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CHARACTERIZATIONS OF COMMON FIXED POINTS FOR A PAIR OF MAPPINGS IN 2-METRIC SPACES

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Abstract: In this paper a few sufficient and necessary conditions to ensure the existence of common fixed points for a pair of arbitrary mappings in a complete 2-metric space are established, and a unique common fixed point theorem for two pairs of compatible mappings of type (P) is obtained. Our results improve and generalize several previously known results.

AMS Subject Classification: 54H25
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1. Introduction

In 1976, Jungck [10] provided some sufficient and necessary conditions for a continuous mapping to possess a fixed point in complete metric spaces. Chang [1],

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Fisher [5], Khan-Fisher [14] and Kubiak [16] employed commuting or weakly commuting mappings to establish criteria for the existence of common fixed points for a pair of continuous mappings in complete metric spaces. It is well known that the concepts of compatible mappings, compatible mappings of type (A) and compatible mappings of type (P) were introduced as a generalization of commuting mappings and weakly commuting mappings in metric spaces (see [11], [20]). Kang-Cho-Jungck [12] utilized compatible mappings to give criteria for the existence of common fixed points for a pair of continuous mappings in complete metric spaces. Liu [17] used compatible mappings to provide equivalent conditions which ensure the existence of common fixed points for a pair of arbitrary mappings in complete metric spaces.

In 1963, Gahler [6] introduced and investigated the concept and properties of a 2-metric space, respectively. In particular, he established that a 2-metric d is a continuous function of any one of its three arguments but it need not be continuous in two arguments. If it is continuous in any two arguments, it is continuous in all three arguments. A 2-metric d which is continuous in all of its arguments will be called continuous.

In 1975, Iséki [8], for the first time, developed a fixed point theorem in 2-metric spaces. Afterwards a quite number of authors (see [2], [4], [7]-[9], [13], [15], [18]-[33]) obtained various existence and uniqueness results of fixed points, common fixed points and coincidence points for certain classes of contractive and expansive type mappings in 2-metric spaces. Especially, Cho [2], Constantin [3] and Singh-Tiwari-Gupta [31] provided characterizations for continuous mappings to possess common fixed points by using compatible or commuting mappings in 2-metric spaces. Liu-Zhang [18], Liu-Zhang-Mao [19] and Tan-Liu-Kim [33] discussed criteria for a pair of any mappings to possess common fixed points by using compatible mappings, compatible mappings of type (A) and compatible mappings of type (P), respectively, in 2-metric spaces.

Motivated and inspired by the results in [2]-[4], [7]-[9], [13], [15], [11], [18]-[23], [31], [33], in this paper, we establish a few equivalent conditions of the existence of common fixed points for a pair of arbitrary mappings in a complete 2-metric space by using compatible mappings of type (P). Meanwhile, we obtain a sufficient condition which guarantees the existence and uniqueness of common fixed point for two pairs of compatible mappings of type (P) in a complete 2-metric space. Our results generalize, improve and unify the corresponding results of Chang [1], Fisher [5], Jungck [10], Khan-Fisher [15] and Kubiak [16].

2. Preliminaries

Throughout this paper ω and $\mathbb{N}$ denote the sets of nonnegative and positive integers, respectively. Let $\mathbb{R}^+ = [0, +\infty)$ and

$$W = \{w \mid w : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a continuous mapping such that } 0 < w(t) < t \text{ for all } t > 0\}.$$

Definition 2.1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a 2-metric space (X, d) is said to be *convergent* to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$. The point x is called the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.

Definition 2.2. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* if $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$ for all $a \in X$.

Definition 2.3. A 2-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Note that, in a 2-metric space (X, d), a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X (see [21]).

Definition 2.4. Let f and g be mappings from a 2-metric space (X, d) into itself. f and g are said to be *compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n, a) = 0, \quad \forall a \in X,$$

whenever $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$; f and g are said to be *compatible of type* (P) if

$$\lim_{n\to\infty} d(ffx_n, ggx_n, a) = 0, \quad \forall a \in X,$$

wherever $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.5. A mapping f from a 2-metric space (X, d) into itself is said to be *continuous* at $x \in X$ if for every sequence $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$, $\lim_{n\to\infty} d(fx_n, fx, a) = 0$. f is called *continuous* in X if it is so at all points of X.

The following lemma plays a crucial role in the proof of our main results.

Lemma 2.1. (see [23]) *Let f and g be compatible mappings of type (P) from a 2-metric spaces (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = ggt = gft = fft$.*

3. Characterizations of Common Fixed Points

In this section, we provide some criteria of the existence of common fixed points for a pair of arbitrary mappings in complete 2-metric spaces and establish a common fixed point theorem for two pairs of compatible mappings of type (P) in complete 2-metric spaces.

