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ASYMPTOTIC PROPERTIES OF DIFFERENTIAL EQUATIONS WITH INVOLUTIONS

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Abstract: This paper addresses the asymptotic behavior of a class of functional differential equations whose arguments transforms are involutions. Earlier work on asymptotic behavior involves only those involutions which have fixed points, and are consequently decreasing. In contrast we admit involutions without fixed points, allowing the possibility of non-decreasing. The monotonicity of the argument transform significantly influences the asymptotic behavior in some cases. Stability near equilibrium points is also studied.

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1. Introduction

Knowing the asymptotic behavior of select classes of differential equations is a useful step in developing a general theory. Differential equations with involutions is a case in point. Solutions of modified Wiener equations are introduced in [6]. The asymptotic behavior of the solution for some Wiener equations is predicted by the monotonicity of the argument transform.

A function $f(x) \not\equiv x$, that maps a set, G, of real numbers onto itself and satisfies on G the condition

$$f(f(x)) = x, \qquad \text{or} \qquad f^{-1}(x) = f(x), \tag{1}$$

is called an *involution on G*. The simplest examples of involutions are

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$$\begin{aligned} f(x) = -x \quad &\text{or} \quad f(x) = c - x, \quad x \in (-\infty, \infty), \quad c \in \mathbb{R}, \\ f(x) = 1/x \quad &\text{or} \quad f(x) = -1/x, \quad x \in (-\infty, 0) \cup (0, \infty). \end{aligned} \tag{2}$$

Definition 1. A relation of the form

$$F(x, y(f_1(x)), \ldots, y(f_m(x)), \ldots, y^{(n)}(f_1(x)), \ldots, y^{(n)}(f_m(x))) = 0, \tag{3}$$

where $f_i(f_i(x)) = x$ for every i, and $f_i(x) \not\equiv x$ for some i, is called a *differential equation with involutions*.

We investigate differential equations of the form

$$y'(x) = F(x, y(x), y(f(x))), \tag{4}$$

where $f(x)$ is an involution. The properties of involutions that are key to our discussion are proven in [6].

Property 2. Suppose $f(x)$ is a continuous involution on an open connected set, G. Then f has a unique fixed point on G, and f is decreasing on G.

Definition 3. If $x \in \mathcal{D}$, where $\mathcal{D}$ is the domain of an involution f, then the *component* $\mathcal{K}(x)$ of $\mathcal{D}$ is the maximally connected subset of $\mathcal{D}$ that contains x. Two components, $\mathcal{K}(x_1)$ and $\mathcal{K}(x_2)$, are said to be f-complements if $f(\mathcal{K}(x_1)) = \mathcal{K}(x_2)$.

It is observed that $f(\mathcal{K}(x)) = \mathcal{K}(f(x))$. Furthermore, whenever $f(x_1) = x_2$, then $\mathcal{K}(x_1)$ and $\mathcal{K}(x_2)$ are f-complements. The following statement is a corollary of Property 2. It highlights the fact that when f has no fixed point, the f-complements are disjoint. This is a key idea in understanding the theorems that follow.

Property 4. Suppose $f(x)$ is a continuous involution on $\mathcal{D} \subseteq \mathbb{R}$, and f has no fixed point. Then, for $x_0 \in \mathcal{D}$, $f(\mathcal{K}(x_0)) \cap \mathcal{K}(x_0)$ is empty. □

A consequence of the properties is that the domain of an increasing involution is not connected.

2. Main Results

A theorem from [6] establishes the existence of twice differentiable solutions of a class of differential equations with involution:

Theorem 5. *Assume that in the problem*

$$y'(x) = F(y(f(x))), \quad y(x_0) = y_0, \tag{5}$$

the function $f(x)$ is a continuously differentiable involution on its connected domain, $\mathcal{D}$; and the function F is defined, continuously differentiable, and strictly monotonic on $y(\mathcal{D})$. Then the solution of the ordinary differential equation

$$y''(x) = F'(y(f(x)))F(y(x))f'(x), \quad y(f(x)) = F^{-1}(y'(x)) \tag{6}$$

with the boundary conditions

$$y(x_0) = y_0, \quad y'(f(x_0)) = F(y_0) \tag{7}$$

is a solution of equation (5). □

The simplest example of an equation satisfying the hypothesis of the theorem is Silberstein's equation [5]. Silberstein solves the equation

$$y'(x) = y\left(\frac{1}{x}\right), \quad 0 < x < \infty, \tag{8}$$

by assuming a solution of the form

$$y(x) = x^k + \lambda x^m, \tag{9}$$

where k, m, and λ are constants. Wiener [7] obtains a solution by differentiating (8),

$$y''(x) = -\frac{1}{x^2}y'\left(\frac{1}{x}\right) = -\frac{1}{x^2}y(x), \tag{10}$$

whence,

$$x^2y''(x) + y(x) = 0. \tag{11}$$

This is a Cauchy-Euler equation, with the general solution of the form

$$y(x) = A_1x^{r_1} + A_2x^{r_2}. \tag{12}$$

Hence r_1 and r_2 are the roots of

$$r^2 - r + 1 = 0, \tag{13}$$

which are complex, making equation (12) oscillatory.

