International Journal of Pure and Applied Mathematics
Volume 44 No. 3 2008, 467-483

VIABLE MODELS OF TRAVERSABLE WORMHOLES SUPPORTED BY SMALL AMOUNTS OF EXOTIC MATTER

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Abstract: Wormholes allowed by the general theory of relativity that are simultaneously traversable by humanoid travelers are subject to severe constraints from quantum field theory, particularly the so-called quantum inequalities, here slightly extended. Moreover, self-collapse of such wormholes can only be prevented by the use of "exotic matter," which, being rather problematical, should be used in only minimal quantities. However, making the layer of exotic matter arbitrarily thin leads to other problems, such as the need for extreme fine-tuning. This paper discusses a class of wormhole geometries that strike a balance between reducing the proper distance across the exotic region and the degree of fine-tuning required to achieve this reduction. Surprisingly, the degree of fine-tuning appears to be a generic feature of the type of wormhole discussed. No particular restriction is placed on the throat size, even though the proper thickness of the exotic region can indeed be quite small. Various traversability criteria are shown to be met.

AMS Subject Classification: 83C05
Key Words: traversable wormholes, exotic matter

1. Introduction: Viable Wormhole Models at Last

It is well known that traversable wormholes require exotic matter to prevent self-collapse [7]. Such matter is confined to a small region around the throat, a region in which the weak energy condition is violated. While it is desirable

Received: March 24, 2008

to keep this region as small as possible, the use of arbitrarily small amounts of exotic matter leads to severe problems, as discussed by Fewster and Roman [1, 2]. The discovery by Ford and Roman [3, 4] that quantum field theory places severe constraints on the wormhole geometries has shown that most of the “classical” wormholes could not exist on a macroscopic scale. The wormhole described by Kuhfittig [6] is an earlier attempt to strike a balance between two conflicting requirements, reducing the amount of exotic matter and fine-tuning the values of certain parameters. The purpose of this paper is to extend these ideas to much more general models. The quantum inequalities are generalized to be valid, not only at the throat, but in the entire exotic region. The models discussed will therefore (1) satisfy all the constraints imposed by quantum field theory, (2) strike a reasonable balance between a small proper thickness of the exotic region and the degree of fine-tuning of the metric coefficients, equation (1) below, and (3) minimize the assumptions on these metric coefficients. Another key finding is that the degree of fine-tuning is the same for all of the wormholes models considered.

Problems with arbitrarily small amounts of exotic matter are also discussed in reference [8], but the author states explicitly that the issues discussed here and in reference [6] are beyond the scope of his paper.

2. A General Model

Consider the general line element [5]

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where the units are taken to be those for which $G = c = 1$. The function γ is called the redshift function; this function must be everywhere finite to avoid an event horizon at the throat. The function α is related to the shape function $b = b(r)$:

$$e^{2\alpha(r)} = \frac{1}{1 - b(r)/r}$$

(the shape function determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram). It now follows that

$$b(r) = r(1 - e^{-2\alpha(r)}) \tag{2}$$

and that α has a vertical asymptote at the throat $r = r_0$:

$$\lim_{r\to r_0+} \alpha(r) = +\infty.$$

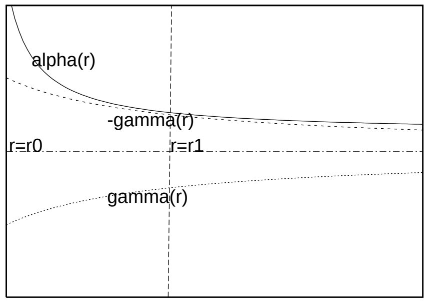

Figure 1: Qualitative features of $\alpha(r)$ and $\gamma(r)$

Also, $\alpha(r) \to 0$ and $\gamma(r) \to 0$ as $r \to \infty$. The qualitative features of $\alpha(r)$, $\gamma(r)$, and $-\gamma(r)$, the reflection of $\gamma(r)$ in the horizontal axis, are shown in Figure 1. It is assumed that α and γ are twice differentiable with $\alpha'(r) < 0$ and $\gamma'(r) > 0$; in addition, $\alpha''(r) > 0$, $\gamma''(r) \leq 0$, and $\alpha''(r) > |\gamma''(r)|$.