Theorem 3.1. *Let (X, d) be a complete 2-metric space with d continuous in X and let h and t be two any mappings from (X, d) into itself. Then the following statements are equivalent:*

(1) *h and t have a common fixed point in X;*

(2) *there exist $r \in (0,1)$, $f : X \to t(X)$ and $g : X \to h(X)$ such that:*

(c_1) *the pairs f, h and g, t are compatible of type (P),*

(c_2) *one of f, g, h and t is continuous,*

(c_3) $d(fx, gy, a)$

$$\leq r \max\Big\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}\big[d(hx, gy, a) + d(ty, fx, a)\big], \frac{1}{2}\big[d(hx, ty, a) + d(fx, gy, a)\big], \frac{d(hx, fx, a)d(ty, gy, a)}{1 + d(fx, gy, a)}, \frac{d(hx, gy, a)d(ty, fx, a)}{1 + d(fx, gy, a)}, \frac{d(hx, fx, a)d(ty, gy, a)}{1 + d(hx, ty, a)}, \frac{d(hx, gy, a)d(ty, fx, a)}{1 + d(hx, ty, a)}\Big\}, \quad \forall x, y, a \in X;$$

(3) *there exist $w \in W, f : X \to t(X)$ and $g : X \to h(X)$ satisfying (c_1), (c_2) and*

(c_4) $d(fx, gy, a)$

$$\leq \max\Big\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}\big[d(hx, gy, a) + d(ty, fx, a)\big], \frac{1}{2}\big[d(hx, ty, a) + d(fx, gy, a)\big], \frac{d(hx, fx, a)d(ty, gy, a)}{1 + d(fx, gy, a)}, \frac{d(hx, gy, a)d(ty, fx, a)}{1 + d(fx, gy, a)}, \frac{d(hx, fx, a)d(ty, gy, a)}{1 + d(hx, ty, a)}, \frac{d(hx, gy, a)d(ty, fx, a)}{1 + d(hx, ty, a)}\Big\}$$

$$
\begin{aligned}
- w\Big(\max \Big\{ & d(hx,ty,a), d(hx,fx,a), d(ty,gy,a), \\
& \frac{1}{2}\big[d(hx,gy,a)+d(ty,fx,a)\big], \frac{1}{2}\big[d(hx,ty,a)+d(fx,gy,a)\big], \\
& \frac{d(hx,fx,a)d(ty,gy,a)}{1+d(fx,gy,a)}, \frac{d(hx,gy,a)d(ty,fx,a)}{1+d(fx,gy,a)}, \\
& \frac{d(hx,fx,a)d(ty,gy,a)}{1+d(hx,ty,a)}, \frac{d(hx,gy,a)d(ty,fx,a)}{1+d(hx,ty,a)} \Big\}\Big), \quad \forall x,y,a \in X.
\end{aligned}
$$

Proof. (1) $\Rightarrow$ (2). Let z be a common fixed point of h and t in X. Define $f : X \to t(X)$ and $g : X \to h(X)$ by $fx = gx = z$ for all $x \in X$. Then (c_1) and (c_2) hold. For any $r \in (0,1)$, (c_3) also holds.

(2) $\Rightarrow$ (3). Let $w(t) = (1-r)t$ for all $t \in \mathbb{R}^+$. Clearly $w \in W$ and (c_3) implies (c_4).

(3) $\Rightarrow$ (1). Let x_0 be any point in X. According to $f(X) \subseteq t(X)$ and $g(X) \subseteq h(X)$, we choose a point $x_1 \in X$ satisfying $y_1 = tx_1 = fx_0$ and, for this point x_1, there exists a point $x_2 \in X$ such that $y_2 = hx_2 = gx_1$ and so on. Inductively, we define sequences $\{x_n\}_{n\in\omega}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X satisfying $y_{2n+1} = tx_{2n+1} = fx_{2n}$, $y_{2n+2} = hx_{2n+2} = gx_{2n+1}$ for all $n \in \omega$. Define $d_n(a) = d(y_n, y_{n+1}, a)$ for any $a \in X$ and $n \in \mathbb{N}$. We first show that

$$
\lim_{n\to\infty} d_n(a) = 0, \quad \forall a \in X. \tag{3.1}
$$

Assume that $d_{2n}(y_{2n+2}) > 0$ for some $n \in \mathbb{N}$. Indeed, by (c_4) we deduce that

$$
\begin{aligned}
& d_{2n}(y_{2n+2}) \\
& = d(fx_{2n}, gx_{2n+1}, y_{2n}) \\
& \le \max\Big\{0,0,d_{2n}(y_{2n+2}),0,\frac{1}{2}d_{2n}(y_{2n+2}),0,0,0,0\Big\} \\
& \quad - w\Big(\max\Big\{0,0,d_{2n}(y_{2n+2}),0,\frac{1}{2}d_{2n}(y_{2n+2}),0,0,0,0\Big\}\Big) \\
& = d_{2n}(y_{2n+2}) - w(d_{2n}(y_{2n+2})) < d_{2n}(y_{2n+2}),
\end{aligned}
$$

