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A COMMON FIXED POINT THEOREM FOR TWO PAIRS OF COMPATIBLE MAPPINGS AND ITS APPLICATION

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Abstract: An existence and uniqueness result of common fixed point for two pairs of compatible mappings in an induced complete metric space is obtained. As application the existence and uniqueness of common solution for a system of functional equations arising in dynamic programming is discussed.

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Key Words: countable family of pseudometrics, induced metric space, compatible mappings, contractive mappings, common fixed point, common solution, system of functional equations, dynamic programming

1. Introduction and Preliminaries

It is well known that the fixed point theory is an important part of nonlin-

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ear analysis, which has formed a very abundant and sound system. Around the fixed point theory, many scholars have studied and founded various conditions of contractive mappings to ensure the existence of fixed points for a long time, see [1-13] and the references thereof. Jungck [3] introduced the concept of compatible mappings as a common generalization of commuting mappings and weakly commuting mappings. Liu [6] and Rhoades, Tiwary and Singh [13] employed compatible mappings to establish criteria for the existence of common fixed points for some contractive type mappings. On the other hand, the applications of the fixed point theory is wide. In particular, Bhakta and Choudhury [1], Bhakta and Mitra [2], Liu [7], Liu and Ume [8], Liu, Agarwal and Kang [9], Pathak and Fisher [11] and others established the existence and uniqueness of solutions for several classes of functional equations or system of functional equations arising in dynamic programming by utilizing fixed or common fixed point theorems.

In this paper, we use the concept of compatible mappings in an induced metric space to prove a common fixed point theorem involving contractive type mappings and establish the existence and uniqueness of common solution for a system of functional equations arising in dynamic programming by using the common fixed point theorem. Our results generalize some results of Jungck [4], Liu [10] and Sessa, Rhoades and Khan [12].

Throughout this paper, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$ and

$$\Phi = \big\{\varphi : \varphi : (\mathbb{R}^+)^8 \to \mathbb{R}^+ \text{ is upper semicontinuous, nondecreasing in each coordinate variable and satisfies } (a)\big\},$$

where

(*a*) $\varphi_1(t) = \max\{\varphi(t,t,t,t,t,t,0,0), \varphi(0,0,t,t,0,0,t,t)\}$ for all $t \in \mathbb{R}^+$ and $\varphi_1(t) < t$ for all $t > 0$.

Assume that $\{d_k\}_{k\geq 1}$ is a countable family of pseudometrics on a nonempty set X such that for any distinct $x, y \in X, d_k(x, y) \neq 0$ for some $k \geq 1$. Define

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x,y)}{1 + d_k(x,y)}, \quad \forall x, y \in X.$$

It is clear that d is a metric on X and the metric d and the metric space (X, d) are called, *an induced metric and induced metric space by the countable pseudometrics family* $\{d_k\}_{k\geq 1}$, respectively. A sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges to a point $x \in X$ if and only if $d_k(x_n, x) \to 0$ as $n \to \infty$ for each $k \geq 1$, and $\{x_n\}_{n\geq 1} \subseteq X$ is a Cauchy sequence if and only if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \geq 1$. The induced metric space (X, d) is called *complete* if each Cauchy sequence in X converges to some point in X. A self mapping f on (X, d)

is said to be continuous in X if $\lim_{n\to\infty} fx_n = fx$ whenever $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{x_n\}_{n\geq 1}$ converges to $x \in X$.

2. A Common Fixed Point Theorem

In this section, we assume that (X, d) is the induced metric space by a countable family of pseudometrics $\{d_k\}_{k\geq 1}$ such that for any distinct $x, y \in X$, $d_k(x, y) \neq 0$ for some $k \geq 1$.

Definition 2.1. (see [10]) A pair of self mappings f and g on the induced metric space (X, d) are called *compatible* if $\lim_{n\to\infty} d_k(fgx_n, gfx_n) = 0$ for $k \geq 1$ whenever $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\{fx_n\}_{n\geq 1}$ and $\{gx_n\}_{n\geq 1}$ converge to some $t \in X$.