The next step is to list the components of the Einstein tensor in the orthonormal frame. From reference [5],

$$G_{\hat{t}\hat{t}} = \frac{2}{r}e^{-2\alpha(r)}\alpha'(r) + \frac{1}{r^2}(1 - e^{-2\alpha(r)}),$$

$$G_{\hat{r}\hat{r}} = \frac{2}{r}e^{-2\alpha(r)}\gamma'(r) - \frac{1}{r^2}(1 - e^{-2\alpha(r)}),$$

and

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = e^{-2\alpha(r)}\left[\gamma''(r) + \alpha'(r)\gamma'(r) \quad +[\gamma'(r)]^2 + \frac{1}{r}\gamma'(r) - \frac{1}{r}\alpha'(r)\right].$$

Since the Einstein field equations $G_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}$ imply that the stress-energy tensor is proportional to the Einstein tensor, the only nonzero components are $T_{\hat{t}\hat{t}} = \rho$, $T_{\hat{r}\hat{r}} = -\tau$, and $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p$. Now recall that the weak energy condition (WEC) requires the mass-energy tensor $T_{\alpha\beta}$ to obey

$$T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$$

for all time-like vectors and, by continuity, all null vectors. Using the radial outgoing null vector $\mu^{\hat{\alpha}} = (1, 1, 0, 0)$, the condition now becomes $T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = \rho - \tau \geq 0$. So if the WEC is violated, then $\rho - \tau < 0$. The field equations $G_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}$ now imply that

$$\rho - \tau = \frac{1}{8\pi}\left[\frac{2}{r}e^{-2\alpha(r)}\left[\alpha'(r) + \gamma'(r)\right]\right]. \tag{3}$$

Sufficiently close to the asymptote, $\alpha'(r) + \gamma'(r)$ is clearly negative (recall that

$\alpha' < 0$ and $\gamma' > 0$). To satisfy the Ford-Roman constraints [3, 4], we would like the WEC to be satisfied outside of some small interval $[r_0, r_1]$. In other words,

$$|\alpha'(r_1)| = \gamma'(r_1), \tag{4}$$

$$\alpha'(r) + \gamma'(r) < 0 \quad \text{for} \quad r_0 < r < r_1, \tag{5}$$

and

$$\alpha'(r) + \gamma'(r) > 0 \quad \text{for} \quad r > r_1 \tag{6}$$

(see Figure 1).

3. Other Constraints

Before discussing additional constraints, we need to list some of the components of the Riemann curvature tensor in the orthonormal frame. From reference [5]

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = e^{-2\alpha(r)}\left(\gamma''(r) - \alpha'(r)\gamma'(r) + \left[\gamma'(r)\right]^2\right), \tag{7}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = \frac{1}{r}e^{-2\alpha(r)}\gamma'(r), \tag{8}$$

and

$$R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = \frac{1}{r}e^{-2\alpha(r)}\alpha'(r). \tag{9}$$

Much of what follows is based on the discussion in reference [7]. In particular, we have for the radial tidal constraint

$$\left|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}\right| = \left|R_{\hat{r}\hat{t}\hat{r}\hat{t}}\right| = e^{-2\alpha(r)}\left|\gamma''(r) - \alpha'(r)\gamma'(r) + \left[\gamma'(r)\right]^2\right| \leq (10^8\,\text{m})^{-2}. \tag{10}$$

The lateral tidal constraints are (reinserting c)

$$\begin{aligned}\left|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}\right| &= \left|R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}\right| = \gamma^2\left|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}\right| + \gamma^2\left(\frac{v}{c}\right)^2\left|R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}}\right| \\ &= \gamma^2\left(\frac{1}{r}e^{-2\alpha(r)}\gamma'(r)\right) + \gamma^2\left(\frac{v}{c}\right)^2\left(\frac{1}{r}e^{-2\alpha(r)}\alpha'(r)\right) \leq (10^8\,\text{m})^{-2};\end{aligned} \tag{11}$$

here $\gamma^2 = 1/\left[1 - (v/c)^2\right]$.

As already noted, wormhole solutions allowed by general relativity may be subject to severe constraints from quantum field theory. Of particular interest to us is equation (95) in reference [4] (returning to geometrized units):

$$\frac{r_m}{r_0} \leq \left(\frac{1}{v^2 - b_0'}\right)^{1/4}\frac{\sqrt{\gamma}}{f}\left(\frac{l_p}{r_0}\right)^{1/2}, \tag{12}$$

where r_m is the smallest of several length scales, v is the velocity of a boosted observer relative to the static frame, $\gamma = 1/\sqrt{1 - v^2}$, l_p is the Planck length,

f is a small scale factor, and $b'_0 = b'(r_0)$. Inequality (12) is trivially satisfied whenever $b'(r_0) \approx 1$. Accordingly, we will impose this condition to help satisfy the tidal constraints and to obtain some of the numerical estimates. The fact remains, however, that according to reference [1], since the WEC is violated arbitrarily close to the throat, the analysis should be extended, but to do so would require additional information about the redshift and shape functions. We will therefore return to this problem in Section 8.

