International Journal of Pure and Applied Mathematics

Volume 44 No. 3 2008, 447-465

ON GENERALIZED EULER POLYNOMIALS IN CLIFFORD ANALYSIS

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Abstract: The aim of this paper is to obtain a multi-dimensional generalization of the well known Euler polynomials in the context of Clifford Analysis. The generalized Euler polynomials are achieved via a generating function. They are special hypercomplex holomorphic functions with properties analogous to those of the classical polynomials. Furthermore, a hypercomplex polynomial Euler matrix is defined using a hypercomplex Pascal matrix.

AMS Subject Classification: 05A19, 11B68, 30G35
Key Words: hypercomplex Euler polynomials, Euler numbers, hypercomplex Euler matrix

1. Introduction

The goal of this article is to introduce Euler type polynomials into the context of Clifford Analysis.

Euler polynomials $E_n(x)$ of one real variable are defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1}$$

Received: March 6, 2008

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They are closely related to Bernoulli polynomials $B_n(x)$, which are implicitly given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Relationships between both polynomials can be found in [1], [5], and [13], for instance.

In [9] hypercomplex Bernoulli polynomials have been introduced and their properties have been studied. In this article we are going to obtain Euler polynomials in a similar way by using generalized power series representations of hypercomplex monogenic functions. For readers not familiar with this type of generalized holomorphic functions in higher dimensions we are referring the details in Subsection 2.1. In Subsection 2.2 are introduced Euler polynomials in n hypercomplex variables. Some of their basic properties are proved and the similarity with the classical Euler polynomials are emphasized (see, for instance, [1], [5], [10], [12], and [13] for their corresponding properties). Finally, in the last section, a matrix of hypercomplex Euler polynomials is defined, based on a hypercomplex Pascal matrix.

2. Hypercomplex Generalizations of Euler Polynomials

2.1. Preliminaries

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product defined by the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$, $1 \leq h_1 \leq \cdots \leq h_n$, $e_\emptyset = e_0 = 1$, forms a base of the Clifford algebra $Cl_{0,n}$, over $\mathbb{R}$, generated by this product.

Identifying the element $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with

$$z = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n \in \mathcal{A} \equiv span_{\mathbb{R}}\{e_0, \ldots, e_n\} \cong \mathbb{R}^{n+1},$$

the real vector space $\mathbb{R}^{n+1}$ will be embedded in $Cl_{0,n}$.

Defining the operator

$$D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \cdots + \frac{\partial}{\partial x_n} e_n,$$

as natural generalization of the complex Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}),$$

the equation

$$Df = 0 \ \ (fD = 0)$$

defines left (right) hypercomplex holomorphic functions (also called left (right) monogenic functions) $f = f(z)$ as Clifford algebra valued functions in the kernel of D (cf. [3]). In the following, for simplicity, we mainly work with left monogenic functions, and the case of right monogenic functions can be treated analogously.

Powers of z, $f(z) = z^k, k = 2, \ldots$, are not monogenic functions. Actually, the function $f(z) = z \in \mathcal{A}$ is only monogenic if $n = 1$, i.e, if $\mathcal{A} = \mathbb{C}$ because $Dz = 1 - n$. This fact means, that powers z^k cannot be considered as hypercomplex generalizations of the complex power $z^k, z \in \mathbb{C}$, and justify the use of another hypercomplex structure for $\mathbb{R}^{n+1}$ (cf. [6]). Such structure is based on an isomorphism between $\mathbb{R}^{n+1}$ and

$$\mathcal{H}^n = \{\vec{z} : \vec{z} = (z_1, \ldots, z_n)\,, z_k = x_k - x_0 e_k, \quad x_0, x_k \in \mathbb{R}, k = 1, \ldots, n\}\,,$$

where the components of the vector $\vec{z}$ (also called hypercomplex variables) are monogenic. Since the ordinary products $z_i z_k, i \neq k$, are not monogenic, in [6] was introduced a n-ary operation which solves the problem.

Definition 2.1. Let $V_{+,\cdot}$ be a commutative or non-commutative ring, $a_k \in V$ $(k = 1, \ldots, n)$, then the *symmetric "×"-product* is defined by

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1,\ldots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_n}\,, \tag{2}$$

where the sum runs over *all* permutations of all $(i_1, \ldots, i_n)$.

Additionally we apply the following (see (4) in [7])

Convention. If the factor a_j occurs σ_j-times in (2), we briefly write

$$\begin{aligned} &\underbrace{a_1 \times \cdots \times a_1}_{\sigma_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\sigma_n} \\ &\quad = {a_1}^{\sigma_1} \times {a_2}^{\sigma_2} \times \cdots \times {a_n}^{\sigma_n} = \vec{a}^{\,\sigma}\,, \end{aligned} \tag{3}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n,)$ and set parentheses if the powers are understood in the ordinary way.

Formulas (2) and (3) allow to obtain the polynomial formula

$$(z_1 + \cdots + z_n)^k = \sum_{|\sigma|=k} \binom{k}{\sigma} \vec{z}^{\sigma}, \quad k \in \mathbb{N}, \tag{4}$$

$\binom{k}{\sigma} = \frac{k!}{\sigma!}, \sigma = (\sigma_1, \ldots, \sigma_n)$ (see [7] and [8]) and to work in the same way as in the case of several commutative variables.

Since all functions of the form $f(z) = \vec{z}^{\sigma}$ are left and right monogenic and $Cl_{0,n}$ - linear independent, they can be used as base for generalized power series. In [7] and [8] it has been shown, that the generalized power series of the form

$$P(\vec{z}) = \sum_{k=0}^{\infty} \left(\sum_{|\sigma|=k} \vec{z}^{\sigma} c_{\sigma} \right), c_{\sigma} \in Cl_{0,n}$$

generates in the neighborhood of the origin a monogenic function $f(\vec{z})$ such that

$$f(\vec{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{|\sigma|=k} \vec{z}^{\sigma} \binom{k}{\sigma} \frac{\partial^{|\sigma|} f(\vec{0})}{\partial \vec{x}^{\sigma}} \right),$$

where $\vec{x} = (x_1, \ldots, x_n)$, which means that $f(\vec{z})$ coincides in the interior of its domain of convergence with its Taylor series.

