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A CHARACTERIZATION OF MINIMAL LAGRANGIAN SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

Yoshio Matsuyama

Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku, Tokyo, 112-8551, JAPAN
e-mail: matuyama@math.chuo-u.ac.jp

Abstract: Let M be a minimal Lagrangian submanifold in a complex projective space $CP^n(c)$ and x any point of M. Then there exists a neighborhood U of x which a local field ξ of any normal vector and the second fundamental form A_ξ in the direction of ξ are defined on U. In the present paper we will give a characterization of minimal Lagrangian submanifolds in $CP^n(c)$ which satisfy $(R(X,Y)A_\xi)Z = 0$ for all X, Y and Z tangent to M and A_ξ in the direction of any normal vector ξ.

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1. Introduction

Ryan [9] proved: Let M^n be a hypersurface of a real space form $\overline{M}^{n+1}(c)$ which satisfies for any $x \in M$

$$(R(X,Y)A)(Z) = 0,$$

where R, A and X, Y and Z denote the curvature tensor of M, the second fundamental form of M which is defined on a neighborhood U of x and vector fields on U, respectively. For a brevity of the notation, we denote by RA the tensor of type $(1,3)$ defined by $(RA)(Z,X,Y) = (R(X,Y)A)(Z)$. Then the condition holds if and only if M has at most two principal curvatures at $x \in M$. Concerning to real hypersurfaces of the complex projective space $CP^n(c)$ with constant holomorphic sectional curvature c, Maeda [6] showed:

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There exist no real hypersurfaces of $CP^n(c)$ which satisfy $n \geq 3$ and $RA = 0$ for any vector fields on M. Also, Kimura and Maeda [5] obtained: If a real hypersurface M of $CP^n(c)(n \geq 2)$ satisfies $(R(X,Y)A)(Z) + (R(Y,Z)A)(X) + (R(Z,X)A)(Y) = 0$ for any X, Y and $Z \in TM$, then M is locally congruent to a geodesic hypersphere of $CP^n(c)$ or $n = 2$ and the structure vector $\xi = -J\nu$ is a principal vector, where ν and J denote a unit local normal vector field on M and the complex structure of $CP^n(c)$, respectively. Moreover, Gotoh [4] gave: Let M be a real hypersurface of $CP^n(c)(n \geq 3)$. If M satisfies $RA = 0$ for any vector fields on $\xi^\perp$, then M is locally congruent to a geodesic hypersphere of $CP^n(c)$, where $\xi^\perp$ is the orthogonal complement of ξ in TM.

Now, let M be an n-dimensional Lagrangian submanifold immersed in $CP^n(c)$. Then totally umbilical submanifolds, if there exists, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. Chen and Ogiue [2] showed a complex space form of nonzero constant holomorphic sectional curvature of complex dimension ≥ 2 admits no totally umbilical, Lagrangian submanifolds except the totally geodesic ones. On the other hand, the author [7] proved: Let M^n be a minimal Lagrangian submanifold of $CP^n(c)$ which has at most two principal curvatures in the direction of any normal. Then if M^n is not totally geodesic, then M^n has parallel second fundamental form and is isotropic $(n \geq 4)$, or isotropic minimal surface in $CP^2(c)$. In the former case, if n is even (resp. odd), then M^n is Einstein and is locally congruent to the following: $SU(3), n = 8; SU(6)/S_p(3), n = 14$ or $E_6/F_4, n = 26$ (resp. M^n does not exist).

Considering the condition of $RA = 0$ for a hypersurfaces of a real space form and the condition of at most two principal curvatures in the direction of any normal vector for a Lagrangian submanifold of $CP^n(c)$, we think of the following condition. That is, the purpose of this paper is to prove:

Theorem. *Let M^n be a minimal Lagrangian submanifold in $CP^n(c)$ which satisfies*

$$RA_\xi = 0$$

for the shape operator A_ξ in the direction of any normal ξ. Then M is totally geodesic or locally congruent to a flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form.