Returning to equation (2), we have for the shape function,

$$b'(r_0) = \frac{d}{dr}\left[r(1 - e^{-2\alpha(r)})\right]_{r=r_0} = 2r_0 e^{-2\alpha(r_0)}\alpha'(r_0) + 1 - e^{-2\alpha(r_0)}. \qquad (13)$$

To obtain $b'(r_0) = 1$, we require that

$$\lim_{r\to r_0} e^{-2\alpha(r)}\alpha'(r) = 0.$$

As a consequence, the radial tidal constraint (10) is satisfied at the throat, while the lateral tidal constraints (11) merely constrain the velocity of the traveler in the vicinity of the throat.

The condition $b'(r_0) \approx 1$ has two other consequences: since the inequality (12) is satisfied, there is no particular restriction on the size r_0 of the radius of the throat. On the other hand, $b'(r_0) \approx 1$ implies that the wormhole will flare out very slowly, so that the coordinate distance from $r = r_0$ to $r = r_1$ will be much less than the proper distance (this behavior can be seen from Figure 2).

4. The Exotic Region

We saw in the last section that α has to go to infinity fast enough so that $\lim_{r\to r_0} e^{-2\alpha(r)}\alpha'(r) = 0$. At the same time, α has to go to infinity slowly enough so that the proper distance

$$\ell(r) = \int_{r_0}^{r} e^{\alpha(r')}dr'$$

is finite. Then by the mean-value theorem, there exists a value $r = r_2$ such that

$$\ell(r) = e^{\alpha(r_2)}(r - r_0), \quad r_0 < r_2 < r.$$

In particular, $\ell(r_0) = 0$ and

$$\ell(r_1) = e^{\alpha(r_2)}(r_1 - r_0). \qquad (14)$$

With this information we can examine the radial tidal constraint at $r = r_1$. From equation (7)

$$|R_{\hat{r}\hat{t}\hat{r}\hat{t}}| = e^{-2\alpha(r_1)}\left|\gamma''(r_1) - \alpha'(r_1)\gamma'(r_1) + \left[\gamma'(r_1)\right]^2\right|$$

$$= e^{-2\alpha(r_1)} \left|\gamma''(r_1) - \alpha'(r_1)[-\alpha'(r_1)] + \left[\alpha'(r_1)\right]^2\right|$$

by equation (4). So by inequality (10),

$$|R_{\hat{r}\hat{t}\hat{r}\hat{t}}| = e^{-2\alpha(r_1)} \left|\gamma''(r_1) + \alpha'(r_1)\alpha'(r_1) + \left[\alpha'(r_1)\right]^2\right| \leq (10^8\text{m})^{-2}.$$

Since $e^{-2\alpha(r)}$ is strictly increasing, it follows that

$$e^{-2\alpha(r_2)} \left|\gamma''(r_1) + 2\left[\alpha'(r_1)\right]^2\right| < 10^{-16}\text{m}^{-2}.$$

From equation (14), we now get the following:

$$\frac{(r_1 - r_0)^2}{[\ell(r_1)]^2} \left|\gamma''(r_1) + 2\left[\alpha'(r_1)\right]^2\right| < 10^{-16}\text{m}^{-2}$$

and

$$\left|\gamma''(r_1) + 2\left[\alpha'(r_1)\right]^2\right| < \frac{[\ell(r_1)]^2}{10^{16}(r_1 - r_0)^2}. \tag{15}$$

As a consequence,

$$\gamma''(r_1) + 2\left[\alpha'(r_1)\right]^2 < \frac{[\ell(r_1)]^2}{10^{16}(r_1 - r_0)^2} \tag{16}$$

or

$$\gamma''(r_1) + 2\left[\alpha'(r_1)\right]^2 > -\frac{[\ell(r_1)]^2}{10^{16}(r_1 - r_0)^2}. \tag{17}$$

So if either condition (16) or condition (17) is satisfied, then so is condition (15).

To estimate the size of the exotic region, we need some idea of the magnitude of $\alpha(r_1)$ and $\alpha'(r_1)$. The only information available is that $\alpha(r)$ increases slowly enough to keep $\int_{r_0}^{r} e^{\alpha(r')}dr'$ finite. One way to accomplish this is to assume that for computational purposes $\alpha(r)$ is roughly logarithmic, at least near $r = r_1$, as in reference [6]:

$$\alpha(r) \sim \ln\frac{K}{(r - r_0)^A}, \quad 0 < A < 1. \tag{18}$$

While this form of $\alpha(r)$ may be just a computational convenience, there is no guarantee that $\alpha(r)$ is drastically different from the ln-form, so that one must proceed with caution. In particular, we now have to assume that

$$\alpha'(r_1) \sim -\frac{A}{r_1 - r_0} \quad \text{and} \quad \alpha''(r_1) \sim \frac{A}{(r_1 - r_0)^2}. \tag{19}$$