which is absurd. Hence $d_{2n}(y_{2n+2}) = 0$ for any $n \in \mathbb{N}$. In a similar manner, it can be shown that $d_{2n-1}(y_{2n+1}) = 0$ for all $n \in \mathbb{N}$. Consequently, $d_n(y_{n+2}) = 0$ for all $n \in \mathbb{N}$. Note that

$$
\begin{aligned}
d(y_n, y_{n+2}, a) &\le d_n(y_{n+2}) + d_n(a) + d_{n+1}(a) \\
&= d_n(a) + d_{n+1}(a), \quad \forall a \in X,\ n \in \mathbb{N}.
\end{aligned} \tag{3.2}
$$

Using (c_4) and (3.2) we conclude that

$$\begin{aligned}
&d_{2n+1}(a)\\
&= d(fx_{2n}, gx_{2n+1}, a)\\
&\le \max\Big\{d_{2n}(a), d_{2n}(a), d_{2n+1}(a), \frac{1}{2}d(y_{2n}, y_{2n+2}),\\
&\qquad \frac{1}{2}\big[d_{2n}(a)+d_{2n+1}(a)\big], \frac{d_{2n}(a)d_{2n+1}(a)}{1+d_{2n+1}(a)}, 0, \frac{d_{2n}(a)d_{2n+1}(a)}{1+d_{2n}(a)}, 0\Big\}\\
&\quad - w\Big(\max\Big\{d_{2n}(a), d_{2n}(a), d_{2n+1}(a), \frac{1}{2}d(y_{2n}, y_{2n+2}),\\
&\qquad \frac{1}{2}\big[d_{2n}(a)+d_{2n+1}(a)\big], \frac{d_{2n}(a)d_{2n+1}(a)}{1+d_{2n+1}(a)}, 0, \frac{d_{2n}(a)d_{2n+1}(a)}{1+d_{2n}(a)}, 0\Big\}\Big)\\
&\le \max\Big\{d_{2n}(a), d_{2n+1}(a), \frac{1}{2}\big[d_{2n}(a)+d_{2n+1}(a)\big]\Big\}\\
&\quad - w\Big(\max\Big\{d_{2n}(a), d_{2n+1}(a), \frac{1}{2}\big[d_{2n}(a)+d_{2n+1}(a)\big]\Big\}\Big)\\
&= \max\{d_{2n}(a), d_{2n+1}(a)\} - w(\max\{d_{2n}(a), d_{2n+1}(a)\}), \quad \forall a \in X,\ n \in \mathbb{N}.
\end{aligned}$$

If $d_{2n+1}(a) > d_{2n}(a)$ for some $a \in X$ and $n \in \mathbb{N}$ in the above inequalities, it is easy to verify that

$$d_{2n+1}(a) \le d_{2n+1}(a) - w(d_{2n+1}(a)) < d_{2n+1}(a),$$

a contradiction. Hence, we infer that $d_{2n+1}(a) \le d_{2n}(a)$ for any $a \in X$ and $n \in \mathbb{N}$ and so $d_{2n+1}(a) \le d_{2n}(a) - w(d_{2n}(a))$ for any $a \in X$ and $n \in \mathbb{N}$. Similarly, by (c_4) and (3.2) we also have $d_{2n}(a) \le d_{2n-1}(a) - w(d_{2n-1}(a))$ for any $a \in X$ and $n \in \mathbb{N}$. Consequently, $d_{n+1}(a) \le d_n(a) - w(d_n(a))$ for all $a \in X$ and $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
\sum_{i=1}^{n} w(d_i(a)) &\le \sum_{i=1}^{n}[d_i(a) - d_{i+1}(a)]\\
&= d_1(a) - d_{n+1}(a) \le d_1(a), \quad \forall a \in X,\ n \in \mathbb{N}
\end{aligned}$$

and

$$d_{n+1}(a) \le d_n(a) \le d_{n-1}(a) \le \cdots \le d_1(a), \quad \forall a \in X,\ n \in \mathbb{N}. \tag{3.3}$$

Further, the series $\sum_{n=1}^{\infty} w(d_n(a))$ and the sequence $\{d_n(a)\}_{n\in\mathbb{N}}$ are convergent. Thus $\lim_{n\to\infty} w(d_n(a)) = 0$ and there exists some point $p \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n(a) = p$. Invoking the continuity of w, we derive that

$$\lim_{n\to\infty} w(d_n(a)) = w(p) = 0,$$

which shows that $p = 0$. Hence (3.1) holds.