Lemma 2.1. (see [5]) *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing upper semicontinuous function. Then for every $t > 0$, $f(t) < t$ if and only if $\lim_{n\to\infty} f^n(t) = 0$, where $f^n(t)$ denotes the composition of $f(t)$ with itself n times.*

Lemma 2.2. (see [10]) *Let f and g be compatible mappings from the induced metric space (X, d) into itself. Then:*

(a) *$ffa = gga = fga = gfa$ if $fa = ga$ for some $a \in X$;*

(b) *$\{gfx_n\}_{n\geq 1}$ converges to fa if f is continuous and $\{fx_n\}_{n\geq 1}$ and $\{gx_n\}_{n\geq 1}$ converge to some $a \in X$.*

Theorem 2.1. *Let A, B, S and T be mappings from the induced metric space (X, d) into itself such that:*

(a) *$A(X) \subseteq T(X), B(X) \subseteq S(X)$;*

(b) *both A, S and B, T are compatible;*

(c) *one of A, B, S, T is continuous in X;*

(d) *one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X.*

If there exists some $\varphi \in \Phi$ satisfying

$$\begin{aligned} d_k(Ax, By) \leq \varphi\Big(& d_k(Ax, Sx), d_k(By, Ty), \\ & d_k(Sx, Ty), \frac{1}{2}\big[d_k(Ax, Ty) + d_k(By, Sx)\big], \frac{d_k(Ax, Sx)d_k(By, Ty)}{d_k(Ax, By) + 1}, \\ & \frac{d_k(Ax, Sx)d_k(By, Ty)}{d_k(Sx, Ty) + 1}, \frac{d_k(Ax, Ty)d_k(By, Sx)}{d_k(Ax, By) + 1}, \frac{d_k(Ax, Ty)d_k(By, Sx)}{d_k(Sx, Ty) + 1}\Big) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and $k \geq 1$, then A, B, S and T have a unique common fixed *point in X.*

Proof. Let x_0 be any point of X. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, we can choose $x_1 \in X$ such that $y_1 = Tx_1 = Ax_0$ and for this point x_1, there exists a point $x_2 \in X$ such that $y_2 = Sx_2 = Bx_1$ and so on. Inductively, we define sequences $\{y_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 1}$ in X such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ for $n \geq 0$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \geq 1$. Define $d_{kn} = d_k(y_n, y_{n+1})$ for $k \geq 1$ and $n \geq 1$. We now show that

$$\lim_{n\to\infty} d_{kn} = 0, \quad \forall k \geq 1. \tag{2.2}$$

From (2.1) we get that

$$\begin{aligned}
&d_k(Ax_{2n}, Bx_{2n+1})\\
&\qquad \leq \varphi\Big(d_k(Ax_{2n}, Sx_{2n}), d_k(Bx_{2n+1}, Tx_{2n+1}), d_k(Sx_{2n}, Tx_{2n+1}),\\
&\qquad\qquad \frac{1}{2}\big[d_k(Ax_{2n}, Tx_{2n+1}) + d_k(Bx_{2n+1}, Sx_{2n})\big],\\
&\qquad \frac{d_k(Ax_{2n}, Sx_{2n})d_k(Bx_{2n+1}, Tx_{2n+1})}{d_k(Ax_{2n}, Bx_{2n+1}) + 1}, \frac{d_k(Ax_{2n}, Sx_{2n})d_k(Bx_{2n+1}, Tx_{2n+1})}{d_k(Sx_{2n}, Tx_{2n+1}) + 1},\\
&\qquad \frac{d_k(Ax_{2n}, Tx_{2n+1})d_k(Bx_{2n+1}, Sx_{2n})}{d_k(Ax_{2n}, Bx_{2n+1}) + 1}, \frac{d_k(Ax_{2n}, Tx_{2n+1})d_k(Bx_{2n+1}, Sx_{2n})}{d_k(Sx_{2n}, Tx_{2n+1}) + 1}\Big)\\
&\qquad = \varphi\Big(d_{k2n}, d_{k2n+1}, d_{k2n}, \frac{1}{2}[0 + d_k(y_{2n+2}, y_{2n})],\\
&\qquad\qquad \frac{d_{k2n}d_{k2n+1}}{d_{k2n+1} + 1}, \frac{d_{k2n}d_{k2n+1}}{d_{k2n} + 1}, 0, 0\Big), \quad \forall k \geq 1,\ n \geq 1.
\end{aligned} \tag{2.3}$$