2. Preliminaries

Let M^n be an n-dimensional submanifold isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\overline{M}^{n+p}$. To each ξ in the normal space $T_x^{\perp}M$ at $x \in M$ is associated a linear transformation of the tangent space T_xM in the following way. Extend ξ to a normal vector field defined in a neighborhood of x and define $-A_\xi X$ to be the tangential component of $\overline{\nabla}_X\xi$ for $X \in T_xM$, where $\overline{\nabla}$ denotes the covariant differentiation in $\overline{M}$. $A_\xi X$ depends only on ξ at x and X. Given an orthonormal basis $\{\xi_1, \cdots, \xi_p\}$ of $T_x^{\perp}M$, we write $A_\alpha = A_{\xi_\alpha}$ and call A_α's the second fundamental forms in the directions of $\xi_1, \cdots, \xi_{p-1}$ and ξ_p. If $\xi_1, \cdots, \xi_{p-1}$ and ξ_p are now orthonormal normal vector fields in a neighborhood of x, they determine normal connection forms $s_{\alpha\beta}$ in a neighborhood of x, by

$$D_X\xi_\alpha = \sum_\beta s_{\alpha\beta}(X)\xi_\beta$$

for X tangent to M, where D denotes the covariant differentiation in the normal bundle. A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form. Let $\overline{M}^n(c)$ be an n-dimensional complex space form of constant holomorphic sectional curvature c. Then we have (see, for example, [1])

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} = \ & \tfrac{c}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y} + \overline{g}(J\overline{Y},\overline{Z})J\overline{X} \\ & -\overline{g}(J\overline{X},\overline{Z})J\overline{Y} + 2\overline{g}(\overline{X},J\overline{Y})J\overline{Z}\},\end{aligned} \tag{1}$$

where $\overline{R}$, $\overline{g}$ and J denote the curvature tensor field, the Riemannian metric and the complex sturucture of $\overline{M}^n(c)$, respectively, and $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are vector fields on $\overline{M}^n(c)$.

Let M^n be an n-dimensional totally real submanifold isometrically immersed in $\overline{M}^n(c)$. If σ is the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated to a normal vector ξ, then it is well known that

$$J(T_xM) \subset T_x^{\perp}M, \tag{2}$$

$$A_{JX}Y = -J\sigma(X,Y), \tag{3}$$

where T_xM and $T_x^{\perp}M$ denote the tangent space and the normal space of M, respacetively. A submanifold M of a Kaehler manifold is Lagrangian if the complex structure of the ambient space J carries each tangent space of M onto the corresponding normal space of M, i.e., $J(T_xM) = T_x^{\perp}$ for any point $x \in M$. Hence an n-dimensional totally rea submanifold immersed in $\overline{M}^n(c)$ is a Lagrangian submanifold in $\overline{M}^n(c)$. Let R be the curvature tensor field of M. Then we have

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \sum_\alpha g(A_\alpha Y, Z)A_\alpha X - \sum_\alpha g(A_\alpha X, Z)A_\alpha Y, \end{aligned} \tag{4}$$

where g is the induced Riemannian metric on M. Let S be the Ricci tensor of M. Then we have

$$S(X,Y) = \frac{1}{4}(n-1)cg(X,Y) + \sum_\alpha (\mathrm{trace} A_\alpha) g(A_\alpha X, Y) - \sum_\alpha g(A_\alpha X, A_\alpha Y). \tag{5}$$

Let ρ be the scalar curvature of M. Then we have

$$\rho = \frac{1}{4}n(n-1)c + \sum_\alpha (\mathrm{trace} A_\alpha)^2 - ||\sigma||^2, \tag{6}$$

where $||\sigma||$ is the length of the second fundamental form σ of the immersion so that

$$||\sigma||^2 = \sum_\alpha \mathrm{trace} A_\alpha^2. \tag{7}$$

If M is a minimal submanifold so that $\mathrm{trace} A_\alpha = 0$ for all α, then we have

$$S(X,Y) = \frac{1}{4}(n-1)cg(X,Y) - \sum_\alpha g(A_\alpha X, A_\alpha Y), \tag{8}$$

$$\rho = \frac{1}{4}n(n-1)c - ||\sigma||^2. \tag{9}$$

Let ∇^* denote the sum of the tangential and the normal connections. ∇^* is the connection in the Whitney sum of the tangent bundle of M and the normal bundle of M induced by ∇ and D. Then we have

$$\nabla^*_X A_\alpha = \nabla_X A_\alpha - \sum_\beta s_{\alpha\beta}(X) A_\beta. \tag{10}$$

