International Journal of Pure and Applied Mathematics
Volume 44 No. 4 2008, 493-507

MULTIPLICITY OF SOLUTIONS FOR QUASILINEAR BREZIS-NIRENBERG TYPE PROBLEMS INVOLVING SINGULAR WEIGHTS

Xiaoming He[1 §], Wenming Zou[2]
[1,2]Department of Mathematical Sciences
Tsinghua University
Beijing, 100084, P.R. CHINA
[1]e-mail: xmhe923@126.com
[2]e-mail: wzou@math.tsinghua.edu.cn

Abstract: In this paper we investigate the existence and multiplicity of solutions for quasilinear Brezis-Nirenberg type problem with critical Sobolev exponents and singular coefficients by using the cohomological index theory of Fadell and Rabinawitz [9].

AMS Subject Classification: 35J60, 35J25
Key Words: Brezis-Nirenberg problem, critical Sobolev exponent, existence and multiplicity, cohomological index

1. Introduction and Main Results

In this paper we consider the quasilinear Brezis-Nirenberg type problem with singular weights

$$\begin{cases} -\operatorname{div}\left(\frac{|Du|^{p-2}Du}{|x|^{pa}}\right) = \beta\frac{|u|^{p^*-2}u}{|x|^{bp^*}} + \lambda\frac{|u|^{p-2}u}{|x|^{p(a+1)-c}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < N$ and $0 \in \Omega, -\infty < a < \frac{N-p}{p}, a \le b < a+1, p^* = \frac{Np}{N-dp}, d = a+1-b \in (0,1], c > pd$, and $\lambda \in \mathbb{R}, \beta > 0$ are two parameters.

When $\beta = 1$ and $f(x,u) \equiv 0$, Xuan [1] recently studied the existence and

Received: February 1, 2008
§Correspondence author

non-existence of nontrivial solutions to problem (1.1). In [2] He considered the singular equation.

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|Du|^{p-2}Du) = \lambda|x|^{-p(a+1)+c}|u|^{p-2}u + |x|^{-bq}f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < p < N, 0 \leq a < (N-p)/p, a \leq b \leq a+1,\ q < p^* = Np/(N-dp), d = 1+a-b \in [0,1], c > 0$. Using the mountain pass lemma and linking arguments [12], [16], He obtained a nontrivial solution for (1.2) with $\lambda \in (0, \lambda_1)$ or $\lambda \in (\lambda_1, \lambda_2)$, where λ_1, λ_2 is the first eigenvalue and the second one of the nonlinear eigenvalue problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|Du|^{p-2}Du) = \lambda|x|^{-p(a+1)+c}|u|^{p-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

respectively (cf. [1]).

For $a = b = 0, c = p$, there are many existence results for such type of problems, we refer the readers to [4], [8], [12]-[14], [16] for $p = 2$, and [2], [3], [5]-[7], [15] for $p \neq 2$ and the references therein. We note that, in the case $a = b = 0, c = p$ and $\lambda = 0$, Silva and Xavier [6] considered problem (1.1) and established the multiplicity of solutions via the symmetric mountain pass theorem [12], [16] and the concentration-compactness principle [10], [11].

Motivated by the papers mentioned above, the main goal of the present paper is to establish the existence and multiplicity of solutions for the more general Caffarelli-Kohn-Nirenberg type problem (1.1) involving critical exponent growth by the cohomological index theory and minimax methods. Our results are different from those of [1], [2], [6].

Before stating the main results, we say a few words for the working space and variational setting. The starting point of the variational approach to the problem above is the following weighted Sobolev-Hardy inequality due to Caffarelli et al [5], which is called the Caffarelli-Kohn-Nirenberg inequality. For all $u \in C_0^\infty(\mathbb{R}^N)$, there is a constant $C_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bq}|u|^q dx\right)^{\frac{p}{q}} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|Du|^p dx, \tag{1.4}$$

where

$$-\infty < a < \frac{N-p}{p}, a \leq b \leq a+1, q = p* = \frac{Np}{N-dp}, d = 1+a-b. \tag{1.5}$$

Let $D_a^{1,p}(\Omega)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $||\cdot||$

defined by

$$||u|| = \left(\int_\Omega |x|^{-ap} |Du|^p dx \right)^{\frac{1}{p}}.$$

From the boundednes of Ω and the standard approximation arguments, it is easy to see that (1.4) holds for any $u \in D_a^{1,p}(\Omega)$ in the sense:

$$\left(\int_\Omega |x|^{-\alpha} |u|^r dx \right)^{\frac{p}{r}} \leq C \int_\Omega |x|^{-ap} |Du|^p dx, \tag{1.6}$$

for $1 \leq r \leq \frac{Np}{N-p}, \alpha \leq (1+a)r + N(1 - r/p)$, here and in the sequel, the letter C will be used to denote various generic positive constants.