Suppose that $d_{k2n} < d_{k2n+1}$ for some $k \geq 1$ and $n \geq 1$ in (2.3). Because φ is non-decreasing in each coordinate variable, from (2.3) we obtain that

$$\begin{aligned}
d_{k2n+1} &\leq \varphi(d_{k2n+1}, d_{k2n+1}, d_{k2n+1}, d_{k2n+1}, d_{k2n+1}, d_{k2n+1}, 0, 0)\\
&\leq \varphi_1(d_{k2n+1}) < d_{k2n+1},
\end{aligned}$$

which is impossible and hence $d_{k2n+1} \leq \varphi_1(d_{k2n})$ for all $k \geq 1, n \geq 1$. Similarly we gain that $d_{k2n} \leq \varphi_1(d_{k2n-1})$ for all $k \geq 1$ and $n \geq 1$. It follows that

$$d_{kn} \leq \varphi_1(d_{kn-1}) \leq \varphi_1^2(d_{kn-2}) \leq \cdots \leq \varphi_1^{n-1}(d_{k1}), \quad \forall n \geq 1,\ k \geq 1. \tag{2.4}$$

Obviously, (2.4) together with Lemma 2.1 gives that $\lim_{n\to\infty} d_{kn} = 0$, that is, (2.2) holds.

In order to prove that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. If not, there exist some k and a positive number M such that for each even integer $2N$, there are even integers $2m(N)$

and $2n(N)$ such that $2m(N) > 2n(N) > 2N$ and

$$d_k(y_{2m(N)}, y_{2n(N)}) > M. \tag{2.5}$$

For each even integer $2N$, let $2m(N)$ be the least even integer exceeding $2n(N)$ satisfying (2.5), so that

$$d_k(y_{2m(N)-2}, y_{2n(N)}) \le M. \tag{2.6}$$

It follows that for each even integer $2N$,

$$d_k(y_{2n(N)}, y_{2m(N)}) \le d_k(y_{2n(N)}, y_{2m(N)-2}) + d_{k2m(N)-2} + d_{k2m(N)-1}.$$

In view of (2.2), (2.5) and (2.6) we arrive at

$$\lim_{N\to\infty} d_k(y_{2n(N)}, y_{2m(N)}) = M. \tag{2.7}$$

From the triangular inequality, we obtain that

$$|d_k(y_{2m(N)}, y_{2n(N)+1}) - d_k(y_{2m(N)}, y_{2n(N)})| \le d_{k2n(N)}$$

and

$$|d_k(y_{2m(N)+1}, y_{2n(N)}) - d_k(y_{2n(N)}, y_{2m(N)})| \le d_{k2m(N)}.$$

From (2.7), (2.2) and the above inequalities we get that

$$\lim_{N\to\infty} d_k(y_{2m(N)}, y_{2n(N)+1}) = M = \lim_{N\to\infty} d_k(y_{2m(N)+1}, y_{2n(N)}). \tag{2.8}$$

Applying the triangular inequality, we have

$$|d_k(y_{2m(N)+1}, y_{2n(N)+1}) - d_k(y_{2n(N)+1}, y_{2m(N)})| \le d_{k2m(N)}$$

and

$$|d_k(y_{2n(N)+2}, y_{2m(N)}) - d_k(y_{2m(N)}, y_{2n(N)+1})| \le d_{k2n(N)+1},$$

which together with (2.2) and (2.8) ensures that

$$\lim_{N\to\infty} d_k(y_{2m(N)+1}, y_{2n(N)+1}) = M = \lim_{N\to\infty} d_k(y_{2n(N)+2}, y_{2m(N)}). \tag{2.9}$$

Note that

$$|d_k(y_{2m(N)+1}, y_{2n(N)+2}) - d_k(y_{2n(N)+2}, y_{2m(N)})| \le d_{k2m(N)}.$$