Setting $f_1 = \mathrm{trace} A_1^2$ on M, f_1 satisfies a differential equation for an orthonormal basis $\{E_1, \cdots E_n\}$ of $T_x M$:

Lemma 1. *Let M^n be an n-dimensional minimal Lagrangian submanifold immersed in $\overline{M}^n(c)$. Then*

$$\begin{aligned} \frac{1}{2}\Delta f_1 &= ||\nabla A_1||^2 + \sum_\beta \mathrm{trace}(A_1 A_\beta - A_\beta A_1)^2 \\ &- \sum_\beta (\mathrm{trace} A_1 A_\beta)^2 + \frac{n+1}{4} c f_1 \end{aligned} \tag{11}$$

$$
\begin{aligned}
&+ \sum_{i,\beta}(\nabla_{E_i}s_{1\beta})(E_i)\mathrm{trace}A_1A_\beta + 2\sum_{i,\beta}s_{1\beta}(E_i)\mathrm{trace}(\nabla_{E_i}A_\beta)A_1\\
&- \sum_{i,\beta,\gamma}s_{1\beta}(E_i)s_{\beta\gamma}(E_i)\mathrm{trace}A_\gamma A_1\,.
\end{aligned}
$$

Proof. Since M is a minimal submanifold of $\overline{M}^n(c)$, the following holds [3]:

$$
\begin{aligned}
\frac{1}{2}\Delta f_1 &= ||\nabla A_1||^2 + \sum_\beta \mathrm{trace}(A_1A_\beta - A_\beta A_1)^2 - \sum_\beta(\mathrm{trace}A_1A_\beta)^2\\
&+ \sum_{i,\beta}(\nabla_{E_i}s_{1\beta})(E_i)\mathrm{trace}A_1A_\beta + 2\sum_{i,\beta}s_{1\beta}(E_i)\mathrm{trace}(\nabla_{E_i}A_\beta)A_1\\
&- \sum_{i,\beta,\gamma}s_{1\beta}(E_i)s_{\beta\gamma}(E_i)\mathrm{trace}A_\gamma A_1\\
&+ \sum_{i,j,k,\beta}(4\overline{g}(\overline{R}(E_j,E_i)\xi,\xi_\beta)g(A_\xi E_j,E_k)g(A_\beta E_i,E_k)\\
&- \overline{g}(\overline{R}(E_k,\xi_\beta)\xi,E_k)g(A_\xi E_i,E_j)g(A_\beta E_i,E_j))\\
&+ \sum_{i,j,k}(2\overline{g}(\overline{R}(E_j,E_k)E_i,E_j)g(A_\xi E_i,E_\ell)g(A_\xi E_k,E_\ell)\\
&+ 2\overline{g}(\overline{R}(E_\ell,E_k)E_i,E_j)g(A_\xi E_i,E_\ell)g(A_\xi E_j,E_k)).
\end{aligned}
$$

By (1) the last two terms of the right side of the above equation is equal to $\frac{n+1}{4}cf_1$. □

The second fundamental form σ of the immersion satisfies a differential equation.

Lemma 2. (see [1]) *Let M^n be an n-dimensional minimal Lagrangian submanifold immersed in $\overline{M}^n(c)$. Then*

$$
\begin{aligned}
\frac{1}{2}\Delta||\sigma||^2 &= ||\nabla^*\sigma||^2 + \sum_{\alpha,\beta}\mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 \qquad (12)\\
&- \sum_{\alpha,\beta}(\mathrm{trace}A_\alpha A_\beta)^2 + \frac{1}{4}(n+1)c||\sigma||^2\\
&= ||\nabla^*\sigma|| + 2\sum_{\alpha,\beta}\mathrm{trace}(A_\alpha A_\beta)^2 - 2\sum_{\alpha,\beta}\mathrm{trace}(A_\alpha^2A_\beta^2)\\
&- \sum_{\alpha,\beta}(\mathrm{trace}A_\alpha A_\beta)^2 + \frac{1}{4}(n+1)c||\sigma||^2.
\end{aligned}
$$