The partial derivatives of $\vec{z}^{\sigma}$ with respect to x_k are obtained as

$$\frac{\partial \vec{z}^{\sigma}}{\partial x_k} = \sum_{\sigma} \sigma_k \vec{z}^{\sigma - \tau_k}, \tag{5}$$

where τ_k is the multiindex with 1 at place k and zero otherwise (cf. [7]).

A real differentiable function is left (right) hypercomplex derivable in $\Omega \subset \mathcal{H}^n$ if and only if f is left (right) monogenic in $\Omega \subset \mathcal{H}^n$. If the hypercomplex derivative exists, is given by (cf. [8])

$$\frac{1}{2}\overline{D}f \ \ (\text{resp. } \frac{1}{2}f\overline{D}),$$

where $\overline{D}$ is the conjugated generalized Cauchy-Riemann operator

$$\overline{D} = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}e_1 - \cdots - \frac{\partial}{\partial x_n}e_n.$$

Furthermore, the left(right) hypercomplex derivative of f at $\vec{z}$ is exactly

$$\frac{1}{2}\overline{D}f = \frac{\partial f}{\partial x_0} \ \ (\text{resp. } \frac{1}{2}f\overline{D} = \frac{\partial f}{\partial x_0}). \tag{6}$$

These results are analogous to the well known for the complex holomorphic function $f : \mathbb{C} \to \mathbb{C}$, where if the complex derivative $f' = \frac{df}{dz}$ exists then it is given by the complex partial derivative

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$$

and satisfies

$$f' = \frac{df}{dz} = \frac{\partial f}{\partial x}.$$

2.2. Hypercomplex Euler Polynomials

Motivated by (1), written in the form

$$2e^{tx} = 2\sum_{j=0}^{\infty} \frac{1}{j!}(xt)^j = \left(1 + \sum_{r=0}^{\infty} \frac{1}{r!}t^r\right) \sum_{n=0}^{\infty} \frac{1}{n!}E_n(x)t^n, \tag{7}$$

we can extend the definition of Euler polynomials in terms of its generating function to the hypercomplex case. With this regard, we start by defining a hypercomplex exponential function in the following way:

$$\mathbf{Exp(t,z)} := \exp(t_1 z_1, \dots, t_n z_n) = \sum_{k=0}^{\infty} \frac{1}{k!}(t_1 z_1 + \cdots + t_n z_n)^k,$$

where $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$, $\mathbf{z} = (z_1, \dots, z_n) \in \mathcal{H}^n$.

Definition 2.2. The *hypercomplex Euler polynomials* $E_{s_1,\dots,s_n}(z_1, \dots, z_n)$, $s_k \in \mathbb{N}_0,\ k = 1, \dots, n$ are defined by the relation:

$$2\mathbf{Exp(t,z)} = \left(1 + \sum_{r=0}^{\infty} \frac{1}{r!}(t_1 + \cdots + t_n)^r\right) \sum_{|s|=0}^{\infty} \frac{1}{s!}E_{s_1,\dots,s_n}(z_1, \dots, z_n)t_1^{s_1} \dots t_n^{s_n}.$$

By virtue of (4), this equality can be written in the equivalent form

$$2\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!}z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} t_1^{\sigma_1} \cdots t_n^{\sigma_n} = \sum_{|s|=0}^{\infty} \frac{1}{s!}E_{s_1,\dots,s_n}(z_1, \dots, z_n)t_1^{s_1} \cdots t_n^{s_n} + \left(\sum_{|j|=0}^{\infty} \frac{1}{j!}t_1^{j_1} \cdots t_n^{j_n}\right)\left(\sum_{|s|=0}^{\infty} \frac{1}{s!}E_{s_1,\dots,s_n}(z_1, \dots, z_n)t_1^{s_1} \cdots t_n^{s_n}\right),$$

or

$$2\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!}z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} t_1^{\sigma_1} \cdots t_n^{\sigma_n} = \sum_{|\sigma|=0}^{\infty} \left(\frac{1}{\sigma!}E_{\sigma_1,\dots,\sigma_n}(z_1, \dots, z_n)t_1^{\sigma_1} \cdots t_n^{\sigma_n}\right.$$

$$+ \sum_{j+s=\sigma} \frac{1}{j!s!} E_{s_1,\ldots,s_n}(z_1,\ldots,z_n) \Bigg) t_1^{\sigma_1} \cdots t_n^{\sigma_n}.$$

Thus, comparing both sides we arrive to

$$\frac{E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)}{\sigma!} + \sum_{j+s=\sigma} \frac{E_{s_1,\ldots,s_n}(z_1,\ldots,z_n)}{j!s!} = \frac{2}{\sigma!} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} \tag{8}$$

or, equivalently,

$$E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) + \sum_{s_1=0}^{\sigma_1} \cdots \sum_{s_n=0}^{\sigma_n} \binom{\sigma_1}{s_1} \cdots \binom{\sigma_n}{s_n} E_{s_1,\ldots,s_n}(z_1,\ldots,z_n)$$
$$= 2 z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}, \tag{9}$$

for $\sigma_k = 0, 1, \ldots, k = 1, \ldots, n$, which relates the hypercomplex Euler polynomials to the generalized powers and provides a way to compute those polynomials.