Since we also want $\alpha''(r) > |\gamma''(r)|$ [or $\alpha''(r) > -\gamma''(r)$], we have in view of inequality (16),

$$\frac{A}{(r_1 - r_0)^2} > -\gamma''(r_1) > \frac{2A^2}{(r_1 - r_0)^2} - \frac{[\ell(r_1)]^2}{10^{16}(r_1 - r_0)^2}. \tag{20}$$

Conversely, the inequality

$$\frac{2A^2}{(r_1-r_0)^2}-\frac{[\ell(r_1)]^2}{10^{16}(r_1-r_0)^2}=2[\alpha'(r_1)]^2-\frac{[\ell(r_1)]^2}{10^{16}(r_1-r_0)^2}<-\gamma''(r_1)$$

implies condition (16). Since $-\gamma''(r)<\alpha''(r)$, we conclude that inequality (20) is valid if, and only if, condition (16) is met.

Inequality (20) now implies that

$$2A^2-A-\frac{[\ell(r_1)]^2}{10^{16}}<0. \tag{21}$$

Equality is achieved whenever

$$A=\frac{1\pm\sqrt{1+\frac{8[\ell(r_1)]^2}{10^{16}}}}{4}.$$

Returning to the condition $b'(r_0)=1$ for a moment, if the ln-approximation is used again, then A must exceed 1/2. It follows that

$$\frac{1}{2}<A<\frac{1+\sqrt{1+\frac{8[\ell(r_1)]^2}{10^{16}}}}{4}. \tag{22}$$

This solution shows that considerable fine-tuning is required. We will return to this point in Section 7.

Finally, observe that with the extra condition $|\gamma''(r_1)|<\alpha''(r_1)$, the qualitative features in Figure 1 are retained, so that no additional assumptions are needed.

Returning to equation (18), the form of $\alpha(r)$ yields the following estimate for $\ell(r)$:

$$\ell(r)\sim\int_{r_0}^{r}e^{\ln[K/(r'-r_0)^A]}dr'=\frac{K}{1-A}(r-r_0)^{1-A},\quad 0<A<1 \tag{23}$$

(see Figure 2). When $A\approx 1/2$, then $\ell(r)\approx 2K(r-r_0)^{1/2}$. We can see from the figure that $\ell(r_1)$ is much larger than r_1-r_0 near the throat, a consequence of the slow flaring out.

5. Numerical Estimates

This section is devoted to numerical calculations. As in reference [7], we start with an estimate of the size of the wormhole, as measured by the position of the space station. According to reference [7], the space station should be far

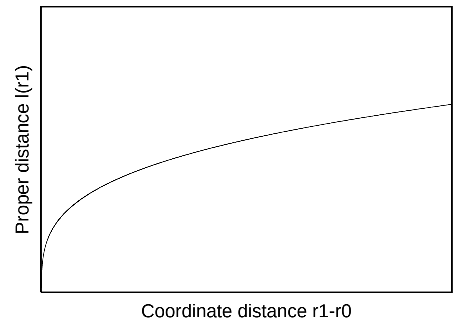

Figure 2: Graph showing the proper thickness $\ell(r_1)$ as a function of the coordinate distance $r_1 - r_0$.

enough away from the throat so that

$$1 - \frac{b(r)}{r} = e^{-2\alpha(r)} \approx 1, \tag{24}$$

making the space nearly flat. Another condition involves the redshift function: at the station we must also have

$$|\gamma'(r)| \leq g_{\oplus} / \left(c^2 \sqrt{1 - b(r)/r} \right). \tag{25}$$

It will be seen below that for our wormhole the first condition, equation (24), is easily satisfied. By condition (6), as well as Figure 1, $|\alpha'(r)| < \gamma'(r)$ for $r > r_1$. So if $1 - b(r)/r \approx 1$, then we have

$$|\alpha'(r)| < 10^{-16}\,\mathrm{m}^{-1} \tag{26}$$

at the station. This inequality should give us at least a rough estimate of the distance to the station, since for large r, $\alpha(r) \sim \gamma(r)$. It must be kept in mind, however, that the inequality $|\alpha'(r)| < \gamma'(r)$ implies that this procedure does underestimate the distance, perhaps by quite a bit. The main reason for using α in the first place is to avoid making additional assumptions involving γ. Instead, γ can be left to its more obvious role, adjusted if necessary, to help meet the tidal constraint in equation (10) for $r > r_1$. We will return to this point after discussing α (very close to the throat the redshift function may also have to be fine-tuned to help meet the quantum inequalities, as will be seen in Section 8).