Let n, m be in $\mathbb{N}$. It follows from (3.3) that $0 = d_n(y_m)$ for $n \ge m$. Clearly we get that for $n < m$

$$\begin{aligned} d_n(y_m) &= d(y_n, y_{n+1}, y_m) \\ &\le d(y_n, y_{n+1}, y_{m-1}) + d(y_n, y_{m-1}, y_m) + d(y_{m-1}, y_{n+1}, y_m) \\ &= d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1}) \\ &\le d_n(y_{m-1}) + d_n(y_n) + d_n(y_{n+1}) \\ &= d_n(y_{m-1}) \le d_n(y_{m-2}) \le \cdots \le d_n(y_{n+1}) = 0. \end{aligned}$$

Thus, for all $n, m \in \mathbb{N}$, $d_n(y_m) = 0$.

Next we prove that for all $i, j, k \in \mathbb{N}$

$$d(y_i, y_j, y_k) = 0. \tag{3.4}$$

Without loss of generality, we may assume that $i \le j$. It follows that

$$\begin{aligned} d(y_i, y_j, y_k) &\le d(y_i, y_j, y_{i+1}) + d(y_i, y_{i+1}, y_k) + d(y_{i+1}, y_j, y_k) \\ &= d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k) \\ &= d(y_{i+1}, y_j, y_k) \le d(y_{i+2}, y_j, y_k) \\ &\le \cdots \le d(y_{j-1}, y_j, y_k) = d_{j-1}(y_k) = 0. \end{aligned}$$

Therefore (3.4) holds.

In order to show that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, it is sufficient to prove that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Given $\epsilon > 0$ and $a \in X$ such that, for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(y_{2m(k)}, y_{2n(k)}, a) \ge \epsilon$. Further, let $2m(k)$ denote the least even integer exceeding $2n(k)$ which satisfies that

$$d(y_{2m(k)-2}, y_{2n(k)}, a) \le \epsilon \quad \text{and} \quad d(y_{2m(k)}, y_{2n(k)}, a) > \epsilon. \tag{3.5}$$

Following (3.4) and (3.5), we deduce that

$$\begin{aligned} \epsilon &< d(y_{2m(k)}, y_{2n(k)}, a) \\ &\le d(y_{2m(k)-2}, y_{2n(k)}, a) + d(y_{2m(k)}, y_{2m(k)-2}, a) + d(y_{2m(k)}, y_{2n(k)}, y_{2m(k)-2}) \\ &\le \epsilon + d(y_{2m(k)}, y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2m(k)-1}, a) \\ &\quad + d(y_{2m(k)-1}, y_{2m(k)-2}, a) \\ &= \epsilon + d_{2m(k)-1}(a) + d_{2m(k)-2}(a), \end{aligned}$$

which means that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}, a) = \epsilon, \quad \forall a \in X. \tag{3.6}$$

Notice that for any $k \in \mathbb{N}$ and $a \in X$

$$|d(y_{2n(k)}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)}, a)| \leq d_{2m(k)-1}(a) + d_{2m(k)-1}(y_{2n(k)}),$$

$$|d(y_{2n(k)+1}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)}, a)| \leq d_{2n(k)}(a) + d_{2n(k)}(y_{2m(k)}),$$

$$|d(y_{2n(k)+1}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)-1}, a)| \leq d_{2n(k)}(a) + d_{2n(k)}(y_{2m(k)-1}).$$

It is easy to check that

$$\begin{aligned}&\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}, a) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}, a)\\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}, a) = \epsilon, \quad \forall a \in X.\end{aligned} \tag{3.7}$$

Using (c_4) and (3.4) again, we obtain that, $\forall a \in X$, $k \in \mathbb{N}$