(2.2) and (2.9) guarantee that

$$\lim_{N\to\infty} d_k(y_{2m(N)+1}, y_{2n(N)+2}) = M. \tag{2.10}$$

Using (2.1), we get that

$$\begin{aligned} & d_k(Ax_{2m(N)}, Bx_{2n(N)+1}) \\ & \quad \le \varphi\Big(d_k(Ax_{2m(N)}, Sx_{2m(N)}), d_k(Bx_{2n(N)+1}, Tx_{2n(N)+1}), \\ & \qquad\qquad d_k(Sx_{2m(N)}, Tx_{2n(N)+1}), \\ & \quad \frac{1}{2}\big[d_k(Ax_{2m(N)}, Tx_{2n(N)+1}) + d_k(Bx_{2n(N)+1}, Sx_{2m(N)})\big], \end{aligned}$$

$$\frac{d_k(Ax_{2m(N)}, Sx_{2m(N)})d_k(Bx_{2n(N)+1}, Tx_{2n+1(N)})}{d_k(Ax_{2m(N)}, Bx_{2n(N)+1})+1},$$
$$\frac{d_k(Ax_{2m(N)}, Sx_{2m(N)})d_k(Bx_{2n(N)+1}, Tx_{2n+1(N)})}{d_k(Sx_{2m(N)}, Tx_{2n(N)+1})+1},$$
$$\frac{d_k(Ax_{2m(N)}, Tx_{2n(N)+1})d_k(Bx_{2n(N)+1}, Sx_{2m(N)})}{d_k(Ax_{2m(N)}, Bx_{2n(N)+1})+1},$$
$$\frac{d_k(Ax_{2m(N)}, Tx_{2n(N)+1})d_k(Bx_{2n(N)+1}, Sx_{2m(N)})}{d_k(Sx_{2m(N)}, Tx_{2n(N)+1})+1}\Big). \quad (2.11)$$

Letting $N \to \infty$ in (2.11), we infer that

$$M \le \varphi\Big(0, 0, M, M, 0, 0, \frac{M^2}{M+1}, \frac{M^2}{M+1}\Big) \le \varphi_1(M) < M,$$

which is a contradiction. Thus $\{y_n\}_{n\ge1}$ is a Cauchy sequence. In light of (d), there exists some point $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Suppose that A is continuous. Since A and S are compatible. Lemma 2.2 infers that AAx_{2n} and $SAx_{2n} \to Az$ as $n \to \infty$. In view of (2.1) we get that

$$\begin{aligned}
&d_k(AAx_{2n}, Bx_{2n-1})\\
&\le \varphi\Big(d_k(AAx_{2n}, SAx_{2n}), d_k(Bx_{2n-1}, Tx_{2n-1}), d_k(SAx_{2n}, Tx_{2n-1}),\\
&\frac{1}{2}\big[d_k(AAx_{2n}, Tx_{2n-1}) + d_k(Bx_{2n-1}, SAx_{2n})\big],\\
&\frac{d_k(AAx_{2n}, SAx_{2n})d_k(Bx_{2n-1}, Tx_{2n-1})}{d_k(AAx_{2n}, Bx_{2n-1})+1}, \frac{d_k(AAx_{2n}, SAx_{2n})d_k(Bx_{2n-1}, Tx_{2n-1})}{d_k(SAx_{2n}, Tx_{2n-1})+1},\\
&\frac{d_k(AAx_{2n}, Tx_{2n-1})d_k(Bx_{2n-1}, SAx_{2n})}{d_k(AAx_{2n}, Bx_{2n-1})+1}, \frac{d_k(AAx_{2n}, Tx_{2n-1})d_k(Bx_{2n-1}, SAx_{2n})}{d_k(SAx_{2n}, Tx_{2n-1})+1}\Big),\\
&\forall k \ge 1,\ n \ge 1. \quad (2.12)
\end{aligned}$$