By modifying the domain of the involution in Silberstein's equation, we obtain an illustrative example. Consider the differential equation

$$y'(x) = y(f(x)), \quad f(x) = \frac{1}{x}, \qquad x \in (0,1) \cup (1,\infty), \tag{14}$$

with the initial conditions

$$y(2) = y_2, \quad y'(2) = \tilde{y}_2. \tag{15}$$

It is important that the two components of the domain are f-complements. That means that compatible conditions generated from the initial conditions are in the other component and hence do not effect the continuity of the solution in the original component. Therefore, we seek a solution of the form

$$y(x) = \begin{cases} C_1 x^{r_1} + C_2 x^{r_2}, & x \in (0,1) \\ D_1 x^{r_1} + D_2 x^{r_2}, & x \in (1,\infty). \end{cases} \tag{16}$$

The initial conditions uniquely determine D_1 and D_2 by the system of equations

$$\begin{aligned} y(2) = y_2 &= 2^{r_1} D_1 + 2^{r_2} D_2\,, \\ y'(2) = \tilde{y}_2 &= r_1 2^{-r_2} D_1 + r_2 2^{-r_1} D_2. \end{aligned} \tag{17}$$

The determinant of this system is $r_2 - r_1$ hence there is a unique solution for D_1 and D_2. The differential equation implies that $y'(1/x) = y(x)$, therefore the initial conditions in (15) generate compatible conditions

$$y(1/2) = \tilde{y}_2, \quad y'(1/2) = y_2 \tag{18}$$

which produce the system of equations

$$\begin{aligned} y(1/2) = \tilde{y}_2 &= 2^{-r_1} C_1 + 2^{-r_2} C_2\,, \\ y'(1/2) = y_2 &= r_1 2^{r_2} C_1 + r_2 2^{r_1} C_2, \end{aligned} \tag{19}$$

which also has a unique solution. It is clear that the solution is oscillatory in the $(1,\infty)$ component as $x \longrightarrow \infty$ and in the $(0,1)$ component as $x \longrightarrow 0$. This problem hints at a more general theorem.

Theorem 6. *If $a > 0$ and $f(x)$ is an involution which is continuosly differentiable on $[a,\infty)$; and for every $b > a$, $x > b$ implies*

$$f(x) - f(b) < \frac{1}{x} - \frac{1}{b}. \tag{20}$$

Then

$$y'(x) = y(f(x)) \tag{21}$$

has an oscillatory solution.

Proof. Because $f(x)$ is an involution (and hence monotone on components of its domain) and the right hand side of (20) is negative whenever $x > b$, we see that $f(x)$ is decreasing on $[a,\infty)$. Furthermore, (20) implies

$$\frac{f(x) - f(b)}{x - b} < \frac{1/x - 1/b}{x - b} \tag{22}$$

or

$$f'(x) = \lim_{b \longrightarrow x} \frac{f(x) - f(b)}{x - b} \leq \lim_{b \longrightarrow x} \frac{1/x - 1/b}{x - b} = \frac{-1}{x^2}. \tag{23}$$

Differentiating equation (21) yields $y''(x) = f'(x) y'(f(x))$. And because $y'(f(x)) = y(x)$ this reduces to a second order ordinary differential equation

$$y''(x) - f'(x) y(x) = 0. \tag{24}$$

The continuity of $f'(x)$ guarantees a unique solution, $y(x)$ on (a,∞). The

Cauchy-Euler equation

$$y''(x) + y(x)/x^2 = 0 \tag{25}$$

oscillates and has the property that $1/x^2 \leq -f'(x)$, therefore the solution of (21), $y(x)$, oscillates by the Oscillation Theorem [3, Appendix A.7, p. 654-655]. □

In contrast to the preceding theorem, conditions that cause $f'(x)$ to be positive guarantee the solutions of equation (21) do not oscillate. We have the following theorem.

Theorem 7. *Suppose $f(x)$ is a differentiable involution defined on $(\infty, a) \cup (a, \infty)$, f does not extend to a continuous function of $\mathbb{R}$, and f has no fixed point. Then non-trivial solutions of*

$$y'(x) = y(f(x)) \tag{26}$$

have at most one zero in (a, ∞).