$$\begin{aligned}
&d(y_{2n(k)}, y_{2m(k)}, a)\\
&\leq d(y_{2n(k)}, y_{2m(k)}, y_{2n(k)+1}) + d(y_{2n(k)}, y_{2n(k)+1}, a)\\
&\quad + d(y_{2n(k)+1}, y_{2m(k)}, a)\\
&= d_{2n(k)}(a) + d(fx_{2n(k)}, gx_{2m(k)-1}, a)\\
&\leq d_{2n(k)}(a) + \max\Big\{d(y_{2n(k)}, y_{2m(k)-1}, a), d_{2n(k)}(a), d_{2m(k)-1}(a),\\
&\qquad \frac{1}{2}\big[d(y_{2n(k)}, y_{2m(k)}, a) + d(y_{2m(k)-1}, y_{2n(k)+1}, a)\big],\\
&\qquad \frac{1}{2}\big[d(y_{2n(k)}, y_{2m(k)-1}, a) + d(y_{2n(k)+1}, y_{2m(k)}, a)\big],\\
&\qquad \frac{d_{2n(k)}(a)d_{2m(k)-1}(a)}{1 + d(y_{2n(k)+1}, y_{2m(k)}, a)}, \frac{d(y_{2n(k)}, y_{2m(k)}, a)d(y_{2m(k)-1}, y_{2n(k)+1}, a)}{1 + d(y_{2n(k)+1}, y_{2m(k)}, a)},\\
&\qquad \frac{d_{2n(k)}(a)d_{2m(k)-1}(a)}{1 + d(y_{2n(k)}, y_{2m(k)-1}, a)}, \frac{d(y_{2n(k)}, y_{2m(k)}, a)d(y_{2m(k)-1}, y_{2n(k)+1}, a)}{1 + d(y_{2n(k)}, y_{2m(k)-1}, a)}\Big\}\\
&\quad - w\Big(\max\Big\{d(y_{2n(k)}, y_{2m(k)-1}, a), d_{2n(k)}(a), d_{2m(k)-1}(a),\\
&\qquad \frac{1}{2}\big[d(y_{2n(k)}, y_{2m(k)}, a) + d(y_{2m(k)-1}, y_{2n(k)+1}, a)\big],\\
&\qquad \frac{1}{2}\big[d(y_{2n(k)}, y_{2m(k)-1}, a) + d(y_{2n(k)+1}, y_{2m(k)}, a)\big],\\
&\qquad \frac{d_{2n(k)}(a)d_{2m(k)-1}(a)}{1 + d(y_{2n(k)+1}, y_{2m(k)}, a)}, \frac{d(y_{2n(k)}, y_{2m(k)}, a)d(y_{2m(k)-1}, y_{2n(k)+1}, a)}{1 + d(y_{2n(k)+1}, y_{2m(k)}, a)},\\
&\qquad \frac{d_{2n(k)}(a)d_{2m(k)-1}(a)}{1 + d(y_{2n(k)}, y_{2m(k)-1}, a)}, \frac{d(y_{2n(k)}, y_{2m(k)}, a)d(y_{2m(k)-1}, y_{2n(k)+1}, a)}{1 + d(y_{2n(k)}, y_{2m(k)-1}, a)}\Big\}\Big).
\end{aligned}$$

Letting $k \to \infty$, by (3.1), (3.6) and (3.7) we immediately deduce that

$$\begin{aligned}\epsilon &\le \max\Big\{\epsilon, 0, 0, \epsilon, \epsilon, 0, \frac{\epsilon^2}{1+\epsilon}, 0, \frac{\epsilon^2}{1+\epsilon}\Big\}\\ &\quad - w\Big(\Big\{\epsilon, 0, 0, \epsilon, \epsilon, 0, \frac{\epsilon^2}{1+\epsilon}, 0, \frac{\epsilon^2}{1+\epsilon}\Big\}\Big)\\ &= \epsilon - w(\epsilon) < \epsilon,\end{aligned}$$

from this contradiction we obtain that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. It follows from completeness of (X,d) that $\{y_n\}_{n\in\mathbb{N}}$ converges to a point $u \in X$. Thus fx_{2n}, hx_{2n}, gx_{2n+1}, and $tx_{2n+1} \to u$ as $n \to \infty$. Now, suppose that f is continuous. Since f and h are compatible of type (P), by Lemma 2.1 we gain that $fhx_{2n}, hhx_{2n} \to fu$ as $n \to \infty$. In light of (c_4), we have

$$\begin{aligned}&d(fhx_{2n}, gx_{2n+1}, a)\\ &\le \max\Big\{d(hhx_{2n}, tx_{2n+1}, a), d(hhx_{2n}, fhx_{2n}, a), d(tx_{2n+1}, gx_{2n+1}, a),\\ &\qquad \frac{1}{2}\big[d(hhx_{2n}, gx_{2n+1}, a) + d(tx_{2n+1}, fhx_{2n}, a)\big],\\ &\qquad \frac{1}{2}\big[d(hhx_{2n}, tx_{2n+1}, a) + d(fhx_{2n}, gx_{2n+1}, a)\big],\\ &\qquad \frac{d(hhx_{2n}, fhx_{2n}, a)d(tx_{2n+1}, gx_{2n+1}, a)}{1 + d(fhx_{2n}, gx_{2n+1}, a)},\\ &\qquad \frac{d(hhx_{2n}, gx_{2n+1}, a)d(tx_{2n+1}, fhx_{2n}, a)}{1 + d(fhx_{2n}, gx_{2n+1}, a)},\\ &\qquad \frac{d(hhx_{2n}, fhx_{2n}, a)d(tx_{2n+1}, gx_{2n+1}, a)}{1 + d(hhx_{2n}, tx_{2n+1}, a)},\\ &\qquad \frac{d(hhx_{2n}, gx_{2n+1}, a)d(tx_{2n+1}, fhx_{2n}, a)}{1 + d(hhx_{2n}, tx_{2n+1}, a)}\Big\}\\ &\quad - w\Big(\max\Big\{d(hhx_{2n}, tx_{2n+1}, a), d(hhx_{2n}, fhx_{2n}, a), d(tx_{2n+1}, gx_{2n+1}, a),\\ &\qquad \frac{1}{2}\big[d(hhx_{2n}, gx_{2n+1}, a) + d(tx_{2n+1}, fhx_{2n}, a)\big],\\ &\qquad \frac{1}{2}\big[d(hhx_{2n}, tx_{2n+1}, a) + d(fhx_{2n}, gx_{2n+1}, a)\big],\\ &\qquad \frac{d(hhx_{2n}, fhx_{2n}, a)d(tx_{2n+1}, gx_{2n+1}, a)}{1 + d(fhx_{2n}, gx_{2n+1}, a)},\\ &\qquad \frac{d(hhx_{2n}, gx_{2n+1}, a)d(tx_{2n+1}, fhx_{2n}, a)}{1 + d(fhx_{2n}, gx_{2n+1}, a)},\end{aligned}$$