For example, some hypercomplex Euler polynomials given by (8) or (9), with $i, j \in \mathbb{N}$ and $i, j \leq n$, are shown below:

$$\begin{aligned}
E_{0,\ldots,0}(z_1,\ldots,z_n) &= 1\,, \\
E_{0,\ldots,\underbrace{1}_{i},\ldots,0}(z_1,\ldots,z_n) &= z_i - \tfrac{1}{2}\,, \\
E_{0,\ldots,\underbrace{1}_{i},\ldots,\underbrace{1}_{j},\ldots,0}(z_1,\ldots,z_n) &= z_i \times z_j - \tfrac{1}{2}(z_i + z_j)\,, \\
E_{0,\ldots,\underbrace{2}_{i},\ldots,0}(z_1,\ldots,z_n) &= z_i^2 - z_i\,, \\
E_{0,\ldots,\underbrace{2}_{i},\ldots,\underbrace{1}_{j},\ldots,0}(z_1,\ldots,z_n) &= z_i^2 \times z_j - z_i \times z_j - \tfrac{1}{2} z_i^2 + \tfrac{1}{4}\,, \\
E_{0,\ldots,\underbrace{3}_{i},\ldots,0}(z_1,\ldots,z_n) &= z_i^3 - \tfrac{3}{2} z_i^2 + \tfrac{1}{4}\,.
\end{aligned}$$

The set of classical Euler polynomials is included in the set of hypercomplex Euler polynomials. Each classical polynomial can be obtained when all the indices $\sigma_k, k = 1, \ldots, n$ in (8) or (9) are equal to zero or only one of them is different from zero.

It is well known that Euler numbers $E_n,\ n \in \mathbb{N}_0$ are defined by the generating function (cf. [1] and [10])

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

or, equivalently, by

$$2e^t = (1 + e^{2t}) \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{10}$$

Then, following the same ideas used above for the hypercomplex Euler polynomials and taking as starting point

$$2\exp(t_1,\ldots,t_n)=\Big(1+\exp(2t_1,\ldots,2t_n\Big)\sum_{|\sigma|=0}^{\infty}\frac{1}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}t_1^{\sigma_1}\cdots t_n^{\sigma_n},$$

it is possible to arrive to

$$\frac{E_{\sigma_1,\ldots,\sigma_n}}{\sigma!}+\sum_{j+s=\sigma}2^{|j|}\frac{E_{s_1,\ldots,s_n}}{j!s!}=\frac{2}{\sigma!}$$

or

$$E_{\sigma_1,\ldots,\sigma_n}+\sum_{s_1=0}^{\sigma_1}\cdots\sum_{s_n=0}^{\sigma_n}\binom{\sigma_1}{s_1}\cdots\binom{\sigma_n}{s_n}2^{|\sigma-s|}E_{s_1,\ldots,s_n}=2.$$

Thereby we can compute the numbers $E_{\sigma_1,\ldots,\sigma_n}$ which coincides with the classical Euler numbers. For instance:

$$\begin{array}{lcllcl} E_{\sigma_1,\ldots,\sigma_n} & = & 1, & |\sigma| & = 0\,, \\ E_{\sigma_1,\ldots,\sigma_n} & = & 0, & |\sigma| & = 2k+1, k=0,\ldots\,, \\ E_{\sigma_1,\ldots,\sigma_n} & = & -1, & |\sigma| & = 2\,, \\ E_{\sigma_1,\ldots,\sigma_n} & = & 5, & |\sigma| & = 4\,,\vdots \end{array}$$

The idea developed above is already used in [9] for the implicit definition of hypercomplex Bernoulli polynomials by

$$\sum_{j+s=\sigma}\frac{B_{s_1,\ldots,s_n}(z_1,\ldots,z_n)}{(|j|+1|)j!s!}=\frac{1}{\sigma!}z_1^{\sigma_1}\times\cdots\times z_n^{\sigma_n} \tag{11}$$

for $\sigma_k=0,1,\ldots,k=1,\ldots,n$. This formula is a consequence of the definition of hypercomplex Bernoulli polynomials by

$$\frac{(t_1+\cdots+t_n)\exp\left(t_1z_1,\ldots,t_nz_n\right)}{\exp\left(t_1,\ldots,t_n\right)-1}=\sum_{|\sigma|=0}^{\infty}\frac{1}{\sigma!}B_{\sigma_1,\ldots,\sigma_n}\left(z_1,\ldots,z_n\right)t_1^{\sigma_1}\cdots t_n^{\sigma_n}. \tag{12}$$

Before concluding this section we are going to emphasize some of the most interesting relations and properties of hypercomplex Euler polynomials and Euler numbers. Most of them are analogous to the ones in the classical case.

Property 2.1.

$$E_{\sigma_1,\ldots,\sigma_n}(1-z_1,\ldots,1-z_n)=(-1)^{|\sigma|}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n), \tag{13}$$

$\sigma_i\in\mathbb{N}_0, i=1,\ldots,n.$

Proof. Starting by the definition of hypercomplex Euler polynomials in terms of its generating function we have,

$$\frac{2\exp((1-z_1)t_1,\ldots,(1-z_n)t_n)}{\exp(t_1,\ldots,t_n)+1} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(1-z_1,\ldots,1-z_n) t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$

or

$$\frac{2\exp(t_1,\ldots,t_n)\exp(z_1(-t_1),\ldots,z_n(-t_n))}{\exp(t_1,\ldots,t_n)+1} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(1-z_1,\ldots,1-z_n) t_1^{\sigma_1}\cdots t_n^{\sigma_n}.$$

From this results that

$$\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)(-t_1)^{\sigma_1}\cdots(-t_n)^{\sigma_n} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(1-z_1,\ldots,1-z_n) t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$

and consequently

$$\sum_{|\sigma|=0}^{\infty} \frac{(-1)^{|\sigma|}}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) t_1^{\sigma_1}\cdots t_n^{\sigma_n} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(1-z_1,\ldots,1-z_n) t_1^{\sigma_1}\cdots t_n^{\sigma_n}. \quad (14)$$

Comparing coefficients of $t_1^{\sigma_1}\cdots t_n^{\sigma_n}$ in (14), (13) holds. □

Remark 2.2. Property 2.1 generalizes the symmetry relation known in the classical case (see, e.g., [1] and [10]) $E_k(1-x) = (-1)^k E_k(x)$, $k \in \mathbb{N}_0$.