Proof. The function f has an asymptote at $x = a$, otherwise

$$\lim_{x \longrightarrow a^-} f(x) = \lim_{x \longrightarrow a^+} f(x) = a \tag{27}$$

and hence f extends to a continuous function on $\mathbb{R}$ with a fixed point at $x = a$. Furthermore, f is increasing on (a, ∞), otherwise $\lim_{x \longrightarrow a^+} = \infty$ and since involutions are monotone on components, f then has a fixed point–again a contradiction. Therefore $f'(x) > 0$ on (a, ∞). Differentiating and and simplifying reduces the equation to the Cauchy-Euler equation

$$y''(x) - f'(x)y(x) = 0, \tag{28}$$

which has at most one zero on (a, ∞) because $f'(x) > 0$, again by the Oscillation Theorem [3]. □

Equations of the form

$$x^m y'(x) = ay\left(\frac{-1}{x}\right), \tag{29}$$

with the integer $m > 0$ are studied in [6]. Differentiating and simplifying reduces to a second order ordinary differential equation

$$x^2 y''(x) + mxy'(x) - (-1)^m a^2 y(x) = 0, \tag{30}$$

which has the characteristic equation

$$r^2 + (m-1)r - (-1)^m a^2 = 0. \tag{31}$$

In the case that m is even, the roots of (31) are real, hence all twice differentiable solutions of (29) do not oscillate. On the other hand, when m is odd and $(m-1)/2 < |a|$ the solutions oscillate.

3. Sharkovskii Equations

Following Sharkovskii [4] (also see [8, p. 221]) we investigate

$$\frac{dx}{dt} = ax(t) + bx(-t) + h(x(t), x(-t)). \tag{32}$$

Let,

$$u_1(t) = x(t) \text{ and } u_2(t) = x(-t), \tag{33}$$

then

$$\frac{du_1}{dt} = au_1 + bu_2 + h(u_1, u_2) \text{ and } \frac{du_2}{dt} = -bu_2 - au_1 - h(u_2, u_2). \tag{34}$$

It will be convenient to write this as the vector equation

$$\frac{d}{dt}u = Au\,, \tag{35}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}. \tag{36}$$

Cronin [2] summarizes the solutions of such systems. She categorizes the solutions according the eigenvalues of matrix A. The essential idea is to decouple the system by finding matrix B such that BAB^{-1} is a diagonal matrix, then the system

$$\frac{d}{dt}v = BAB^{-1}v \tag{37}$$

reveals the qualitative behavior of the system (35).

Ours is a very special case of a Cronin equation. In fact, the solution, when it exists, must have the property that $u_1(-t) = u_2(t)$. From this restriction we discover that the only equations of the form equation (32) yielding solutions via the Sharkovskii method, are those where $a = b$ or $a = -b$.

Assume $a^2 \neq b^2$, i.e. A is nonsingular, then the eigenvalues of A are $\pm\lambda$ where

$$\lambda = \sqrt{a^2 - b^2}. \tag{38}$$

Then

$$v(t) = \begin{pmatrix} ke^{t\lambda} \\ ke^{-t\lambda} \end{pmatrix} \tag{39}$$

is the general solution of

$$\frac{dv}{dt} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} v\,. \tag{40}$$

Representative eigenvectors of A associated with λ and $-\lambda$ are $(-b, a-\lambda)^T$ and

$(-b, a+\lambda)^T$ respectively. The matrix

$$B = \frac{1}{\lambda}\begin{pmatrix} -b & b \\ a-\lambda & -a-\lambda \end{pmatrix} \tag{41}$$

maps $(\lambda, 0)^T$ and $(0, -\lambda)^T$ to $(-b, a-\lambda)^T$ and $(-b, a+\lambda)^T$ respectively. Therefore

$$u(t) = Bv(t) \tag{42}$$

is the solution of equation (35). Therefore $u_1 = \frac{k}{\lambda}(-be^{t\lambda} + be^{-t\lambda})$ and $u_2 = \frac{k}{\lambda}((a-\lambda)e^{t\lambda} - (a+\lambda)e^{-t\lambda})$. If $u_1(-t) = u_2(t)$ then $b = a-\lambda = a+\lambda$, hence $\lambda = 0$, which contradicts the assumption that A is nonsingular.

On the other hand, if A is singular, then $a^2 = b^2$ leaving two choices for A. Either

$$A = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \text{ or } A = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}. \tag{43}$$

For each of these choices $A^2 = 0$ and $u = p + Apt$ is the solution of equation (35).

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