$$\frac{d(hhx_{2n}, fhx_{2n}, a)d(tx_{2n+1}, gx_{2n+1}, a)}{1 + d(hhx_{2n}, tx_{2n+1}, a)},$$
$$\frac{d(hhx_{2n}, gx_{2n+1}, a)d(tx_{2n+1}, fhx_{2n}, a)}{1 + d(hhx_{2n}, tx_{2n+1}, a)}\Big\}\Big), \quad \forall a \in X,\ n \in \mathbb{N}.$$

Letting $n \to \infty$, we infer that

$$\begin{aligned} d(fu, u, a) &\le \max\Big\{d(fu, u, a), 0, 0, d(fu, u, a), d(fu, u, a), \\ &\qquad 0, \frac{d^2(fu, u, a)}{1 + d(fu, u, a)}, 0, \frac{d^2(fu, u, a)}{1 + d(fu, u, a)}\Big\} \\ &\qquad - w\Big(\max\Big\{d(fu, u, a), 0, 0, d(fu, u, a), d(fu, u, a), \\ &\qquad 0, \frac{d^2(fu, u, a)}{1 + d(fu, u, a)}, 0, \frac{d^2(fu, u, a)}{1 + d(fu, u, a)}\Big\}\Big) \\ &= d(fu, u, a) - w(d(fu, u, a)), \quad \forall a \in X, \end{aligned}$$

which implies that $fu = u$. Notice that $f(X) \subseteq t(X)$. Thus there exists a point $v \in X$ with $u = tv$. It follows from (c_4) that, $\forall a \in X$, $n \in \mathbb{N}$

$$\begin{aligned} & d(fx_{2n}, gv, a) \\ & \le \max\Big\{d(hx_{2n}, tv, a), d(hx_{2n}, fx_{2n}, a), d(tv, gv, a), \\ & \qquad \frac{1}{2}\big[d(hx_{2n}, gv, a) + d(tv, fx_{2n}, a)\big], \\ & \qquad \frac{1}{2}\big[d(hx_{2n}, tv, a) + d(fx_{2n}, gv, a)\big], \\ & \qquad \frac{d(hx_{2n}, fx_{2n}, a)d(tv, gv, a)}{1 + d(fx_{2n}, gv, a)}, \frac{d(hx_{2n}, gv, a)d(tv, fx_{2n}, a)}{1 + d(fx_{2n}, gv, a)}, \\ & \qquad \frac{d(hx_{2n}, fx_{2n}, a)d(tv, gv, a)}{1 + d(hx_{2n}, tv, a)}, \frac{d(hx_{2n}, gv, a)d(tv, fx_{2n}, a)}{1 + d(hx_{2n}, tv, a)}\Big\} \\ & \quad - w\Big(\max\Big\{d(hx_{2n}, tv, a), d(hx_{2n}, fx_{2n}, a), d(tv, gv, a), \\ & \qquad \frac{1}{2}\big[d(hx_{2n}, gv, a) + d(tv, fx_{2n}, a)\big], \\ & \qquad \frac{1}{2}\big[d(hx_{2n}, tv, a) + d(fx_{2n}, gv, a)\big], \\ & \qquad \frac{d(hx_{2n}, fx_{2n}, a)d(tv, gv, a)}{1 + d(fx_{2n}, gv, a)}, \frac{d(hx_{2n}, gv, a)d(tv, fx_{2n}, a)}{1 + d(fx_{2n}, gv, a)}, \\ & \qquad \frac{d(hx_{2n}, fx_{2n}, a)d(tv, gv, a)}{1 + d(hx_{2n}, tv, a)}, \frac{d(hx_{2n}, gv, a)d(tv, fx_{2n}, a)}{1 + d(hx_{2n}, tv, a)}\Big\}\Big). \end{aligned}$$