As immediate consequence of Property 2.1 we arrive to

$$E_{\sigma_1,\ldots,\sigma_n}(1,\ldots,1) = (-1)^{|\sigma|} E_{\sigma_1,\ldots,\sigma_n}(0,\ldots,0)$$

which has a correspondence in the classical case, too (see, for instance [4]):

$$E_k(1) = (-1)^k E_k(0), \quad k \in \mathbb{N}_0.$$

Concerning the differences relations involving classical Euler polynomials it is known that (see, e.g., [1], [5], and [10])

$$\nabla E_k(x) = 2x^k, \ k \in \mathbb{N}_0\,,$$

where

$$\nabla E_k(x) = E_k(x+1) + E_k(x).$$

Similarly, considering $\nabla := (T + I)$ with T as the shift operator which acts simultaneously on all variables, i. e.,

$$Tf(z_1, \ldots, z_n) = f(z_1 + 1, \ldots, z_n + 1), \tag{15}$$

we have for the generalized polynomials:

Property 2.3.

$$\nabla E_{\sigma_1,\ldots,\sigma_n}(z_1, \ldots, z_n) = 2z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}, \tag{16}$$

$\sigma_i \in \mathbb{N}_0, i = 1, \ldots, n.$

Proof. Noting that

$$\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} \nabla E_{\sigma_1,\ldots,\sigma_n}(z_1, \ldots, z_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n}$$
$$= \frac{2\exp\left(t_1\left(z_1+1\right), \ldots, t_n\left(z_n+1\right)\right) + 2\exp\left(t_1 z_1, \ldots, t_n z_n\right)}{\exp\left(t_1, \ldots, t_n\right) + 1}$$
$$= 2\exp\left(t_1 z_1, \ldots, t_n z_n\right) = \sum_{|\sigma|=0}^{\infty} \frac{2}{\sigma!} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} t_1^{\sigma_1} \cdots t_n^{\sigma_n},$$

and comparing the coefficients of $t_1^{\sigma_1} \cdots t_n^{\sigma_n}$, we get the desired result. □

Property 2.4.

$$(-1)^{|\sigma|+1} E_{\sigma_1,\ldots,\sigma_n}(-z_1, \ldots, -z_n)$$
$$= E_{\sigma_1,\ldots,\sigma_n}(z_1, \ldots, z_n) - 2z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}, \tag{17}$$

$\sigma_i \in \mathbb{N}_0, i = 1, \ldots, n.$

Proof. Since Property 2.3 implies

$$E_{\sigma_1,\ldots,\sigma_n}(1-z_1, \ldots, 1-z_n) + E_{\sigma_1,\ldots,\sigma_n}(-z_1, \ldots, -z_n) = 2(-1)^{|\sigma|} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}$$

and, in view of the Property 2.1 we get

$$(-1)^{|\sigma|} E_{\sigma_1,\ldots,\sigma_n}(z_1, \ldots, z_n) + E_{\sigma_1,\ldots,\sigma_n}(-z_1, \ldots, -z_n)$$
$$= 2(-1)^{|\sigma|} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}$$

or

$$E_{\sigma_1,\ldots,\sigma_n}(z_1, \ldots, z_n) + (-1)^{|\sigma|} E_{\sigma_1,\ldots,\sigma_n}(-z_1, \ldots, -z_n) = 2z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}.$$

Multiplying both sides by (-1) we obtain (17). □

Remark 2.5. This property generalizes

$$(-1)^{k+1}E_k(-x) = E_k(x) - 2x^k, \; k \in \mathbb{N}_0$$

known in the classical case (see, e.g., [1]).

Property 2.6.

$$E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{k_1=0}^{\sigma_1}\cdots\sum_{k_n=0}^{\sigma_n}\binom{\sigma_1}{k_1}\cdots\binom{\sigma_n}{k_n}\frac{E_{k_1,\ldots,k_n}}{2^{|k|}}.$$
$$(z_1-\frac{1}{2})^{\sigma_1-k_1}\times\cdots\times(z_n-\frac{1}{2})^{\sigma_n-k_n},$$

$\sigma_i \in \mathbb{N}_0, i = 1,\ldots,n$.

Proof. By definition of the hypercomplex Euler polynomials we have

$$\frac{2\exp(2z_1t_1,\ldots,2z_nt_n)}{\exp(2t_1,\ldots,2t_n)+1} = \sum_{|\sigma|=0}^{\infty}\frac{2^{|\sigma|}}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$

which is equivalent to

$$\frac{2\exp((z_1-\frac{1}{2})2t_1,\ldots,(z_n-\frac{1}{2})2t_n)}{\exp(-t_1,\ldots,-t_n)(\exp(2t_1,\ldots,2t_n)+1)}$$
$$= \sum_{|\sigma|=0}^{\infty}\frac{2^{|\sigma|}}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)t_1^{\sigma_1}\cdots t_n^{\sigma_n}.$$

Thus

$$\sum_{|k|=0}^{\infty}\frac{E_{k_1,\ldots,k_n}}{k!}t_1^{k_1}\cdots t_n^{k_n}\sum_{|r|=0}^{\infty}\frac{2^{|r|}}{r!}((z_1-\frac{1}{2})^{r_1}\times\cdots\times(z_n-\frac{1}{2})^{r_n})t_1^{r_1}\cdots t_n^{r_n}$$
$$= \sum_{|\sigma|=0}^{\infty}\frac{2^{|\sigma|}}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$

and

$$\frac{2^{|\sigma|}}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{k+r=\sigma}\frac{E_{k_1,\ldots,k_n}2^{|r|}}{k!r!}(z_1-\frac{1}{2})^{r_1}\times\cdots\times(z_n-\frac{1}{2})^{r_n}.$$

Hence,

$$E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{k_1=0}^{\sigma_1}\cdots\sum_{k_n=0}^{\sigma_n}\binom{\sigma_1}{k_1}\cdots\binom{\sigma_n}{k_n}\frac{E_{k_1,\ldots,k_n}}{2^{|k|}}$$
$$(z_1-\frac{1}{2})^{\sigma_1-k_1}\times\cdots\times(z_n-\frac{1}{2})^{\sigma_n-k_n}. \quad \square$$