3. Proof of Theorem

Let M^n be a minimal Lagrangian submanifold in $\overline{M}^n(c)$. Suppose that

$$(R(X,Y)A_\xi)(Z) = 0 \tag{13}$$

for each $X, Y, Z \in T_x(M)$ and the second fundamental form A_ξ of M in the direction of any normal vector ξ. Then using (4), the equation (13) yields

$$\begin{aligned} & \frac{c}{4}\{g(Y, A_\xi Z)X - g(X, A_\xi Z)Y - g(Y,Z)A_\xi X + g(X,Z)A_\xi Y\} \\ + & \sum_\alpha g(A_\alpha Y, A_\xi Z)A_\alpha X - \sum_\alpha g(A_\alpha X, A_\xi Z)A_\alpha Y \\ - & \sum_\alpha g(A_\alpha Y, Z)A_\xi A_\alpha X + \sum_\alpha g(A_\alpha X, Z)A_\xi A_\alpha Y = 0. \end{aligned} \tag{14}$$

We choose a local field of orthonormal frames $e_1, \cdots, e_n$ such that the Ricci tensor S is diagonalized. Then we see that $\mathrm{trace} A_\alpha A_\beta = 0$ for $\alpha \neq \beta$. Putting $Y = e_j$ and $Z = e_j$ in (14), and taking the summation on j, we obtain

$$-\frac{n}{4}cA_\xi X + \sum_\alpha (\mathrm{trace} A_\xi A_\alpha)A_\alpha X - \sum_\alpha A_\alpha A_\xi A_\alpha X + \sum_\alpha A_\xi A_\alpha^2 X = 0, \tag{15}$$

i.e., we obtain

$$-\frac{n}{4}cA_\xi X + (\mathrm{trace} A_\xi^2)A_\xi X - \sum_\alpha A_\alpha A_\xi A_\alpha X + \sum_\alpha A_\xi A_\alpha^2 X = 0. \tag{16}$$

Hence we have

$$-\frac{n}{4}cA_\xi^2 X + (\mathrm{trace} A_\xi^2)A_\xi^2 X - \sum_\alpha A_\alpha A_\xi A_\alpha A_\xi X + \sum_\alpha A_\xi A_\alpha^2 A_\xi X = 0. \tag{17}$$

Thus we see that the following equation

$$\begin{aligned} & (\mathrm{trace} A_\xi^2 - \frac{n}{4}c)\mathrm{trace} A_\xi^2 \\ = & \frac{1}{2}\sum_\alpha \mathrm{trace}(A_\xi A_\alpha - A_\alpha A_\xi)^2 \leq 0. \end{aligned} \tag{18}$$

Setting $A_\xi X$ in the place of X in (13), we have

$$(R(A_\xi X, Y)A_\xi)(Z) = 0. \tag{19}$$

Putting $X = e_j$ and $Z = e_j$ in (19), and taking the summation on j, from (14), we obtain

$$\begin{aligned} -\frac{c}{4}(\mathrm{trace} A_\xi^2)Y & + (\mathrm{trace} A_\xi^2)A_\xi^2 Y \\ +\sum_\alpha A_\alpha A_\xi^2 A_\alpha Y & - \sum_\alpha A_\xi A_\alpha A_\xi A_\alpha Y \end{aligned} \tag{20}$$

$$-\sum_{\alpha}(\mathrm{trace} A_\alpha A_\xi^2)A_\alpha Y \qquad = 0.$$

From (17) and (20) we have

$$\begin{aligned}\sum_{\beta}(\mathrm{trace} A_\beta^2)A_\beta^2 X \quad - \quad \frac{n}{4}c\sum_{\beta} A_\beta^2 X \\ +\sum_{\alpha,\beta} A_\beta A_\alpha^2 A_\beta X \quad - \quad \sum_{\alpha,\beta} A_\alpha A_\beta A_\alpha A_\beta X = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}-\frac{c}{4}||\sigma||^2 Y \quad + \quad \sum_{\beta}(\mathrm{trace} A_\beta^2)A_\beta^2 Y \\ +\sum_{\alpha,\beta} A_\alpha A_\beta^2 A_\alpha Y \quad - \quad \sum_{\alpha,\beta} A_\beta A_\alpha A_\beta A_\alpha Y - \sum_{\alpha,\beta}(\mathrm{trace} A_\alpha A_\beta^2)A_\alpha Y = 0.\end{aligned} \tag{22}$$