As $n \to \infty$ in the above inequality, we obtain that

$$\begin{aligned} d(u,gv,a) &\le \max\Big\{0,0,d(u,gv,a),\frac{1}{2}d(u,gv,a),\frac{1}{2}d(u,gv,a),0,0,0,0\Big\} \\ &\quad - w\Big(\max\Big\{0,0,d(u,gv,a),\frac{1}{2}d(u,gv,a),\frac{1}{2}d(u,gv,a),0,0,0,0\Big\}\Big) \\ &= d(u,gv,a) - w(d(u,gv,a)), \quad \forall a \in X, \end{aligned}$$

which means that $u = gv = tv = fu$. It follows from Lemma 2.1 that $gu = gtv = tgv = tu$. Using (c_4) we gain that, $\forall a \in X$, $n \in \mathbb{N}$

$$\begin{aligned} &d(fx_{2n},gu,a) \\ &\le \max\Big\{d(hx_{2n},tu,a),d(hx_{2n},fx_{2n},a),d(tu,gu,a), \\ &\quad \frac{1}{2}\big[d(hx_{2n},gu,a)+d(tu,fx_{2n},a)\big], \\ &\quad \frac{1}{2}\big[d(hx_{2n},tu,a)+d(fx_{2n},gu,a)\big], \\ &\quad \frac{d(hx_{2n},fx_{2n},a)d(tu,gu,a)}{1+d(fx_{2n},gu,a)},\frac{d(hx_{2n},gu,a)d(tu,fx_{2n},a)}{1+d(fx_{2n},gu,a)}, \\ &\quad \frac{d(hx_{2n},fx_{2n},a)d(tu,gu,a)}{1+d(hx_{2n},tu,a)},\frac{d(hx_{2n},gu,a)d(tu,fx_{2n},a)}{1+d(hx_{2n},tu,a)}\Big\} \\ &\quad - w\Big(\max\Big\{d(hx_{2n},tu,a),d(hx_{2n},fx_{2n},a),d(tu,gu,a), \\ &\quad \frac{1}{2}\big[d(hx_{2n},gu,a)+d(tu,fx_{2n},a)\big], \\ &\quad \frac{1}{2}\big[d(hx_{2n},tu,a)+d(fx_{2n},gu,a)\big], \\ &\quad \frac{d(hx_{2n},fx_{2n},a)d(tu,gu,a)}{1+d(fx_{2n},gu,a)},\frac{d(hx_{2n},gu,a)d(tu,fx_{2n},a)}{1+d(fx_{2n},gu,a)}, \\ &\quad \frac{d(hx_{2n},fx_{2n},a)d(tu,gu,a)}{1+d(hx_{2n},tu,a)},\frac{d(hx_{2n},gu,a)d(tu,fx_{2n},a)}{1+d(hx_{2n},tu,a)}\Big\}\Big). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we conclude that

$$\begin{aligned} d(u,gu,a) \le \max\Big\{&d(u,gu,a),0,0,d(u,gu,a),d(u,gu,a), \\ &0,\frac{d^2(u,gu,a)}{1+d(u,gu,a)},0,\frac{d^2(u,gu,a)}{1+d(u,gu,a)}\Big\} \end{aligned}$$

$$
\begin{aligned}
&- w\Big(\max\Big\{ d(u,gu,a), 0, 0, d(u,gu,a), d(u,gu,a), \\
&\qquad 0, \frac{d^2(u,gu,a)}{1+d(u,gu,a)}, 0, \frac{d^2(u,gu,a)}{1+d(u,gu,a)} \Big\}\Big) \\
&= d(u,gu,a) - w(d(u,gu,a)), \quad \forall a \in X,
\end{aligned}
$$

which gives that $u = gu$. Since $g(X) \subseteq h(X)$, there is $w \in X$ with $u = hw$. In terms of (C_4), we get that

$$
\begin{aligned}
& d(fw,u,a) \\
&= d(fw,gu,a) \\
&\le \max\Big\{ d(hw,tu,a), d(hw,fw,a), d(tu,gu,a), \\
&\quad \frac{1}{2}\big[d(hw,gu,a)+d(tu,fw,a)\big], \frac{1}{2}\big[d(hw,tu,a)+d(fw,gu,a)\big], \\
&\quad \frac{d(hw,fw,a)d(tu,gu,a)}{1+d(fw,gu,a)}, \frac{d(hw,gu,a)d(tu,fw,a)}{1+d(fw,gu,a)}, \\
&\quad \frac{d(hw,fw,a)d(tu,gu,a)}{1+d(hw,tu,a)}, \frac{d(hw,gu,a)d(tu,fw,a)}{1+d(hw,tu,a)} \Big\} \\
&\quad - w\Big(\max\Big\{ d(hw,tu,a), d(hw,fw,a), d(tu,gu,a), \\
&\qquad \frac{1}{2}\big[d(hw,gu,a)+d(tu,fw,a)\big], \frac{1}{2}\big[d(hw,tu,a)+d(fw,gu,a)\big], \\
&\qquad \frac{d(hw,fw,a)d(tu,gu,a)}{1+d(fw,gu,a)}, \frac{d(hw,gu,a)d(tu,fw,a)}{1+d(fw,gu,a)}, \\
&\qquad \frac{d(hw,fw,a)d(tu,gu,a)}{1+d(hw,tu,a)}, \frac{d(hw,gu,a)d(tu,fw,a)}{1+d(hw,tu,a)} \Big\}\Big) \\
&= \max\Big\{ 0, d(u,fw,a), 0, \frac{1}{2}d(u,fw,a), \frac{1}{2}d(u,fw,a), 0,0,0,0 \Big\} \\
&\quad - w\Big(\max\Big\{ 0, d(u,fw,a), 0, \frac{1}{2}d(u,fw,a), \frac{1}{2}d(u,fw,a), 0,0,0,0 \Big\}\Big) \\
&= d(u,fw,a) - w(d(u,fw,a)), \quad \forall a \in X,
\end{aligned}
$$