The following results are known from [1].

(A) The embedding $D_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous for $1 \leq r \leq \frac{Np}{N-p}, \alpha \leq (1+a)r + N(1-r/p)$.

(B) The embedding $D_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact for $1 \leq r < \frac{Np}{N-p}, \alpha < (1+a)r + N(1-r/p)$,

where $L^r(\Omega, |x|^{-\alpha})$ is the weighted L^r space with the norm

$$||u||_{r,\alpha} = ||u||_{L^r(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha} |u|^r dx \right)^{\frac{1}{r}}.$$

Define the best embedding constant

$$S_{(a,b)} = \inf_{u \in D_a^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx \right)^{\frac{p}{p^*}}}$$

and

$$S_R(a,b) = \inf_{u \in D_{a,R}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx \right)^{\frac{p}{p^*}}},$$

where $D_{a,R}^{1,p}(\mathbb{R}^N) = \{u \in D_a^{1,p}(\mathbb{R}^N) : \ u \text{ is radial}\}$. It is well known that for $a < (N-p)/p$ and $b - a < 1, S_R(a,b)$ is always achieved and the extremal functions are given by

$$U_{a,b}(r) = c \left(\frac{N - p - pa}{1 + r^{\frac{dp(N-p-pa)}{(p-1)(N-dp)}}} \right)^{(N-dp)/dp},$$

where

$$c = \left(\frac{N}{(p-1)^{p-1}(N-dp)} \right)^{(N-dp)/dp^2}.$$

Under some further conditions imposed on parameters a, b, N, p [3] obtains that $S(a,b) < S_R(a,b)$ for $a < 0$. So, it is very difficult to verify the corresponding energy functional satisfies the $(PS)_c$ condition.

Throughout this paper we assume that $f(x,u)$ is a Carathéodary function defined on $\Omega\times\mathbb{R}$ satisfying the following conditions.

(A1) $\lim_{|t|\to\infty}\frac{f(x,t)}{|t|^{p^*-1}}=0$ uniformly in Ω;

(A2) $|f(x,t)|\le C(|t|^{\theta-1}+1)$, $\frac{1}{p}tf(x,t)-F(x,t)\ge -c_1-c_2|t|^{\sigma}$ for some $c_1,c_2>0$, and $0\le\sigma<p<\theta<p^*$;

(A3) $\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^p}=\infty$, $\lim_{|t|\to 0}\frac{F(x,t)}{|t|^p}=0$ uniformly in x;

(A4) $f(x,-t)=-f(x,t)$ on $\Omega\times\mathbb{R}$,

where $F(x,u)=\int_0^u f(x,t)dt, x\in\Omega$.

The main results are the following

Theorem 1.1. *If the parameters N,a,b,c,d,p,σ, satisfy*

$$(A0)\begin{cases} N>\max\left\{p,-\frac{p\sigma(a+1)}{p-\sigma}\right\}\\ -\infty<a<\frac{N-p}{p},\ a\le b<a+1\\ d=a+1-b\in(0,1],\ c>dp,\end{cases}$$

and $\lambda>0$ is not an eigenvalue of (1.3). Moreover, assume that f satisfies $(A1)-(A3)$ and $F(x,t)\ge 0$ for $\forall(x,t)\in\Omega\times\mathbb{R}$. Then there exists a $\beta_1>0$ such that problem (1.1) has a nontrivial solution for all $\beta\in(0,\beta_1)$.

Theorem 1.2. *Suppose the parameters N,a,b,c,d,p,σ satisfy (A0), and f satisfies $(A1)-(A4)$. Then there exist a $\beta_k>0$ such that problem (1.1) has at least k pairs of nontrivial solutions for all $\beta\in(0,\beta_k)$.*

In Section 2, we formulate some properties of the cohomological index and the eigenvalue problem (1.3). In Section 3, we prove the Palais-Smale condition is satisfied under a certain level by the concentration-compactness principle. Finally, we give the proofs of Theorem 1.1 and Theorem 1.2 in Section 4.

2. Cohomological Index and Variational Eigenvalues

Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of W. Fadell and Rabinowitz [9] constructed an index theory $i:\mathcal{A}\to\mathbb{N}\cup\{0,\infty\}$ with the following properties

(i) Definiteness: $i(A)=0\Longleftrightarrow A=\emptyset$.