Subtracting (21) from (22), we get

$$-\frac{c}{4}||\sigma||^2 X - \sum_{\alpha,\beta}(\mathrm{trace} A_\alpha A_\beta^2)A_\alpha X = -\frac{n}{4}c\sum_{\beta} A_\beta^2 X. \tag{23}$$

On the other hand, let Y be an arbitrary tangent vector. Taking the product (16) and $A_\xi Y$, we have

$$g((\mathrm{trace} A_\xi^2 - \frac{n}{4}c)A_\xi X - \sum_{\alpha} A_\alpha A_\xi A_\alpha X + \sum_{\alpha} A_\xi A_\alpha^2 X, A_\xi Y) = 0. \tag{24}$$

Since A_ξ is symmetric, we obtain

$$(\mathrm{trace} A_\xi^2 - \frac{n}{4}c)A_\xi^2 X - \sum_{\alpha} A_\xi A_\alpha A_\xi A_\alpha X + \sum_{\alpha} A_\xi^2 A_\alpha^2 X = 0. \tag{25}$$

By the similar argument, from (17) and (25) we get

$$(\mathrm{trace} A_\xi^2 - \frac{n}{4}c)A_\xi^2 X - \sum_{\alpha} A_\xi A_\alpha A_\xi A_\alpha X + \sum_{\alpha} A_\xi A_\alpha^2 A_\xi X = 0\,, \tag{26}$$

$$(\mathrm{trace} A_\xi^2 - \frac{n}{4}c)A_\xi^2 X - \sum_{\alpha} A_\alpha A_\xi A_\alpha A_\xi X + \sum_{\alpha} A_\alpha^2 A_\xi^2 X = 0. \tag{27}$$

From (17) and (27) we obtain

$$\sum_{\alpha} A_\xi A_\alpha^2 A_\xi X = \sum_{\alpha} A_\alpha^2 A_\xi^2 X. \tag{28}$$

Similarly, from (25) and (26) we have

$$\sum_{\alpha} A_\xi A_\alpha^2 A_\xi X = \sum_{\alpha} A_\xi^2 A_\alpha^2 X. \tag{29}$$

Putting $A_\eta^2 X$ in the place of X in (16), we get

$$(\mathrm{trace} A_\xi^2 - \frac{n}{4}c)A_\xi A_\eta^2 X \quad - \quad \sum_{\alpha} A_\alpha A_\xi A_\alpha A_\eta^2 X \tag{30}$$

$$+ \sum_{\alpha} A_\xi A_\alpha^2 A_\eta^2 X = 0.$$

Taking the summation on η of (30), we have

$$(\mathrm{trace} A_\xi^2 - \frac{n}{4}c) \sum_{\beta} \mathrm{trace} A_\xi A_\beta^2 - \sum_{\alpha,\beta} \mathrm{trace} A_\alpha A_\xi A_\alpha A_\beta^2 + \sum_{\alpha,\beta} \mathrm{trace} A_\xi A_\alpha^2 A_\beta^2 = 0. \tag{31}$$

Using the equation (29), we obtain

$$\sum_{\alpha,\beta} \mathrm{trace} A_\xi A_\alpha^2 A_\beta^2 = \sum_{\alpha,\beta} \mathrm{trace} A_\xi A_\alpha A_\beta^2 A_\alpha = \sum_{\alpha,\beta} \mathrm{trace} A_\alpha A_\xi A_\alpha A_\beta^2. \tag{32}$$