Letting $n \to \infty$ in (2.12), we obtain that

$$\begin{aligned}
d_k(Az, z) &\le \varphi\Big(0, 0, d_k(Az, z), \frac{1}{2}\big[d_k(Az, z) + d_k(z, Az)\big], 0, 0,\\
&\quad \frac{d_k(Az, z)d(z, Az)}{d_k(Az, z)+1}, \frac{d_k(Az, z)d_k(z, Az)}{d_k(Az, z)+1}\Big)\\
&\le \varphi(0, 0, d_k(Az, z), d_k(Az, z), 0, 0, d_k(Az, z), d_k(Az, z))\\
&\le \varphi_1(d_k(Az, z)), \quad \forall k \ge 1,
\end{aligned}$$

which means that $d_k(Az, z) = 0$ for all $k \ge 1$. Therefore, $Az = z$. By virtue of $A(X) \subseteq T(X)$, there exists a point $v \in X$ such that $z = Az = Tv$. Applying (2.1), we conclude that

$$
\begin{aligned}
d_k(AAx_{2n}, Bv) \le \varphi\Big(& d_k(AAx_{2n}, SAx_{2n}), d_k(Bv, Tv),\\
& d_k(SAx_{2n}, Tv), \frac{1}{2}\big[d_k(AAx_{2n}, Tv) + d_k(Bv, SAx_{2n})\big],\\
& \frac{d_k(AAx_{2n}, SAx_{2n})d_k(Bv, Tv)}{d(AAx_{2n}, Bv)+1}, \frac{d_k(AAx_{2n}, SAx_{2n})d_k(Bv, Tv)}{d_k(SAx_{2n}, Tv)+1},\\
& \frac{d_k(AAx_{2n}, Tv)d_k(Bv, SAx_{2n})}{d_k(AAx_{2n}, Bv)+1}, \frac{d_k(AAx_{2n}, Tv)d_k(Bv, SAx_{2n})}{d_k(SAx_{2n}, Tv)+1}\Big)
\end{aligned}
$$

for all $k \ge 1$ and $n \ge 1$. Letting $n \to \infty$ in the above inequality, we arrive at

$$
\begin{aligned}
d_k(Az, Bv) &\le \varphi\Big(0, d_k(Bv, Tv), d_k(Az, Tv), \frac{1}{2}\big[d_k(Az, Tv) + d_k(Bv, Az)\big], 0, 0,\\
&\qquad \frac{d_k(Az, Tv)d_k(Bv, Az)}{d_k(Az, Bv)+1}, \frac{d_k(Az, Tv)d_k(Bv, Az)}{d_k(Az, Tv)+1}\Big)\\
&\le \varphi(0, d_k(Bv, z), 0, d_k(Bv, z), 0, 0, 0, 0)\\
&\le \varphi_1(d_k(Bv, z)), \quad \forall k \ge 1,
\end{aligned}
$$

which yields that $d_k(Bv, z) = 0$ for all $k \ge 1$, thus $z = Bv$. Since B and T are compatible, it follows from Lemma 2.2 that $Tz = TBv = BTv = Bz$. Using (2.1) again, we gain that

$$
\begin{aligned}
d_k(Ax_{2n}, Bz) \le \varphi\Big(& d_k(Ax_{2n}, Sx_{2n}), d_k(Bz, Tz),\\
& d_k(Sx_{2n}, Tz), \frac{1}{2}\big[d_k(Ax_{2n}, Tz) + d_k(Bz, Sx_{2n})\big],\\
& \frac{d_k(Ax_{2n}, Sx_{2n})d_k(Bz, Tz)}{d_k(Ax_{2n}, Bz)+1}, \frac{d_k(Ax_{2n}, Sx_{2n})d_k(Bz, Tz)}{d_k(Sx_{2n}, Tz)+1},\\
& \frac{d_k(Ax_{2n}, Tz)d_k(Bz, Sx_{2n})}{d_k(Ax_{2n}, Bz)+1}, \frac{d_k(Ax_{2n}, Tz)d_k(Bz, Sx_{2n})}{d_k(Sx_{2n}, Tz)+1}\Big)
\end{aligned}
$$

for all $k \ge 1$ and $n \ge 1$. Taking the limit as $n \to \infty$ in the above inequality, we deduce that