(ii) Monotonicity: If there is an odd map $A\to A'$, then $i(A)\le i(A')$. In particular, equality holds if S and S' are homeomorphic.

(iii) Subadditivity: $i(A\cup A')\le i(A)+i(A')$.

(iv) Continuity: If A is closed, then there is a closed neighborhood of $A_0 \in \mathcal{A}$ of A such that

$$i(A_0) = i(A).$$

(v) Neighborhood of zero: If V is a bounded symmetric neighborhood of 0 in W, then

$$i(\partial V) = \dim W.$$

(vi) Stability: If A is closed and $A * Z_2$ is the joint of A with $Z_2 = \{\pm 1\}$, containing of all segments joining ± 1 to points of A, then

$$i(A * Z_2) = i(A) + 1.$$

(vii) Piercing property: If A, A_0, A_1 are closed and $\varphi : A \times [0,1] \to A_0 \cup A_1$ is an odd map such that $\varphi(A \times [0,1])$ is closed, $\varphi(A \times \{0\}) \subset A_0, \varphi(A \times \{1\}) \subset A_1$, then

$$i(\varphi(A \times [0,1]) \cap A_0 \cap A_1) \geq i(A).$$

We see from the Lagrange multiplier rule that the Dirichlet eigenvalues of (1.3) are the critical values of the functional

$$\Psi(u) = \frac{1}{\int_\Omega |x|^{-p(a+1)+c} |u|^p dx}, \quad u \in S = \{u \in D_a^{1,p}(\Omega) : ||u|| = 1\}.$$

Denote by $\mathcal{A}$ the class of compact symmetric subsets of S, let

$$\Gamma_l = \{A \in \mathcal{A} : \ i(A) \geq l\} \tag{2.1}$$

and set

$$\lambda_l = \inf_{A \in \Gamma_l} \max_{u \in A} \Psi(u).$$

As it has been shown in [1], [2] that λ_l is an eigenvalue of (1.3). Moreover, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l \leq \cdots \to \infty$.

3. The Palais-Smale Conditions

The energy functional associated with problem (1.1) is given by

$$\Phi_\beta(u) = \frac{1}{p} \int_\Omega \frac{|Du|^p}{|x|^{ap}} dx - \frac{\beta}{p^*} \int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}} dx - \frac{\lambda}{p} \int_\Omega \frac{|u|^p}{|x|^{p(a+1)-c}} dx - \int_\Omega F(x,u) dx,$$

for every $u \in D_a^{1,p}(\Omega)$. The assumption (A1) and the facts (A), (B) imply that $\Phi_\beta \in C^1(D_a^{1,p}(\Omega), \mathbb{R})$, and the weak solutions of problem (1.1) is equivalent to the critical points of Φ_β in $D_a^{1,p}(\Omega)$.

Next, we enumerate the concentration-compactness principle which is a weighted version of the concentration-compactness principle due to Lions [10], [11].

Lemma 3.1. (see [1]) *Let* $1 < p < N, -\infty < a < \frac{N-p}{p}, a \leq b \leq a+1, q = p^* = \frac{Np}{N-dp}, d = 1+a-b \in [0,1]$ *and* $M(\mathbb{R}^N)$ *be the space of bounded measures on* $\mathbb{R}^N$*. Suppose that* $\{u_m\} \subset D_a^{1,p}(\mathbb{R}^N)$ *be a sequence such that:*

$$\begin{array}{ll} u_m \rightharpoonup u & \text{in } D_a^{1,p}(\mathbb{R}^N), \\ \mu_m := |x|^{-ap}|Du_m|^p dx \rightharpoonup \mu & \text{in } M(\mathbb{R}^N), \\ \nu_m := |x|^{-bq}|u_m|^q dx \rightharpoonup \nu & \text{in } M(\mathbb{R}^N), \\ u_m \to u & \text{a.e. on } \mathbb{R}^N. \end{array}$$

Then there are the following statements:

(1) There exists some at most countable set J*, a family* $\{x^{(j)}: \ j \in J\}$ *of distinct points in* $\mathbb{R}^N$*, and a family* $\{\nu^{(j)}: \ j \in J\}$ *of positive numbers such that*

$$\nu = |x|^{-bq}|u|^q dx + \Sigma_{j\in J}\nu^{(j)}\delta_{x^{(j)}},$$

where δ_x *is the Dirac-mass of mass 1 concentrated at* $x \in \mathbb{R}^N$*.*

(2) The following inequality holds

$$\mu \geq |x|^{-ap}|Du|^p dx + \Sigma_{j\in J}\mu^{(j)}\delta_{x^{(j)}},$$

for some family $\{\mu^{(j)} > 0: \ j \in J\}$ *satisfying*

$$S_{(a,b)}(\nu^{(j)})^{\frac{p}{q}} \leq \mu^{(j)} \quad \text{for all } j \in J.$$