Remark 2.7. An analogous expansion is verified by classical Euler polynomials (see, e.g., [1]) in the form of

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k},\ n \in \mathbb{N}_0.$$

Property 2.8.

$$E_{\sigma_1,\ldots,\sigma_n} = 2^{|\sigma|} E_{\sigma_1,\ldots,\sigma_n} \left(\frac{1}{2}, \ldots, \frac{1}{2}\right), \tag{18}$$

$\sigma_i \in \mathbb{N}_0, i = 1, \ldots, n.$

Proof. Taking $(z_1, \ldots, z_n) = (\frac{1}{2}, \ldots, \frac{1}{2})$ in the previous property, (18) follows immediately. □

Remark 2.9. A similar relation between Euler numbers and the values of the corresponding polynomials in $x = \frac{1}{2}$ is (cf. [1]):

$$E_k = 2^k E_k \left(\frac{1}{2}\right),\ k \in \mathbb{N}_0.$$

Property 2.10. *The connection of $E_{\sigma_1,\ldots,\sigma_n}$ (Euler numbers) with the values of hypercomplex Euler polynomials at the origin is given by*

$$E_{\sigma_1,\ldots,\sigma_n} = \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} 2^{|k|} E_{k_1,\ldots,k_n}(0, \ldots, 0).$$

Proof. Since

$$\frac{2}{\exp(2t_1, \ldots, 2t_n) + 1} \exp(t_1, \ldots, t_n) = \sum_{|\sigma|=0}^{\infty} \frac{E_{\sigma_1,\ldots,\sigma_n}}{\sigma!} t_1^{\sigma_1} \cdots t_n^{\sigma_n}$$

we have

$$\sum_{|k|=0}^{\infty} \frac{2^{|k|} E_{k_1,\ldots,k_n}(0, \ldots, 0)}{k!} t_1^{k_1} \cdots t_n^{k_n} \sum_{|r|=0}^{\infty} \frac{1}{r!} t_1^{r_1} \cdots t_n^{r_n} = \sum_{|\sigma|=0}^{\infty} \frac{E_{\sigma_1,\ldots,\sigma_n}}{\sigma!} t_1^{\sigma_1} \cdots t_n^{\sigma_n}.$$

From this we conclude that

$$\frac{E_{\sigma_1,\ldots,\sigma_n}}{\sigma!} = \sum_{k+r=\sigma} \frac{2^{|k|}}{k!r!} E_{k_1,\ldots,k_n}(0, \ldots, 0)$$

and

$$E_{\sigma_1,\ldots,\sigma_n} = \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} 2^{|k|} E_{k_1,\ldots,k_n}(0, \ldots, 0).$$

□

Property 2.11. *Let $(z_1, \ldots, z_n) \in \mathcal{H}^n$ and $(h_1, \ldots, h_n) \in \mathbb{R}^n$.*

$E_{\sigma_1,\ldots,\sigma_n}(z_1 + h_1, \ldots, z_n + h_n) =$

$$= \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} E_{k_1,\ldots,k_n}(z_1, \cdots, z_n) h_1^{\sigma_1-k_1} \cdots h_n^{\sigma_n-k_n},$$

$\sigma_i \in \mathbb{N}_0, i = 1, \ldots, n.$

Proof. Starting by

$$\frac{2\exp((z_1+h_1)t_1, \ldots, (z_n+h_n)t_n)}{\exp(t_1, \ldots, t_n)+1} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1+h_1, \ldots, z_n+h_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n},$$

that is,

$$\frac{2\exp(z_1t_1, \ldots, z_nt_n)}{\exp(t_1, \ldots, t_n)+1} \exp(h_1t_1, \ldots, h_nt_n) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1+h_1, \ldots, z_n+h_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n},$$

we obtain the following equivalent form

$$\sum_{|k|=0}^{\infty} \frac{1}{k!} E_{k_1,\ldots,k_n}(z_1, \ldots, z_n) t_1^{k_1} \cdots t_n^{k_n} \sum_{|r|=0}^{\infty} \frac{1}{r!} h_1^{r_1} \cdots h_n^{r_n} t_1^{r_1} \cdots t_n^{r_n} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1+h_1, \ldots, z_n+h_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n}$$

or

$$\sum_{|\sigma|=0}^{\infty} \left(\sum_{k+r=\sigma} \frac{1}{k!r!} E_{k_1,\ldots,k_n}(z_1, \ldots, z_n) h_1^{r_1} \cdots h_n^{r_n} \right) t_1^{\sigma_1} \cdots t_n^{\sigma_n} = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1+h_1, \ldots, z_n+h_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n}.$$

It follows that

$$E_{\sigma_1,\ldots,\sigma_n}(z_1+h_1, \ldots, z_n+h_n) = \sum_{k+r=\sigma} \frac{\sigma!}{k!r!} E_{k_1,\ldots,k_n}(z_1, \ldots, z_n) h_1^{r_1} \cdots h_n^{r_n}$$

i.e.

$$E_{\sigma_1,\ldots,\sigma_n}(z_1+h_1, \ldots, z_n+h_n) = \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} E_{k_1,\ldots,k_n}(z_1, \ldots, z_n) h_1^{\sigma_1-k_1} \cdots h_n^{\sigma_n-k_n}.$$

□

Remark 2.12. This property generalizes the well known in the classical case (cf. [1] and [12]):

$$E_n(x+h) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) h^{n-k}, \quad n \in \mathbb{N}_0.$$

As a particular case of Property 2.11, we have

$$E_{\sigma_1,\ldots,\sigma_n}(z_1+1,\ldots,z_n+1) = \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} E_{k_1,\ldots,k_n}(z_1,\ldots,z_n),$$

and by Property 2.3

$$\sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} E_{k_1,\ldots,k_n}(z_1,\ldots,z_n) + E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \\ = 2 z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}$$

which generalizes, for several hypercomplex variables, the identity obtained by Cheon(see (7) in [5]).