To make use of condition (26), we will again assume that $\alpha(r)$ is similar to $\ln[K/(r - r_0)^A]$ near $r = r_1$, to be denoted by $\alpha_{\text{left}}(r)$. To ensure asymptotic flatness, this function will be joined smoothly at some $r = r_3 > r_1$ to a function

$\alpha_{\text{right}}(r)$, assumed to have the form $\alpha_{\text{right}}(r) = C/(r-r_0)^n$. Thus

$$\alpha_{\text{left}}(r_3) = \alpha_{\text{right}}(r_3) \quad \text{and} \quad \alpha'_{\text{left}}(r_3) = \alpha'_{\text{right}}(r_3).$$

From

$$\alpha'_{\text{left}}(r_3) = -\frac{A}{r_3 - r_0} = -\frac{nC}{(r_3-r_0)^{n+1}} = \alpha'_{\text{right}}(r_3)$$

we obtain $C = (A/n)(r_3-r_0)^n$. Thus

$$\alpha_{\text{right}}(r) = \frac{(A/n)(r_3-r_0)^n}{(r-r_0)^n}. \tag{27}$$

From $\alpha_{\text{left}}(r_3) = \alpha_{\text{right}}(r_3)$, we get

$$e^{\ln[K/(r_3-r_0)^A]} = e^{[(A/n)(r_3-r_0)^n]/(r_3-r_0)^n} = e^{A/n}$$

or

$$K = e^{A/n}(r_3-r_0)^A. \tag{28}$$

The desired distance $r = r_s$ to the space station can now be estimated using equation (27):

$$\left|\alpha'_{\text{right}}(r_s)\right| = \frac{A(r_3-r_0)^n}{(r_s-r_0)^{n+1}} = 10^{-16}\,\text{m}^{-1},$$

which implies that (since $r_s - r_0 \approx r_s$)

$$r_s \approx \left[10^{16}A(r_3-r_0)^n\right]^{1/(n+1)}. \tag{29}$$

For convenience we now restate equation (23) for $r = r_1$:

$$\ell(r_1) \approx \frac{K}{1-A}(r_1-r_0)^{1-A}, \tag{30}$$

where $K = e^{A/n}(r_3-r_0)^A$.

Given the resulting infinite number of solutions, how should the various parameters be chosen? We know that the wormhole flares out very slowly at the throat, which suggests assigning a small coordinate distance to the exotic region, at least initially. A good choice is $r_1 - r_0 = 0.000001\,\text{m}$, as in reference [6]. The distance $r_3 - r_0$ can be much larger; so to fix ideas, we arbitrarily choose $r_3 - r_0 = 1\,\text{mm} = 0.001\,\text{m}$. Using equations (29) and (30) with $A = 1/2$ for the calculations, the values of $\ell(r_1)$ and r_s for various choices of n are given in the accompanying table.

While our choices are necessarily somewhat arbitrary and the calculated values only approximate, the table allows a conservative estimate for both the proper thickness of the exotic region and the size of the wormhole, as measured by r_s, the distance to the space station. Judging from the middle of the table, the proper thickness of the exotic region is only about 0.1 mm; r_s need not be

	A=0.50
n=0.60	0.0146 cm 490 000 km
n=0.65	0.0136 cm 215 000 km
n=0.70	0.0129 cm 100 000 km
n=0.75	0.0123 cm 48 000 km
n=0.80	0.0118 cm 24 000 km
n=0.85	0.0114 cm 13 000 km

Table 1: The top and bottom values in each cell are $\ell(r_1)$ and r_s, respectively

more than about 100 000 km — and could even be much less.

Returning to the radial tidal constraint, based on experience with specific functions (as in reference [5]), $|R_{\hat{r}\hat{t}\hat{r}\hat{t}}|$ is likely to reach its peak just to the right of $r = r_1$. The simplest way to handle this problem is to tighten the constraint in equation (10) at $r = r_1$ by reducing the right side. This change increases the degree of fine-tuning in condition (22).

A final consideration is the time dilation near the throat. Denoting the proper distance by ℓ and the proper time by τ, as usual, we let $v = d\ell/d\tau$, so that $d\tau = d\ell/v$, assuming now that $\gamma \approx 1$. Since $d\ell = e^{\alpha(r)}dr$ and $d\tau = e^{\gamma(r)}dt$, we have for any coordinate interval Δt:

$$\Delta t = \int_{t_a}^{t_b} dt = \int_{\ell_a}^{\ell_b} e^{-\gamma(r)}\frac{d\ell}{v} = \int_{r_a}^{r_b} \frac{1}{v}e^{-\gamma(r)}e^{\alpha(r)}dr.$$

If we assume, once again, that $\alpha(r) \sim \ln[K/(r-r_0)^A]$, then on the interval $[r_0, r_1]$,

$$\Delta t = \int_{r_0}^{r_1} \frac{1}{v}e^{-\gamma(r)}\frac{K}{(r-r_0)^A}dr.$$

Since $\gamma(r)$ is finite, the small size of the interval $[r_0, r_1]$ implies that Δt is relatively small.

