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ISOGONALITY AND INVERSION IN AN ISOTROPIC PLANE

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Abstract: The isogonality with respect to the triangle and inversion with respect to the circle will be defined in an isotropic plane. The images of some lines and points with respect to these mappings will be studied.

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1. Introduction

Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, into the so-called *standard position*, i.e. that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$,

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$B = (b, b^2)$, $C = (c, c^2)$, where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ the equalities $q = bc - a^2$, $q = -(b^2 + bc + c^2)$ are valid (see [2]).

2. Isogonality in an Isotropic Plane

Theorem 1. *Let $T = (x, y)$ and $T' = (x', y')$ be such points which do not lie on the lines BC, CA, AB. The pairs of the lines AT, AT'; BT, BT'; CT, CT' have for bisectors the bisectors of the angles A, B, C of the triangle ABC if and only if the equalities*

$$yx' + xy' + p = 0, \quad xx' + y + y' + q = 0 \tag{1}$$

are valid.

Proof. The lines AT and AT' have the slopes

$$k = \frac{y - a^2}{x - a}, \quad k' = \frac{y' - a^2}{x' - a}, \tag{2}$$

and the lines AC and AB have the slopes $-b$ and $-c$ with the sum a. Therefore the pairs of the lines AT, AT' and AC, AB have the same angle bisector under the condition $k + k' = a$, which has the form

$$(y - a^2)(x' - a) + (y' - a^2)(x - a) - a(x - a)(x' - a) = 0$$

because of (2), i.e. because of

$$a^3 = a(bc - q) = p - aq$$

it gets the form

$$-axx' + yx' + xy' - a(y + y') + p - aq = 0.$$

If we set

$$P = yx' + xy' + p, \quad Q = xx' + y + y' + q,$$

then our condition becomes the first of the three analogous equalities

$$P - aQ = 0, \quad P - bQ = 0, \quad P - cQ = 0, \tag{3}$$

while the remaining two equalities (3) are the analogous conditions for the angles B and C. The equalities (3) are obviously equivalent to the equalities $P = 0$ and $Q = 0$, and they are the equalities (1). □

The points T and T' with the properties from Theorem 1 will be called *isogonal points* with respect to the triangle ABC.

The equalities (1) understandable as the equations for x' and y' have the solutions

$$x' = \frac{xy + qx - p}{y - x^2}, \quad y' = \frac{px - qy - y^2}{y - x^2}. \tag{4}$$

The mapping $T \to T'$, where with $T = (x, y)$ the point $T' = (x', y')$ is given by the formulae (4), will be called *isogonality* with respect to the triangle ABC. Thus we have proved:

Theorem 2. *Isogonality with respect to the triangle ABC in standard position is given by the formulae (4).*

The isogonality is an involutory mapping.

If $x = a$ and $y = a^2$ we get $y - x^2 = 0$ and

$$\begin{aligned} xy + qx - p = a^3 + aq - abc = a(a^2 + q - bc) &= 0, \\ px - qy - y^2 = ap - a^2q - a^4 = a^2(bc - q - a^2) &= 0\,, \end{aligned}$$

so the point isogonal to the point A is not defined, and the same is valid for the points B and C.

Theorem 3. *The mapping $T \to T'$ from Theorem 1 maps any point on the line BC (except B and C) to the point A. Analogously this is valid for the lines CA and AB.*

Proof. According to [2] the line BC has the equation $y = -ax - bc$, and with this relationship of the coordinates it follows

$$\begin{aligned} y - x^2 &= -(x^2 + ax + bc), \\ xy + qx - p &= -ax^2 - bcx + qx - p = -ax^2 - a^2x - p = -a(x^2 + ax + bc), \\ px - qy - y^2 &= px + aqx + bcq - a^2x^2 - 2abcx - b^2c^2 \\ &= -a^2x^2 - a(bc - q)x - bc(bc - q) \\ &= -a^2x^2 - a^3x - a^2bc = -a^2(x^2 + ax + bc), \end{aligned}$$

and because of that in this case from (1) we get $x' = a,\ y' = a^2$. □

If we exclude the points on the lines BC, CA, AB, then the isogonality is a bijection.