Property 2.13.

$$E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) \\ = \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \binom{\sigma_1}{k_1} \cdots \binom{\sigma_n}{k_n} E_{k_1,\ldots,k_n}(0,\ldots,0) z_1^{\sigma_1-k_1} \times \cdots \times z_n^{\sigma_n-k_n},$$

$\sigma_i \in \mathbb{N}_0, i = 1,\ldots,n$.

Proof.

$$\frac{2}{\exp(t_1,\ldots,t_n)+1} \exp(z_1 t_1,\ldots,z_n t_n) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n}$$

can be written in the form

$$\sum_{|k|=0}^{\infty} \frac{1}{k!} E_{k_1,\ldots,k_n}(0,\ldots,0) t_1^{k_1} \cdots t_n^{k_n} \sum_{|j|=0}^{\infty} \frac{1}{j!} z_1^{j_1} \times \cdots \times z_n^{j_n} t_1^{j_1} \cdots t_n^{j_n} \\ = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) t_1^{\sigma_1} \cdots t_n^{\sigma_n},$$

i.e.

$$\sum_{|\sigma|=0}^{\infty}\left(\sum_{k+j=\sigma}\frac{1}{k!j!}E_{k_1,\ldots,k_n}(0,\ldots,0)z_1^{j_1}\times\cdots\times z_n^{j_n}\right)t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$
$$=\sum_{|\sigma|=0}^{\infty}\frac{1}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)t_1^{\sigma_1}\cdots t_n^{\sigma_n}.$$

Hence,

$$\frac{1}{\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)=\sum_{k+j=\sigma}\frac{1}{k!j!}E_{k_1,\ldots,k_n}(0,\ldots,0)z_1^{j_1}\times\cdots\times z_n^{j_n}$$

which is equivalent to

$$E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$
$$=\sum_{k_1=0}^{\sigma_1}\cdots\sum_{k_n=0}^{\sigma_n}\binom{\sigma_1}{k_1}\cdots\binom{\sigma_n}{k_n}E_{k_1,\ldots,k_n}(0,\cdots,0)z_1^{\sigma_1-k_1}\times\cdots\times z_n^{\sigma_n-k_n}.\quad\square$$

Concerning the derivative of classical Euler polynomials we know that (see, e.g., [1] and [10])

$$E_k'(x)=kE_{k-1}(x),\quad k\in\mathbb{N}.$$

Similarly, we have for our case:

Property 2.14.

$$\frac{\partial}{\partial x_i}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)=\sigma_i E_{\sigma_1,\ldots,\sigma_i-1,\ldots,\sigma_n}(z_1,\ldots,z_n),\quad \sigma_i\in\mathbb{N},\ i\in\mathbb{N}.$$

Proof. Using Property 2.6 and differentiating both sides with respect to x_i we get

$$\frac{\partial}{\partial x_i}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$
$$=\sum_{k_1=0}^{\sigma_1}\cdots\sum_{k_i=0}^{\sigma_i-1}\cdots\sum_{k_n=0}^{\sigma_n}\binom{\sigma_1}{k_1}\cdots\binom{\sigma_i}{k_i}\cdots\binom{\sigma_n}{k_n}(\sigma_i-k_i)\frac{E_{k_1,\ldots,k_n}}{2^{|k|}}$$
$$(z_1-\frac{1}{2})^{\sigma_1-k_1}\times\cdots\times(z_i-\frac{1}{2})^{\sigma_i-k_i-1}\times\cdots\times(z_n-\frac{1}{2})^{\sigma_n-k_n}$$
$$=\sum_{k_1=0}^{\sigma_1}\cdots\sum_{k_i=0}^{\sigma_i-1}\cdots\sum_{k_n=0}^{\sigma_n}\binom{\sigma_1}{k_1}\cdots\binom{\sigma_i-1}{k_i}\cdots\binom{\sigma_n}{k_n}\sigma_i\frac{E_{k_1,\ldots,k_n}}{2^{|k|}}$$
$$(z_1-\frac{1}{2})^{\sigma_1-k_1}\times\cdots\times(z_i-\frac{1}{2})^{\sigma_i-k_i-1}\times\cdots\times(z_n-\frac{1}{2})^{\sigma_n-k_n}$$
$$=\sigma_i E_{\sigma_1,\ldots,\sigma_i-1,\ldots,\sigma_n}(z_1,\ldots,z_n),\quad \sigma_i\in\mathbb{N},$$

which proves the desired identity. □

A relationship between the classical Euler and Bernoulli polynomials (see [1], [12], and [13]) is

$$\begin{aligned} E_{k-1}(x) &= \frac{2^k}{k}\left(B_k\left(\frac{x+1}{2}\right) - B_k\left(\frac{x}{2}\right)\right) \\ &= \frac{2}{k}\left(B_k(x) - 2^k B_k\left(\frac{x}{2}\right)\right), \qquad k \in \mathbb{N} \end{aligned}$$

and has its correspondence in the hypercomplex case expressed by:

Property 2.15.

$$\begin{aligned} &\sum_{k=1}^{n} \sigma_k E_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n) \\ &\quad = 2^{|\sigma|}\left(B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1+1}{2},\ldots,\frac{z_n+1}{2}\right) - B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1}{2},\ldots,\frac{z_n}{2}\right)\right), \end{aligned}$$

$\sigma_i \in \mathbb{N}, \;\; i \in \mathbb{N}$.