$$
\begin{aligned}
d_k(z, Bz) &\le \varphi\Big(0, 0, d_k(z, Tz), \frac{1}{2}\big[d_k(z, Tz) + d_k(Bz, z)\big], 0, 0,\\
&\qquad \frac{d_k(z, Tz)d_k(Bz, z)}{d_k(z, Bz)+1}, \quad \frac{d_k(z, Tz)d_k(Bz, z)}{d_k(z, Tz)+1}\Big)\\
&\le \varphi(0, 0, d_k(z, Tz), d_k(z, Tz), 0, 0, d_k(z, Tz), d_k(z, Tz))\\
&\le \varphi_1(d_k(z, Bz)), \quad \forall k \ge 1,
\end{aligned}
$$

which leads to $d_k(z, Bz) = 0$ for all $k \ge 1$, therefore $z = Bz$. In view of $B(X) \subseteq S(X)$, there exists a point $w \in X$ such that $z = Bz = Sw$. It follows from (2.1) that

$$\begin{aligned} d_k(Aw,Bz) \le \varphi\Big(& d_k(Aw,Sw), d_k(Bz,Tz), d_k(Sw,Tz), \\ & \frac{1}{2}\big[d_k(Aw,Tz)+d_k(Bz,Sw)\big], \\ & \frac{d_k(Aw,Sw)d_k(Bz,Tz)}{d_k(Aw,Bz)+1}, \frac{d_k(Aw,Sw)d_k(Bz,Tz)}{d_k(Sw,Tz)+1}, \\ & \frac{d_k(Aw,Tz)d_k(Bz,Sw)}{d_k(Aw,Bz)+1}, \frac{d_k(Aw,Tz)d_k(Bz,Sw)}{d_k(Sw,Tz)+1}\Big) \\ = \varphi\Big(& d_k(Aw,z), 0, 0, \frac{1}{2}\big[d_k(Aw,z)+0\big], 0,0,0,0\Big) \\ \le \varphi_1&(d_k(Aw,z)), \quad \forall k \ge 1, \end{aligned}$$

which means that $d_k(Aw,Bz)=0$ for all $k \ge 1$, that is, $Aw=z$. Since A and S are compatible, it follows that $Sz=SAw=ASw=Az$ by Lemma 2.2. Therefore z is a common fixed point of A, B, S and T.

At last, we show that z is a unique common fixed point of A,B,S and T. Otherwise there exists another common fixed point $q \in X \setminus \{z\}$ of A,B,S and T. In term of (2.1), we know that

$$\begin{aligned} & d_k(q,z) = d_k(Aq,Bz) \\ & \le \varphi\Big(d_k(Aq,Sq), d_k(Bz,Tz), d_k(Sq,Tz), \frac{1}{2}\big[d_k(Aq,Tz)+d_k(Bz,Sq)\big], \\ & \quad \frac{d_k(Aq,Sq)d_k(Bz,Tz)}{d_k(Aq,Bz)+1}, \frac{d_k(Aq,Sq)d(Bz,Tz)}{d_k(Sq,Tz)+1}, \\ & \quad \frac{d_k(Aq,Tz)d_k(Bz,Sq)}{d_k(Aq,Bz)+1}, \frac{d_k(Aq,Tz)d_k(Bz,Sq)}{d_k(Sq,Tz)+1}\Big) \\ & = \varphi\Big(0,0,d_k(q,z), \frac{1}{2}\big[d_k(q,z)+d_k(z,q)\big], 0, 0, \frac{d_k(q,z)d_k(z,q)}{d_k(q,z)+1}, \frac{d_k(q,z)d_k(z,q)}{d_k(q,z)+1}\Big) \\ & \le \varphi(0,0,d_k(q,z),d_k(q,z),0,0,d_k(q,z),d_k(q,z)) \le \varphi_1(d_k(q,z)), \quad \forall k \ge 1, \end{aligned}$$

which implies that $d_k(q,z)=0$ for all $k \ge 1$. Hence $q=z$, which is absurd, therefore z is a unique common fixed point of A,B,S and T. Similarly, we can also complete the proof when B or S or T is continuous. □

Remark 2.1. Theorem 3.1 generalizes and improves Theorem 3.1 in [4] and Theorems 4 and 5 in [12].