which implies that $fw = u$. Hence $fw = hw$. Lemma 2.1 ensures that $u = fu = fhw = hfw = hu$. That is, f, g, h and t have a common fixed point $u \in X$. Similarly, we can complete the proof when g or h or t is continuous. This completes the proof. □

Clearly we have the following results.

Theorem 3.2. *Let f, g, h and t be mappings from a complete 2-metric space (X, d) into itself, where the 2-metric d is continuous in X, satisfying conditions (c_1) and (c_2). If at least one of conditions (c_3) and (c_4) holds, then*

f, g, h and t have a unique common fixed point in X.

Theorem 3.3. *Let (X, d) be a complete 2-metric space with d continuous in X and let h and t be two arbitrary mappings from (X, d) into itself. Then (1) is equivalent to each of the following conditions:*

(4) there exist $r \in (0, 1)$, $f : X \to t(X) \cap h(X)$ such that

(c_5) the pairs f, h and f, t are compatible of type (P),

(c_6) one of f, h and t is continuous,

(c_7) $d(fx, fy, a)$

$$\begin{aligned}&\leq r\max\Big\{d(hx,ty,a), d(hx,fx,a), d(ty,fy,a),\\ &\quad \frac{1}{2}\big[d(hx,fy,a)+d(ty,fx,a)\big], \frac{1}{2}\big[d(hx,ty,a)+d(fx,fy,a)\big],\\ &\quad \frac{d(hx,fx,a)d(ty,fy,a)}{1+d(fx,fy,a)}, \frac{d(hx,fy,a)d(ty,fx,a)}{1+d(fx,fy,a)},\\ &\quad \frac{d(hx,fx,a)d(ty,fy,a)}{1+d(hx,ty,a)}, \frac{d(hx,fy,a)d(ty,fx,a)}{1+d(hx,ty,a)}\Big\}, \quad \forall x,y,a \in X;\end{aligned}$$

(5) there exist $w \in W$ and $f : X \to t(X) \cap h(X)$ satisfying (c_5), (c_6) and

(c_8) $d(fx, fy, a)$

$$\begin{aligned}&\leq \max\Big\{d(hx,ty,a), d(hx,fx,a), d(ty,fy,a),\\ &\quad \frac{1}{2}\big[d(hx,fy,a)+d(ty,fx,a)\big], \frac{1}{2}\big[d(hx,ty,a)+d(fx,fy,a)\big],\\ &\quad \frac{d(hx,fx,a)d(ty,fy,a)}{1+d(fx,fy,a)}, \frac{d(hx,fy,a)d(ty,fx,a)}{1+d(fx,fy,a)},\\ &\quad \frac{d(hx,fx,a)d(ty,gy,a)}{1+d(hx,ty,a)}, \frac{d(hx,fy,a)d(ty,fx,a)}{1+d(hx,ty,a)}\Big\}\\ &\quad - w\Big(\max\Big\{d(hx,ty,a), d(hx,fx,a), d(ty,fy,a),\\ &\quad \frac{1}{2}\big[d(hx,fy,a)+d(ty,fx,a)\big], \frac{1}{2}\big[d(hx,ty,a)+d(fx,fy,a)\big],\\ &\quad \frac{d(hx,fx,a)d(ty,fy,a)}{1+d(fx,fy,a)}, \frac{d(hx,fy,a)d(ty,fx,a)}{1+d(fx,fy,a)},\\ &\quad \frac{d(hx,fx,a)d(ty,fy,a)}{1+d(hx,ty,a)}, \frac{d(hx,fy,a)d(ty,fx,a)}{1+d(hx,ty,a)}\Big\}\Big), \quad \forall x,y,a \in X.\end{aligned}$$

Remark 3.1. Both Theorem 2 of Khan-Fisher [13] and Theorem 1 of Kubiak [15] are all the special cases of Theorem 3.2. Theorem 3.1 and Theorem 3.3 are generalizations and extensions of Theorem 1 of [1], Theorem 2 of [5] and

Theorem of [10].

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