In particular $\Sigma_{j\in J}(\nu^{(j)})^{\frac{p}{q}} < \infty$*.*

In the following we verify the functional Φ_β satisfies the $(PS)_c$ condition below a given level when $\beta > 0$ is small enough. The main idea is taken from [6]. By (A1) we may choose a constant $C_\varepsilon > 0$ such that:

$$|f(x,t)t| \leq C_\varepsilon + \varepsilon|x|^{-bp^*}|t|^{p^*} \text{ for every } t \in \mathbb{R}, \text{ uniformly in } x \in \Omega, \tag{3.1}$$

$$|F(x,t)| \leq C_\varepsilon + \frac{\varepsilon}{p^*}|x|^{-bp^*}|t|^{p^*} \text{ for every } t \in \mathbb{R}, \text{ uniformly in } x \in \Omega. \tag{3.2}$$

Lemma 3.2. *Suppose* f *satisfies* $(A1)$*. Let* $\{u_n\} \subset D_a^{1,p}(\Omega)$ *be a bounded sequence. Then there is* $u \in D_a^{1,p}(\Omega)$ *such that, up to a subsequence,*

$$\int_\Omega |f(x,u_n)u_n - f(x,u)u|dx \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Proof. Taking a subsequence if necessary, we may suppose $u_n \rightharpoonup u$ weakly in $D_a^{1,p}(\Omega)$, and $u_n \to u$ a.e. in Ω. Since f is a Carathéodory function, $f(x,u_n)u_n \to f(x,u)u$ a.e. in Ω. Furthermore, it follows from the embedding

$D_a^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega, |x|^{-bp^*})$ that

$$||u||_{p^*,bp^*}^{p^*} \le C \text{ and } ||u_n||_{p^*,bp^*}^{p^*} \le C \quad \text{for } \forall n \in \mathbb{N}, \tag{3.4}$$

Now given $\delta > 0$, we choose $0 < \varepsilon < \delta/4C$ and apply the Egorov's Theorem to obtain a measurable set $\Omega_0 \subset \Omega$ such that $f(x, u_n)u_n \to f(x, u)u$ uniformly on Ω_0 and $|\Omega \backslash \Omega_0| < \frac{\delta}{4C_\varepsilon}$, where C_ε is the constant in (3.1). Therefore, using (3.1) and (3.4), we have

$$0 \le \int_\Omega |f(x, u_n)u_n - f(x, u)u|dx \le \int_{\Omega_0} |f(x, u_n)u_n - f(x, u)u|dx + \delta.$$

Taking $n \to \infty$ and considering $\delta > 0$ is arbitrarily chosen, we complete the proof. □

By a similar arguments as that in Lemma 3.2, we may verify that if $\{u_n\}$ is a bounded sequence, then there is $u \in D_a^{1,p}(\Omega)$ such that

$$\int_\Omega f(x, u_n)vdx \to \int_\Omega f(x, u)vdx \ \text{ as } n \to \infty, \tag{3.5}$$

and

$$\int_\Omega |x|^{-bp^*}|u_n|^{p^*-2}u_n vdx \to \int_\Omega |x|^{-bp^*}|u|^{p^*-2}uvdx \quad \text{as } n \to \infty, \tag{3.6}$$

for every $v \in D_a^{1,p}(\Omega)$. Furthermore, using Lemma 3.1, Lemma 3.2 we may prove the following results.

Lemma 3.3. *Suppose f satisfies $(A1)$. Let $\{u_n\} \subset D_a^{1,p}(\Omega)$ be a bounded sequence satisfying $\Phi'_\beta(u_n) \to 0$ in $(D_a^{1,p}(\Omega))'$ as $n \to \infty$. Then considering $\nu_j, j \in J$, given by Lemma 3.1, we have either $\nu_j \ge \left(\frac{S_{(a,b)}}{\beta}\right)^{\frac{N}{dp}}$ or $\nu_j = 0$.*

Proof. Since $\{u_n\}$ is bounded in $D_0^{1,p}(\Omega)$, we can extract a subsequence, still denoted by $\{u_n\}$, so that for some $u \in D_a^{1,p}(\Omega)$,

$u_n \rightharpoonup u$ weakly in $D_a^{1,p}(\Omega)$,
$u_n \to u$ a.e. in Ω,
$u_n \to u$ in $L^r(\Omega, |x|^{-\alpha}), \forall 1 \le r < \frac{Np}{N-p}, \frac{\alpha}{r} < (1 + a) + N(\frac{1}{r} - \frac{1}{p})$,
$u_n \rightharpoonup u$ weakly in $L^{p^*}(\Omega, |x|^{-bp^*})$.