3. An Application

Now we utilize the common fixed point theorem in this paper to deduce the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming. Throughout this section, let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ be real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-value mappings on S that are bounded on bounded subsets of S. It is easy to verify that $BB(S)$ is a linear space over $\mathbb{R}$ under usual definitions of addition and multiplication by scalars. For $k \geq 1$ and $a, b \in BB(S)$, let

$$d_k(a,b) = \sup\{|a(x) - b(x)| : x \in \overline{B}(0,k)\},$$

$$d(a,b) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a,b)}{1 + d_k(a,b)},$$

where $\overline{B}(0,k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k \geq 1}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space.

Theorem 3.1. *Let $u : S \times D \to S, T : S \times D \to S$ and H_1, H_2, H_3, $H_4 : S \times D \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

$(C1)$ for given $k \geq 1$ and $a \in BB(S)$, there exist $p(k,a) > 0$ with

$$|u(x,y)| + |H_i(x,y,a(T(x,y)))| \leq p(k,a) \tag{3.1}$$

for all $(x,y) \in \overline{B}(0,k) \times D$, $i \in \{1,2,3,4\}$;

$(C2)$ there exists $\varphi \in \Phi$ satisfying

$$\begin{aligned} |H_1(x,y,a(t)) - H_2(x,y,b(t))| \leq \varphi\Big(& d_k(f_1a, f_3a), d_k(f_2b, f_4b), d_k(f_3a, f_4b), \\ & \frac{1}{2}\big[d_k(f_1a, f_4b) + d_k(f_2b, f_3a)\big], \\ & \frac{d_k(f_1a, f_3a)d_k(f_2b, f_4b)}{d_k(f_1a, f_2b) + 1}, \frac{d_k(f_1a, f_3a)d_k(f_2b, f_4b)}{d_k(f_3a, f_4b) + 1}, \\ & \frac{d_k(f_1a, f_4b)d_k(f_2b, f_3a)}{d_k(f_1a, f_2b) + 1}, \frac{d_k(f_1a, f_4b)d_k(f_2b, f_3a)}{d_k(f_3a, f_4b) + 1}\Big) \end{aligned} \tag{3.2}$$

for $k \geq 1, (x,y,t) \in \overline{B}(0,k) \times D \times S$ and $a, b \in BB(S)$, where f_1, f_2, f_3 and f_4

are defined as follows:

$$\begin{aligned} f_1a(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_1(x,y,a(T(x,y)))\},\\ f_2a(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_2(x,y,a(T(x,y)))\},\\ f_3a(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_3(x,y,a(T(x,y)))\},\\ f_4a(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_4(x,y,a(T(x,y)))\} \end{aligned} \tag{3.3}$$

for $x\in S$ *and* $a\in BB(S)$*;*

$(C3)$ *for any* $\{a_n\}_{n\geq 1}\subset BB(S)$ *and* $k\geq 1$*, if there exist* $a,b\in BB(S)$ *such that*

$$\begin{aligned} \lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|f_1a_n(x)-a(x)| &= \lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|f_3a_n(x)-a(x)|=0,\\ \lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|f_2a_n(x)-b(x)| &= \lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|f_4a_n(x)-b(x)|=0, \end{aligned}$$

then

$$\lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|f_if_{i+2}a_n(x)-f_{i+2}f_ia_n(x)|=0,\ \ i=1,2;$$

$(C4)$ *one of* $f_1(BB(S)),f_2(BB(S)),f_3(BB(S))$ *and* $f_4(BB(S))$ *is a complete subspace of* $BB(S)$*;*

$(C5)$ *there exists some* $f_i\in\{f_1,f_2,f_3,f_4\}$ *such that for any sequence* $\{a_n\}_{n\geq 1}\subseteq BB(S)$*,* $a\in BB(S)$ *and* $k\geq 1$*,*

$$\lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|a_n(x)-a(x)|=0\Rightarrow\lim_{n\to\infty}\sup_{x\in\overline{B}(0,k)}|f_ia_n(x)-f_ia(x)|=0;$$