6. Additional Models

We assumed in the previous section that for computational purposes, $\alpha(r)$ is roughly logarithmic. In this section we consider a more complicated class of functions for α:

$$\alpha(r) = a \ln \left(\frac{1}{(r-r_0)^b} + \sqrt{\frac{1}{(r-r_0)^{nb}} + 1} \right). \tag{31}$$

The main advantage of this model is that $\alpha(r) \to 0$ as $r \to \infty$, so that no modification is needed. For now we will concentrate on the special case $n = 2$ and return to equation (31) later. For $n = 2$, the equation becomes

$$\alpha(r) = a \sinh^{-1} \frac{1}{(r-r_0)^b}, \quad b > \frac{1}{2a}.$$

The need for the assumption $b > 1/(2a)$ comes from the shape function

$$b(r) = r \left(1 - e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \right) :$$

$$b'(r) = 1 - e^{-2a \sinh^{-1}[1/(r-r_0)^b]} + r \left(-e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \right) \frac{2ab}{(r-r_0)\sqrt{(r-r_0)^{2b}+1}};$$

$b'(r) \to 1$ as $r \to r_0$, as long as $b > 1/(2a)$. To see this, it is sufficient to examine

$$e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \frac{1}{r-r_0}$$

as $r \to r_0$:

$$\frac{1}{\left[\frac{1}{(r-r_0)^b} + \sqrt{\frac{1}{(r-r_0)^{2b}} + 1} \right]^{2a}} \frac{1}{r-r_0} = \frac{1}{\frac{1}{(r-r_0)^{2ab}} \left[1 + (r-r_0)^b \sqrt{\frac{1}{(r-r_0)^{2b}} + 1} \right]^{2a}}$$

$$\times \frac{1}{r-r_0} = \frac{1}{\frac{1}{(r-r_0)^{2ab-1}} \left[1 + \sqrt{1 + (r-r_0)^{2b}} \right]^{2a}}$$

$$\sim (r-r_0)^{2ab-1}, \text{ whence } 2ab - 1 > 0.$$

For computational purposes, however, we will simply let $b = 1/(2a)$. Consider next,

$$\alpha'(r) = -\frac{ab}{(r-r_0)\sqrt{(r-r_0)^{2b}+1}}, \quad r > r_0,$$

and

$$\alpha''(r) = \frac{ab\left[(1+b)(r-r_0)^{2b} + 1 \right]}{(r-r_0)^2 \left[(r-r_0)^{2b} + 1 \right]^{3/2}}.$$

Given that $r_1 - r_0 = 0.000001\,\text{m}$ from Section 5, we get

$$\alpha'(r_1) \approx -\frac{ab}{r_1 - r_0}$$

and

$$\alpha''(r_1) \approx \frac{ab}{(r_1 - r_0)^2}.$$

Comparing these results to equation (19), we conclude, in view of inequality (16) and $|\gamma''(r)| < \alpha''(r)$, that ab is subject to exactly the same fine-tuning as A in inequality (22):

$$\frac{1}{2} < ab < \frac{1 + \sqrt{1 + \frac{8[\ell(r_1)]^2}{10^{16}}}}{4}. \tag{32}$$

The left inequality confirms that $b > 1/(2a)$.

Letting $b = 1/(2a)$, we now have

$$\ell(r_1) = \int_{r_0}^{r_0+0.000001} e^{a \sinh^{-1}[1/(r-r_0)^{1/(2a)}]}\,dr.$$

These values change very little with a. For example, if a ranges from 0.1 to 0.5, then $\ell(r_1)$ ranges from 0.0021 m to 0.0028 m. These values are larger than our previous values, unless we reduce the coordinate distance. Thus for $r_1 - r_0 = 0.000000001\,\text{m}$ and $a = 0.5$, we get $\ell(r_1) = 0.000089\,\text{m} < 0.1\,\text{mm}$, corresponding to $r_s \approx 70\ 000$ km .