From (1) it follows immediately that isogonality maps the points on the circumscribed circle to the points at infinity. More precisely, if $y = x^2$ then it follows

$$\frac{px - qy - y^2}{xy + qx - p} = \frac{px - qx^2 - x^4}{x^3 + qx - p} = -x,$$

i.e. the following theorem is valid.

Theorem 4. *Isogonality with respect to the triangle ABC in the standard position maps the point (x, x^2), on its circumscribed circle, to the point at infinity of the line with the slope $-x$.*

From Theorem 4 it follows that the point $(0, 0)$ is isogonal to the point at infinity of the lines with the slope 0, and they are for example: orthic axis, inertial axis and the Feuerbach line of the triangle ABC (see [1] and [2]).

Theorem 5. *Isogonality with respect to the triangle ABC maps any line to a circumscribed conic of that triangle.*

Proof. By the substitutions

$$x \to \frac{xy + qx - p}{y - x^2}, \quad y \to \frac{px - qy - y^2}{y - x^2}, \tag{5}$$

the equation of the line $y = kx + l$ changes into the equation

$$px - qy - y^2 = k(xy + qx - p) + l(y - x^2),$$

i.e. the equation of the conic

$$lx^2 - kxy - y^2 + (p - kq)x - (q + l)y + kp = 0. \tag{6}$$

Let us find the intersections of that conic with the circumscribed circle of the triangle ABC. If we put x^2 in (6) instead of y, then we get the equation of x

$$x^4 + kx^3 + qx^2 + kqx - px - kp = 0$$

which can be written in the form $(x^3 + qx - p)(x + k) = 0$, i.e. because of

$$(x - a)(x - b)(x - c) = x^3 + qx - p,$$

in the form

$$(x - a)(x - b)(x - c)(x + k) = 0.$$

Because of that, the intersections we are looking for are the points A, B, C and the point $D = (-k, k^2)$, i.e. the conic (6) passes through the points A, B, C and D. By the substitutions (5) the equation $x = m$ of an isotropic line becomes the equation of the conic

$$mx^2 + xy + qx - my - p = 0, \tag{7}$$

which because of $y = x^2$ becomes the equation $x^3 + qx - p = 0$, i.e. $(x - a)(x - b)(x - c) = 0$. Because of that, conic (7) passes through the points A, B, C. From (7) follows

$$y = -\frac{mx^2 + qx - p}{x - m}, \tag{8}$$

and so the image of the line $x = m$ is the special hyperbola (8) with the asymptote $x = m$, which has with the circumscribed circle the absolute point for the fourth intersection. □

Corollary 6. *Isogonality with respect to the triangle ABC in the standard position maps nonisotropic line with the equation $y = kx + l$ to a circumscribed conic of the triangle ABC with the equation (3), which has with the circumscribed circle of the triangle ABC, the fourth intersection $D = (-k, k^2)$.*

Corollary 7. *Isogonality with respect to the triangle ABC in the standard position maps an isotropic line to a special hyperbola circumscribed to the triangle ABC whose isotropic asymptote is that line. If the line has the equation $x = m$, then the special hyperbola has the equation (8).*

From (8) follows

$$y = -mx - m^2 - q - \frac{m^3 + mq - p}{x - m}$$

and with $x \to \infty$ the right side tends to the expression $-mx - m^2 - q$. For $x = m$ this expression equals $-2m^2 - q$. Because of that we get:

Corollary 8. *The special hyperbola from Corollary 7 with the equation (8) has a non-isotropic asymptote whose equation is $y = -mx - m^2 - q$ and the center $(m, -2m^2 - q)$.*