$(C6)$ $f_1(BB(S))\subseteq f_4(BB(S)),f_2(BB(S))\subseteq f_3(BB(S))$*.*

Then the system of functional equations

$$\begin{aligned} f_1(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_1(x,y,f_1(T(x,y)))\},\ \ x\in S,\\ f_2(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_2(x,y,f_2(T(x,y)))\},\ \ x\in S,\\ f_3(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_3(x,y,f_3(T(x,y)))\},\ \ x\in S,\\ f_4(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_4(x,y,f_4(T(x,y)))\},\ \ x\in S \end{aligned} \tag{3.4}$$

possesses a unique common solution in $BB(S)$*.*

Proof. It is easy to verify that $(C1)$ and (3.3) yield that f_1,f_2,f_3 and f_4 map $BB(S)$ into itself. $(C3)$ infers that f_1,f_3 and f_2,f_4 are compatible, and $(C5)$ means that f_i is continuous. Assume that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. Given $a, b \in BB(S),\ k \geq 1, x \in \overline{B}(0,k)$ and $\varepsilon > 0$. In terms of (3.3), we deduce that there exist $y, z \in D$ such that

$$f_1a(x) > u(x,y) + H_1(x,y,a(T(x,y))) - \varepsilon, \tag{3.5}$$

$$f_2b(x) > u(x,z) + H_2(x,z,b(T(x,z))) - \varepsilon. \tag{3.6}$$

It is easy to see that

$$f_1a(x) \leq u(x,z) + H_1(x,z,a(T(x,z))), \tag{3.7}$$

$$f_2b(x) \leq u(x,y) + H_2(x,y,b(T(x,y))). \tag{3.8}$$

By virtue of (3.6) and (3.7), we infer that

$$\begin{aligned} &f_1a(x) - f_2b(x) \\ &< H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z))) + \varepsilon \\ &\leq |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))| + \varepsilon. \end{aligned} \tag{3.9}$$

In terms of (3.5) and (3.8), we obtain that

$$\begin{aligned} &f_1a(x) - f_2b(x) \\ &> H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y))) - \varepsilon \\ &\geq -|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))| - \varepsilon. \end{aligned} \tag{3.10}$$

It follows from (3.2), (3.9) and (3.10) that

$$\begin{aligned} |f_1a(x) - f_2b(x)| \leq \varepsilon + \varphi\Big(& d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ & d_k(f_3a, f_4b), \frac{1}{2}\big[d_k(f_1a, f_4b) + d_k(f_2b, f_3a)\big], \\ & \frac{d_k(f_1a, f_3a)d_k(f_2b, f_4b)}{d_k(f_1a, f_2b)+1}, \frac{d_k(f_1a, f_3a)d_k(f_2b, f_4b)}{d_k(f_3a, f_4b)+1}, \\ & \frac{d_k(f_1a, f_4b)d_k(f_2b, f_3a)}{d_k(f_1a, f_2b)+1}, \frac{d_k(f_1a, f_4b)d_k(f_2b, f_3a)}{d_k(f_3a, f_4b)+1}\Big). \end{aligned} \tag{3.11}$$

Letting $\varepsilon \to 0$ in (3.11), we easily infer that

$$\begin{aligned} d_k(f_1a, f_2b) \leq \varphi\Big(& d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ & d_k(f_3a, f_4b), \frac{1}{2}\big[d_k(f_1a, f_4b) + d_k(f_2b, f_3a)\big], \\ & \frac{d_k(f_1a, f_3a)d_k(f_2b, f_4b)}{d_k(f_1a, f_2b)+1}, \frac{d_k(f_1a, f_3a)d_k(f_2b, f_4b)}{d_k(f_3a, f_4b)+1}, \\ & \frac{d_k(f_1a, f_4b)d_k(f_2b, f_3a)}{d_k(f_1a, f_2b)+1}, \frac{d_k(f_1a, f_4b)d_k(f_2b, f_3a)}{d_k(f_3a, f_4b)+1}\Big). \end{aligned} \tag{3.12}$$

Similarly, we conclude that (3.12) also holds for $\text{opt}_{y\in D} = \sup_{y\in D}$. Theorem 2.1 makes sure that f_1, f_2, f_3 and f_4 have a unique common fixed point $w \in BB(S)$. That is, w is a unique common solution of the system of functional equations (3.4). This completes the proof. □

Remark 3.1. Theorem 3.1 extends Theorem 3.1 in [10].

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