ in H^∞ such that $S_\phi = A$ and $\|\phi + \mathcal{K}\| = \|S_\phi\|$. Thus A belongs to $M(H^p(W)/K)$. Therefore $M(H^p(W)/K)' = M(H^p(W)/K)$ and $M(H^p(W)/\mathcal{K} = M(H^p(W)/K)$. □

Corollary 3. *For $1 \leq p \leq \infty$, $M(H^p(W)/QH^p(W)) = H^\infty/QH^\infty$ for any inner function Q.*

Acknowledgements

This research was partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

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