In [1] it is shown that Euler circle of the triangle ABC has the equation $y = -2x^2 - q$. For this reason, from Corollary 8 we get the following:

Corollary 9. *The centers of the special hyperbolas circumscribed to an allowable triangle lie on its Euler circle.*

In [1] it is also shown that the inscribed circle of the triangle ABC has the equation $y = \frac{1}{4}x^2 - q$. From this equation and the equation of the non-isotropic asymptote $y = -mx - m^2 - q$ follows the equation $\frac{1}{4}x^2 + mx + m^2 = 0$ which has a double solution $x = -2m$. Therefore it follows:

Corollary 10. *Non-isotropic tangents of the special hyperbola circumscribed to an allowable triangle envelope its inscribed circle (see [4], p. 94).*

The Euler line of the triangle ABC has the equation $x = 0$, and with $m = 0$ the equation (7) gets the form

$$xy + qx - p = 0. \tag{9}$$

We shall call the obtained special hyperbola, by analogy with the Euclidean case, *Jerabek hyperbola* of the triangle ABC. It has the non-isotropic asymptote with the equation $y = -q$ and the center $(0, -q)$, and these are according to [1] the Feuerbach line and the Feuerbach point of the triangle ABC.

The Brocard diameter of the triangle ABC has the equation $x = \frac{3p}{2q}$ (see [3]), and with $m = \frac{3p}{2q}$ the equation (7), after multiplication by $2q$, gets the form

$$3px^2 + 2qxy + 2q^2x - 3py - 2pq = 0. \tag{10}$$

This special hyperbola will be called the *Kiepert hyperbola* of the triangle ABC. It passes through the centroid G of that triangle, what is in accordance with the fact that the point G is isogonal to the symmedian center $K = (\frac{3p}{2q}, -\frac{q}{3})$ of the triangle ABC (see [3]). Really, with $x = 0$, $y = -\frac{2}{3}q$ from (4) we get

$$x' = -\frac{p}{y} = \frac{3p}{2q}, \quad y' = -q - y = -\frac{q}{3}.$$

This hyperbola has the center $S = \left(\frac{3p}{2q}, -\frac{3p^2}{2q^2} - q\right)$.

Theorem 11. *If the points T and T' are two isogonal points with respect to the triangle ABC, and none of them lies on its circumscribed circle, and if δ_a, δ_b, δ_c and δ'_a, δ'_b δ'_c are distances of these points from the lines BC, CA, AB, then these equalities*

$$\delta_a\delta'_a = \delta_b\delta'_b = \delta_c\delta'_c$$

are valid.

Proof. With the labels from Theorem 1 we have $\delta_a = y + ax + bc$ and $\delta'_a = y' + ax' + bc$, so it follows

$$\begin{aligned}
\delta'_a(y - x^2) &= px - qy - y^2 + a(xy + qx - p) + bc(y - x^2) \\
&= -y^2 + axy - bcx^2 + (p + aq)x + (bc - q)y - ap \\
&= -y^2 - (b + c)xy - bcx^2 - a(b^2 + c^2)x - a(b + c)y - a^2bc \\
&= -(y + bx + ca)(y + cx + ab)
\end{aligned}$$

because of $p + aq = a(q + bc) = -a(b^2 + c^2)$. Therefore we have

$$\delta_a\delta'_a = -\frac{1}{y - x^2}(y + ax + bc)(y + bx + ca)(y + cx + ab)$$

and the symmetry of the right side of this equation by a, b, c proves the statement of the theorem. □

Theorem 12. *Let T and T' be such points which lie on the circumscribed circle of the allowable triangle ABC. The lines AT and AT' are isogonal for the angle A if and only if $TT' \,||\, BC$.*

Proof. Let ABC be standard triangle and $T = (t, t^2)$, $T' = (t', t'^2)$ are given points. The line with the equation $y = (a + t)x - at$ passes obviously through the points $A = (a, a^2)$ and T, so it is the line AT. Its slope is $a + t$. Analogously, the lines AB, AC, AT' have the slopes $a + b$, $a + c$, $a + t'$. The lines AT and AT' are isogonal under the condition $a+t+a+t' = a+b+a+c$, i.e. $t+t' = b+c$, but it is the condition of the parallelism of the lines TT' and BC, which have the slopes $t+t'$ and $b+c$. □

Let us mention that the isogonality in an isotropic plane is studied in [5].