Proof. By formula (12)

$$\begin{aligned} &\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!}\Big(B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1+1}{2},\ldots,\frac{z_n+1}{2}\right) - B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1}{2},\ldots,\frac{z_n}{2}\right)\Big)t_1^{\sigma_1}\cdots t_n^{\sigma_n} \\ &\quad = \frac{(t_1+\cdots+t_n)(\exp(\frac{z_1+1}{2}t_1,\ldots,\frac{z_n+1}{2}t_n) - \exp(\frac{z_1}{2}t_1,\ldots,\frac{z_n}{2}t_n))}{\exp(t_1,\ldots,t_n)-1} \\ &\quad = \frac{(t_1+\cdots+t_n)\exp(\frac{z_1}{2}t_1,\ldots,\frac{z_n}{2}t_n)(\exp(\frac{t_1}{2},\ldots,\frac{t_n}{2})-1)}{\exp(t_1,\ldots,t_n)-1} \\ &\quad = \frac{2\exp(z_1\frac{t_1}{2},\ldots,z_n\frac{t_n}{2})}{\exp(\frac{t_1}{2},\ldots,\frac{t_n}{2})+1}\,\frac{t_1+\cdots+t_n}{2} \end{aligned}$$

and, on the other hand, by definition of hypercomplex Euler polynomials we can write

$$\begin{aligned} &\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!}\Big(B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1+1}{2},\ldots,\frac{z_n+1}{2}\right) - B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1}{2},\ldots,\frac{z_n}{2}\right)\Big)t_1^{\sigma_1}\cdots t_n^{\sigma_n} \\ &\quad = \left(\sum_{|\sigma|=0}^{\infty} \frac{1}{2^{|\sigma|}\sigma!}E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)t_1^{\sigma_1}\cdots t_n^{\sigma_n}\right)\frac{t_1+\cdots+t_n}{2} \end{aligned}$$

$$= \sum_{|\sigma|=0}^{\infty} \frac{1}{2^{|\sigma|+1}\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) t_1^{\sigma_1+1} t_2^{\sigma_2} \cdots t_n^{\sigma_n} + \cdots$$

$$+ \sum_{|\sigma|=0}^{\infty} \frac{1}{2^{|\sigma|+1}\sigma!} E_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) t_1^{\sigma_1} t_2^{\sigma_2} \cdots t_n^{\sigma_n+1}$$

$$= \sum_{\substack{|\sigma|=1 \\ (\sigma_1=1,2,\ldots)}}^{\infty} \frac{1}{2^{|\sigma|}(\sigma_1-1)!\cdots\sigma_n!} E_{\sigma_1-1,\sigma_2,\ldots,\sigma_n}(z_1,\ldots,z_n) t_1^{\sigma_1} t_2^{\sigma_2} \cdots t_n^{\sigma_n}$$

$$+ \cdots + \sum_{\substack{|\sigma|=1 \\ (\sigma_n=1,2,\ldots)}}^{\infty} \frac{1}{2^{|\sigma|}\sigma_1!\cdots(\sigma_n-1)!} E_{\sigma_1,\ldots,\sigma_n-1}(z_1,\ldots,z_n) t_1^{\sigma_1} t_2^{\sigma_2} \cdots t_n^{\sigma_n}.$$

Therefore

$$\sum_{\substack{|\sigma|=n \\ (\sigma_1,\ldots,\sigma_n=1,2,\ldots)}}^{\infty} \frac{1}{\sigma!}\Big(B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1+1}{2},\ldots,\frac{z_n+1}{2}\right)$$

$$- B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1}{2},\ldots,\frac{z_n}{2}\right)\Big) t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$

$$= \sum_{\substack{|\sigma|=n \\ (\sigma_1,\ldots,\sigma_n=1,2,\ldots)}}^{\infty} \frac{1}{2^{|\sigma|}\sigma!}\Big(\sigma_1 E_{\sigma_1-1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$

$$+ \cdots + \sigma_n E_{\sigma_1,\ldots,\sigma_n-1}(z_1,\ldots,z_n)\Big) t_1^{\sigma_1}\cdots t_n^{\sigma_n}$$

and, by equating coefficients of $t_1^{\sigma_1}\ldots t_n^{\sigma_n}$, we get

$$\sigma_1 E_{\sigma_1-1,\ldots,\sigma_n}(z_1,\ldots,z_n) + \cdots + \sigma_n E_{\sigma_1,\ldots,\sigma_n-1}(z_1,\ldots,z_n)$$

$$= 2^{|\sigma|}\left(B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1+1}{2},\ldots,\frac{z_n+1}{2}\right) - B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1}{2},\ldots,\frac{z_n}{2}\right)\right). \quad \square$$

The last property can be expressed in the following form:

$$\sum_{k=1}^{n} \sigma_k E_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$

$$= 2\left(B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) - 2^{|\sigma|} B_{\sigma_1,\ldots,\sigma_n}\left(\frac{z_1}{2},\ldots,\frac{z_n}{2}\right)\right),$$

$\sigma_i \in \mathbb{N}$, $i \in \mathbb{N}$, and, in virtue of this, it is possible to deduce

$$\sum_{k=1}^{n} \sigma_k E_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$

$$= \sum_{k_1=0}^{\sigma_1} \cdots \sum_{k_n=0}^{\sigma_n} \left(2 - 2^{|k|+1}\right) B_{k_1,\ldots,k_n} z_1^{\sigma_1-k_1} \times \cdots \times z_n^{\sigma_n-k_n},$$

where $B_{k_1,\ldots,k_n}$ are Bernoulli numbers, which extends the property referred by Cheon for a single variable (see (2) in [5]) to several hypercomplex variables.

Also, Property 2.15 together with Property 2.3, yields the equality, already obtained by the authors in [9]:

$$\Delta B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{k=1}^{n} \sigma_k z_1^{\sigma_1} \times \cdots \times z_k^{\sigma_k-1} \times \cdots \times z_n^{\sigma_n} \tag{19}$$

$\sigma_i \in \mathbb{N}, i \in \mathbb{N}$. Here $\Delta := (T - I)$, where T is the shift operator (15).

Actually, by Properties 2.3 and 2.15

$$\sum_{k=1}^{n} \sigma_k 2(2z_1)^{\sigma_1} \times \cdots \times (2z_k)^{\sigma_k-1} \times \cdots \times (2z_n)^{\sigma_n}$$

$$= \sum_{k=1}^{n} \sigma_k \Big(E_{\sigma_1,\ldots,\sigma_{k-1},\ldots,\sigma_n}(2z_1+1,\ldots,2z_n+1)$$

$$+ E_{\sigma_1,\ldots,\sigma_{k-1},\ldots,\sigma_n}(2z_1,\ldots,2z_n)\Big)$$

$$= 2^{|\sigma|}\Big(B_{\sigma_1,\ldots,\sigma_n}(\frac{2z_1+2}{2},\ldots,\frac{2z_n+2}{2}) - B_{\sigma_1,\ldots,\sigma_n}(\frac{2z_1}{2},\ldots,\frac{2z_n}{2})\Big)$$

$$= 2^{|\sigma|}\Delta B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$

which leads to (19).