A good alternative is to use equation (31), subject to the condition

$$nab - b + \frac{1}{2}nb > 1$$

(as before, this condition comes from the requirement that $b'(r_0) = 1$; in fact, if $n = 2$, we are back to $2ab > 1$). For example, retaining $r_1 - r_0 = 0.000001\,\text{m}$, if $a = 0.2$ and $b = 1$, then $nb = 2.857$. These values yield $\ell(r_1) \approx 0.0000725\,\text{m} < 0.1\,\text{mm}$. The corresponding distance r_s, obtained from $\alpha'(r)$ (now referring to equation (31)), is about 45 000 km.

Using the equation $nab - b + \frac{1}{2}nb = 1$ to eliminate n in equation (31) shows that further reductions in $\ell(r_1)$ are only significant if a and b get unrealistically close to zero. So practically speaking, a further reduction in the proper distance $\ell(r_1)$ requires a reduction in the coordinate distance $r_1 - r_0$.

7. The Fine-Tuning Problem in General

The almost identical inequalities (22) and (32) suggest that the degree of fine-tuning encountered is a general property of the type of wormhole being considered, namely wormholes for which $b'(r_0) = 1$ and $\alpha(r) = a\ln f(r - r_0)$, where (generalizing from earlier cases) $f(r-r_0)|_{r=r_0}$ is undefined ($+\infty$) and $f(\frac{1}{r-r_0})|_{r=r_0}$ is a constant (possible zero). If we also assume that $g(r - r_0) = f(\frac{1}{r-r_0})$ can be expanded in a Maclaurin series, then we have for $r \approx r_0$,

$$f\left(\frac{1}{r-r_0}\right) = g(r-r_0) = a_0 + a_1(r-r_0) + a_2(r-r_0)^2 + \cdots \approx a_0 + a_1(r-r_0).$$

It follows that

$$f(r-r_0) = a_0 + \frac{a_1}{r-r_0}$$

near the throat. So

$$\alpha(r) = a\ln\left(a_0 + \frac{a_1}{r-r_0}\right),$$

$$\alpha'(r_1) = \frac{-aa_1}{a_0 + \frac{a_1}{r_1-r_0}}\frac{1}{(r_1-r_0)^2} \sim -\frac{a}{r_1-r_0}, \tag{33}$$

and

$$\alpha''(r_1) = \frac{aa_1[2a_0(r_1-r_0)+a_1]}{[a_0(r_1-r_0)^2 + a_1(r_1-r_0)]^2} \sim \frac{a}{(r_1-r_0)^2}. \tag{34}$$

To show that $b'(r_0) = 1$, we need to show that $e^{-2\alpha(r)}\alpha'(r) \to 0$ as $r \to r_0$:

$$e^{-2a\ln[a_0+a_1/(r-r_0)]}\frac{-aa_1}{a_0+\frac{a_1}{r-r_0}}\frac{1}{(r-r_0)^2} = \frac{1}{\left(a_0+\frac{a_1}{r-r_0}\right)^{2a}}\frac{-aa_1}{a_0+\frac{a_1}{r-r_0}}\frac{1}{(r-r_0)^2}.$$

Since a_0 is negligible if r is close to r_0, we obtain

$$e^{-2\alpha(r)}\alpha'(r) \sim (r-r_0)^{2a-1},$$

so that $2a - 1 > 0$ and $a > \frac{1}{2}$. Comparing equations (33) and (34) to equation (19), we conclude that

$$\frac{1}{2} < a < \frac{1+\sqrt{1+\frac{8[\ell(r_1)]^2}{10^{16}}}}{4}. \tag{35}$$

So the amount of fine-tuning required really does appear to be a general property of wormholes of the present type. While the degree of fine-tuning considered so far is quite severe, it is considerably milder than most of the cases discussed in reference [1].

As indicated at the end of Section 6, if equation (31) is used in the model, any further reduction in $\ell(r_1)$ requires a reduction in the coordinate distance $r_1 - r_0$. We can see from condition (35), however, that reducing $\ell(r_1)$ will increase the degree of fine-tuning. While basically presenting us with an engineering challenge, this increase can only be carried so far. In particular, we are confirming the assertion in reference [1] that the amount of exotic matter cannot be made arbitrarily small.

8. The Quantum Inequalities Near the Throat

We know that inequality (12) is trivially satisfied as long as $b'(r_0) \approx 1$. The purpose of this section is to derive an analogous inequality for r close to r_0.