Combining (32) with (31), we get

$$(\mathrm{trace} A_\xi^2 - \frac{n}{4}c) \sum_{\beta} \mathrm{trace} A_\xi A_\beta^2 = 0 \tag{33}$$

for any point x of M. Assume that $\mathrm{trace} A_\xi^2 \equiv \frac{n}{4}c$ for a point x_0 of M. Then from (18) we have

$$\sum_{\alpha} \mathrm{trace}(A_\xi A_\alpha - A_\alpha A_\xi)(A_\alpha A_\xi - A_\xi A_\alpha) = 0 \tag{34}$$

at x_0. Also, we remark that, from (10), we obtain

$$||\nabla^* A_1||^2 = ||\nabla A_1||^2 - 2 \sum_{i,\beta} s_{1\beta}(E_i) \mathrm{trace}(\nabla_{E_i} A_1) A_\beta + \sum_{i,\beta} (s_{1\beta}(E_i))^2 \mathrm{trace} A_\beta^2 \tag{35}$$

at x_0. Setting $f_\xi = \mathrm{trace} A_\xi^2$ at x_0, from (11) and (34), we obtain

$$\begin{aligned} \tfrac{1}{2}\Delta f_\xi &= ||\nabla A_\xi||^2 - f_\xi^2 + \frac{n+1}{4} c f_\xi \\ &-2 \sum_{i,\beta} \overline{g}(D_{E_i}\xi, \xi_\beta) \mathrm{trace}(\nabla_{E_i} A_\xi) A_\beta + \sum_{i,\beta} \overline{g}(D_{E_i}\xi, \xi_\beta)^2 \mathrm{trace} A_\beta^2 \\ &+2 \sum_{i,\beta} \overline{g}(D_{E_i}\xi, \xi_\beta) \mathrm{trace}((\nabla_{E_i} A_\xi) A_\beta + (\nabla_{E_i} A_\beta) A_\xi) \\ &+ \sum_{i,\beta} \overline{g}(D_{E_i}\xi, \xi_\beta)^2 (f_\xi - \mathrm{trace} A_\beta^2) \end{aligned} \tag{36}$$

$$= -f_\xi^2 + \frac{n+1}{4} c f_\xi + ||\nabla^* A_\xi||^2 + \sum_{i,\beta} \overline{g}(D_{E_i}\xi, \xi_\beta)^2 (f_\xi - \mathrm{trace} A_\beta^2),$$

since $\mathrm{trace} A_\alpha A_\beta = 0$ for $\alpha \neq \beta$. Thus by (18) we see that $f_\xi = 0$ at x_0, which is a contradiction. Hence we may assume that $\mathrm{trace} A_\alpha^2 \neq \frac{n}{4} c$ for any α on M. By (23) and (33) we have

$$\sum_\alpha A_\alpha^2 = \frac{||\sigma||^2}{n} I. \tag{37}$$

We remark that M is Einstein from (8) if $n \geq 3$. Therefore from (37) we obtain

$$\begin{aligned} \mathrm{trace} A_{JX}^2 &= \sum_\alpha g(A_{JX} J\xi_\alpha, A_{JX} J\xi_\alpha) \\ &= \sum_\alpha g(A_\alpha X, A_\alpha X) \\ &= \frac{||\sigma||^2}{n} g(X, X) = \frac{||\sigma||^2}{n} \end{aligned}$$

for any unit vector X. By (12), (26), (27) and (37) we obtain

$$\frac{1}{2} \Delta ||\sigma||^2 = ||\nabla^* \sigma||^2 + \frac{||\sigma||^2}{n} (||\sigma||^2 - \frac{n(n-1)}{4} c). \tag{38}$$

In (16) put $\xi = JZ$. Then for any unit vector X we have

$$(2\frac{||\sigma||^2}{n} - \frac{n}{4} c) A_{JZ} X - \sum_\alpha A_\alpha A_{JZ} A_\alpha X = 0. \tag{39}$$

Putting $\xi = JY$ in (14) and taking the contraction with respect to Y, we obtain

$$(\frac{c}{4} - \frac{||\sigma||^2}{n}) A_{JZ} X + \frac{1}{2} (\sum_\alpha A_\alpha A_{JZ} A_\alpha X + \sum_\alpha A_\alpha A_{JX} A_\alpha Z) = 0. \tag{40}$$

From (39) and (40) we get

$$(||\sigma||^2 - \frac{n(n-1)}{4} c) A_{JZ} X = 0.$$

If M is not totally geodesic, then we may assume that $||\sigma||^2 - \frac{n(n-1)}{4} c = 0$, i.e., $\rho = 0$. Since M has parallel second fundamental form by (38), we know that M is of flat and $n = 2$ (see Naitoh [8]).

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