3. Hypercomplex Pascal Matrix and Euler Polynomials

In [2] the properties of the so called Pascal matrix are discussed. The paper [14] established a relationship between the Pascal matrix and the corresponding Bernoulli matrix. In [9] this has been generalized to the case of hypercomplex Pascal and hypercomplex Bernoulli matrices.

Our goal in this section is to obtain a corresponding relation between the hypercomplex Pascal matrix and a matrix whose entries are the hypercomplex Euler polynomials.

With this purpose, we use the hypercomplex Pascal matrix given in the form of a $(n+1)\times(n+1)$-block matrix, $\mathcal{P}(z_1,z_2)=[\mathcal{P}_{ij}^{sr}(z_1,z_2)]$, such that

$$\mathcal{P}_{ij}^{sr}(z_1,z_2)=\begin{cases}\binom{i}{j}\binom{s}{r}z_1^{i-j}\times z_2^{s-r} & ,\ i\geq j\wedge s\geq r\\ 0 & ,\ \text{otherwise,}\end{cases}$$

$i,j,s,r=0,\ldots,n$ (cf. [9]).

Definition 3.1. The *hypercomplex polynomial Euler matrix* is the $(n+1)\times(n+1)$-block matrix, $\mathcal{E}(z_1,z_2)=[\mathcal{E}_{ij}^{sr}(z_1,z_2)]$, such that

$$\mathcal{E}_{ij}^{sr}(z_1,z_2)=\begin{cases}\binom{i}{j}\binom{s}{r}E_{i-j,s-r}(z_1,z_2) & ,\ i\geq j\wedge s\geq r\\ 0 & ,\ \text{otherwise,}\end{cases}$$

$i,j,s,r=0,\ldots,n$.

Theorem 3.1. *Let z_1,z_2 be elements of $\mathcal{H}^n$. Then*

$$\mathcal{E}(z_1,z_2)=\mathcal{P}(z_1,z_2)\mathcal{E}(0,0).$$

Proof. It is easy to conclude that

$$\mathcal{E}_{ij}^{sr}(z_1,z_2)=(\mathcal{P}(z_1,z_2)\mathcal{E}(0,0))_{ij}^{sr}=0,$$

if $i<j\vee s<r$.

Now, consider $i\geq j\wedge s\geq r$. Writing $i=j+l$ and $s=r+m$, $l,m\geq 0$,

$$\begin{aligned}(\mathcal{P}(z_1,z_2)\mathcal{E}(0,0))_{ij}^{sr}&=\sum_{k_1=0}^{l}\sum_{k_2=0}^{m}\mathcal{P}_{j+l,j+k_1}^{r+m,r+k_2}(z_1,z_2)\,(\mathcal{E}(0,0))_{j+k_1,j}^{r+k_2,r}\\ &=\binom{j+l}{j}\binom{r+m}{r}\sum_{k_1=0}^{l}\sum_{k_2=0}^{m}\binom{l}{k_1}\binom{m}{k_2}z_1^{l-k_1}\times z_2^{m-k_2}E_{k_1,k_2}(0,0)\\ &=\binom{i}{j}\binom{s}{r}\sum_{k_1=0}^{i-j}\sum_{k_2=0}^{s-r}\binom{i-j}{k_1}\binom{s-r}{k_2}E_{k_1,k_2}(0,0)z_1^{i-j-k_1}\times z_2^{s-r-k_2}.\end{aligned}$$

In view of Property 2.13 holds

$$(\mathcal{P}(z_1,z_2)\mathcal{E}(0,0))_{ij}^{sr}=\binom{i}{j}\binom{s}{r}E_{i-j,s-r}(z_1,z_2)=\mathcal{E}_{ij}^{sr}(z_1,z_2).$$ □

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1965).

[2] L. Aceto, D. Trigiante, The matrices of Pascal and other greats, *Amer. Math. Monthly*, **108**, No. 3 (2001), 232-245.

[3] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman, Boston-London-Melbourne (1982).

[4] G. Bretti, P.E. Ricci, Euler polynomials and the related quadrature rule, *Georgian Math. J.*, **8**, No. 3 (2001), 447-453.

[5] G.-S. Cheon, A note on the Bernoulli and Euler polynomials, *Appl. Math. Lett.*, **16** (2003), 365-368.

[6] H. Malonek, A new hypercomplex structure of the Euclidean space $\mathbb{R}^{m+1}$ and the concept of hypercomplex differentiability, *Complex Variables*, **14** (1990), 25-33.

[7] H. Malonek, Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product, *Complex Variables*, **15** (1990), 181-191.

[8] H. Malonek, Selected topics in hypercomplex function theory, In: *Clifford Algebras and Potential Theory* (Ed-s: Eriksson, Sirkka-Liisa), University of Joensuu, Report Series, **7** (2004), 111-150.

[9] H. Malonek, G. Tomaz, Bernoulli polynomials and Pascal matrices in the context of Clifford Analysis, *Discrete Appl. Math.* (2007), Submited.

[10] L.M. Milne-Thomson, *The Calculus of Finite Differences*, McMillan, New York (1965).

[11] P. Natalini, A. Bernardini, A generalization of the Bernoulli polynomials, *J. of Appl. Math.* (2003), 155-163.

[12] H. Pan, Z.-W. Sun, New identities involving Bernoulli and Euler polynomials, *J. of Combin. Theory*, Series A, **113** (2006), 156-175.

[13] H.M. Srivastava, Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.*, **17** (2004), 375-380.

[14] Z. Zhang, J.Wang, Bernoulli matrix and its algebraic properties, *Discrete Appl. Math.*, **154**, No. 11 (2006), 1622-1632.