The analysis in reference [4] is based on the inequality

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^4}, \tag{36}$$

where τ is the observer's proper time and τ_0 the duration of the sampling time (see reference [4] for details). Put another way, the energy density is sampled in a time interval of duration τ_0 which is centered around an arbitrary point on the observer's worldline so chosen that $\tau = 0$ at this point. The sampling time itself is usually assumed to be so short that the energy density does not change very much over this time interval and may therefore be taken to be approximately constant:

$$\begin{aligned}\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2} &\approx \langle T_{\mu\nu}u^\mu u^\nu\rangle\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{d\tau}{\tau^2+\tau_0^2}\\ &= \langle T_{\mu\nu}u^\mu u^\nu\rangle = \rho \geq -\frac{3}{32\pi^2\tau_0^4}.\end{aligned} \tag{37}$$

To obtain a condition analogous to inequality (12), it is convenient to use the forms for $T_{\hat{t}\hat{t}}$ and $T_{\hat{r}\hat{r}}$ in reference [4]:

$$T_{\hat{t}\hat{t}} = \rho = \frac{b'(r)}{8\pi r^2} \tag{38}$$

and

$$T_{\hat{r}\hat{r}} = -\tau = p_r = -\frac{1}{8\pi}\left[\frac{b(r)}{r^3} - \frac{2\gamma'(r)}{r}\left(1-\frac{b(r)}{r}\right)\right]. \tag{39}$$

Away from the throat we have the analogous formula for r_m:

$$r_m \equiv \min\left[b(r), \left|\frac{b(r)}{b'(r)}\right|, \frac{1}{|\gamma'(r)|}, \left|\frac{\gamma'(r)}{\gamma''(r)}\right|\right]. \tag{40}$$

Finally, still following reference [4], the energy density in the "boosted frame" is by the Lorentz transformation

$$T_{\hat{0}'\hat{0}'} = \rho' = \gamma^2(\rho + v^2 p_r), \tag{41}$$

where $\gamma = 1/\sqrt{1-v^2}$ (it is stated in reference [4] that in this frame the energy density does not change very much over the sampling time, so that $\rho' \geq -3/(32\pi^2\tau_0^4)$). Substitution yields

$$\rho' = \frac{\gamma^2}{8\pi r^2}\left[b'(r) - v^2\frac{b(r)}{r} + v^2 r(2\gamma'(r))\left(1 - \frac{b(r)}{r}\right)\right].$$

From

$$\frac{3}{32\pi^2\tau_0^4} \geq -\rho'$$

we now have

$$\frac{32\pi^2\tau_0^4}{3} \leq \frac{8\pi r^2}{\gamma^2}\left[v^2\frac{b(r)}{r} - b'(r) - v^2 r(2\gamma'(r))\left(1 - \frac{b(r)}{r}\right)\right]^{-1}.$$

The suggested sampling time is

$$\tau_0 = \frac{f r_m}{\gamma},$$

where f is a scale factor such that $f \ll 1$. After dividing both sides by r^4, we have (disregarding a small coefficient)

$$\frac{f^4 r_m^4}{r^4\gamma^4} \leq \frac{1}{r^2\gamma^2}\left[v^2\frac{b(r)}{r} - b'(r) - 2v^2 r\gamma'(r)\left(1 - \frac{b(r)}{r}\right)\right]^{-1}$$

and, after inserting l_p,

$$\frac{r_m}{r} \leq \left(\frac{1}{v^2\frac{b(r)}{r} - b'(r) - 2v^2\gamma'(r)\frac{r}{l_p}\left(1 - \frac{b(r)}{r}\right)}\right)^{1/4} \times \frac{\sqrt{\gamma}}{f}\left(\frac{l_p}{r}\right)^{1/2}. \tag{42}$$

At the throat, where $b(r_0) = r_0$, inequality (42) reduces to inequality (12).

It is understood that in inequality (42), r is close to r_0, so that $b'(r)$ is close to unity. But since $b'(r) < 1$, inequality (12) is not necessarily satisfied for $r > r_0$. Inequality (42), however, $\gamma'(r)$ can be fine-tuned so that the condition is satisfied in the interval $[r_0, r_1]$.

9. Conclusion

This paper discusses a class of wormhole geometries that, finally, satisfy the constraints from quantum field theory, while striking a balance between reducing the proper thickness of the exotic region as much as possible, while trying to keep the fine-tuning requirement within reasonable bounds. The assumptions on the metric coefficients in line element (1) have been kept to a minimum. An unexpected finding is that the degree of fine-tuning is a generic property of the type of wormhole discussed.

The wormholes are macroscopic and satisfy various traversability criteria.

There are many possible choices for the parameters and hence many solutions. The particular choices discussed are fairly conservative, leading to the following promising results: approximately 0.1 mm for the proper thickness of the exotic region, corresponding roughly to a distance of 100 000 km to the space station, possibly much less. The proper thickness of 0.1 mm should not be viewed as the final outcome, however. By decreasing the coordinate distance, it is theoretically possible to decrease the thickness of the exotic region indefinitely. While this decrease may be thought of as an engineering problem, the fact remains that the concomitant increase in the degree of fine-tuning would eventually exceed any practical limit.

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