3. Inversion in an Isotropic Plane

The inverse image of the given point T with regard to the circle $\mathcal{K}$ is called the point T' in which the diameter of $\mathcal{K}$ through the point T intersects with the polar of the point T for the circle $\mathcal{K}$.

Let the circle $\mathcal{K}$ be given with the equation $y = x^2$. The given point $T = (x_o, y_o)$ has, with respect to the circle, the polar line with the equation $y + y_o = 2x_o x$. The diameter of $\mathcal{K}$ through the point T has the equation $x = x_o$. From these two equations it follows $y = 2x_o^2 - y_o$. Therefore we get $T' = (x_o, 2x_o^2 - y_o)$. Thus, we have proved the following theorem.

Theorem 13. *The inversion with regard to the circle with the equation $y = x^2$ is given by the substitutions*

$$x \to x, \quad y \to 2x^2 - y. \tag{11}$$

The inversion is obviously an involutory mapping. The points of the circle $\mathcal{K}$ are obviously the fixed points of that mapping (11). The theorems to follow are analogous to the ones in Euclidean geometry.

Theorem 14. *The inversion maps the lines and circles to the lines and circles again.*

Proof. Any line or circle has the equation of the form $y = ux^2 + vx + w$. By the substitution (11) that equation gets the form $y = (2-u)x^2 - vx - w$ and represents a line or a circle again. □

Theorem 15. *If the inversion maps the lines $\mathcal{P}_1$ and $\mathcal{P}_2$ with the intersection T to the circles ${\mathcal{P}'}_1$ and ${\mathcal{P}'}_2$ through the image T' of the point T, then the following is valid for the oriented angles*

$$\measuredangle({\mathcal{P}'}_1, {\mathcal{P}'}_2) = -\measuredangle(\mathcal{P}_1, \mathcal{P}_2).$$

Proof. Let $T = (x_o, y_o)$. The lines $\mathcal{P}_1$ and $\mathcal{P}_2$ have the equation of the form

$$y = k_i x + y_o - k_i x_o \quad (i = 1, 2)$$

and make the angle $\measuredangle(\mathcal{P}_1, \mathcal{P}_2) = k_2 - k_1$. The inverse image of these lines are the circles $\mathcal{P}'_1$ and $\mathcal{P}'_2$ with the equations

$$y = 2x^2 - k_i x - y_o + k_i x_o \quad (i = 1, 2),$$

which passes through the point $T' = (x_o, 2x_o^2 - y_o)$ and in this point they have the tangents $\mathcal{P}''_1$ and $\mathcal{P}''_2$ with the slopes $2x_o - k_i$. Therefore we get

$$\measuredangle(\mathcal{P}'_1, \mathcal{P}'_2) = \measuredangle(\mathcal{P}''_1, \mathcal{P}''_2) = 2x_o - k_2 - (2x_o - k_1) = k_1 - k_2 = -\measuredangle(\mathcal{P}_1, \mathcal{P}_2).$$

□

From the proof of Theorem 13 it follows that the mutually inverse points with regard to some circle are parallel and its midpoints lie on that circle. Therefore the following theorem is more generally valid than Theorem 13.

Theorem 16. *The inversion with regard to the circle with the equation* $y = ux^2 + vx + w$ *is given by the substitutions*

$$x \to x, \quad y \to 2ux^2 + 2vx + 2w - y.$$

Corollary 17. *Two points are inverse with regard to a circle if and only if they are parallel and if their midpoint lies on that circle.*

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