We define $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \equiv 1$ in $B(x_j, \varepsilon), \psi(x) \equiv 0$ in $\mathbb{R}^N \backslash B(x_j, 2\varepsilon)$, and $|\nabla \psi| \le \frac{2}{\varepsilon}$. Note that $\langle \Phi'_\beta(u_n), u_n\psi \rangle \to 0$, i.e.,

$$\int_\Omega |x|^{-ap}|\nabla u_n|^p \psi dx + \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}u_n \nabla u_n \nabla \psi dx - \beta \int_\Omega |x|^{-bp^*}|u_n|^{p^*}\psi dx$$
$$- \lambda \int_\Omega |x|^{-p(a+1)+c}|u_n|^p \psi dx - \int_\Omega f(x, u_n)u_n \psi dx \to 0. \tag{3.7}$$

By Lemma 3.1, we have

$$\int_\Omega |x|^{-ap}|\nabla u_n|^p \psi dx \to \int_\Omega \psi d\mu; \quad \int_\Omega |x|^{-bp^*}|u_n|^{p^*}\psi dx \to \int_\Omega \psi d\nu.$$

Since the embedding $D_a^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega, |x|^{-bp^*})$ is continuous and $\{u_n\}$ is bounded in $D_a^{1,p}(\Omega)$, then $\{u_n\}$ is also bounded in $L^{p^*}(\Omega, |x|^{-bp^*})$, that is, (3.4) holds.

By Strauss' Lemma [12]

$$\lim_{n\to\infty}\int_\Omega |x|^{-p(a+1)+c}|u_n|^p\psi dx = \int_\Omega |x|^{-p(a+1)+c}|u|^p\psi dx, \tag{3.8}$$

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)u_n\psi dx = \int_\Omega f(x,u)u\psi dx. \tag{3.9}$$

Then let $n \to \infty$ in (3.7), we have

$$\begin{aligned}\int_\Omega \psi d\mu - \beta\int_\Omega \psi d\nu \quad &- \quad \lambda \textstyle\int_\Omega |x|^{-p(a+1)+c}|u|^p\psi dx - \int_\Omega f(x,u)u\psi dx \\ &= \quad -\lim_{n\to\infty}\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}u_n\nabla u_n\nabla\psi dx.\end{aligned}$$

By Hölder inequality and Young inequality, for some $\delta > 0$, there exists $C_\delta > 0$ such that

$$\begin{aligned}\lim_{n\to\infty}\left|\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}u_n\nabla u_n\nabla\psi dx\right| &\le \overline{\lim_{n\to\infty}}\int_\Omega |x|^{-ap}|\nabla u_n|^{p-1}|u_n|\cdot|\nabla\psi|dx \\ &\le \overline{\lim_{n\to\infty}}\left(\int_\Omega |x|^{-ap}|\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |x|^{-ap}|u_n|^p|\nabla\psi|^p dx\right)^{\frac{1}{p}} \\ &\le \overline{\lim_{n\to\infty}}\left(\delta\int_\Omega |x|^{-ap}|\nabla u_n|^p dx + C_\delta\int_\Omega |x|^{-ap}|u_n|^p|\nabla\psi|^p dx\right) \\ &\le \delta C + C_\delta\int_{B(x_j;2\varepsilon)} |x|^{p(b-a)}|\nabla\psi|^p\cdot|x|^{-bp}|u|^p \\ &\le \delta C + C\cdot C_\delta\left(\int_{B(x_j;2\varepsilon)}|x|^{-bp^*}|u|^{p^*}\right)^{\frac{p}{p^*}},\end{aligned} \tag{3.10}$$

where C is independent of δ and ε.

From (3.7)-(3.10), we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega |x|^{-ap}|\nabla u_n|^p\psi dx &\le \beta\lim_{n\to\infty}\int_\Omega |x|^{-bp^*}|u_n|^{p^*}\psi dx \\ &\quad + \lambda\int_\Omega |x|^{-p(a+1)+c}|u|^p\psi dx + \int_\Omega f(x,u)u\psi dx + \delta C\end{aligned}$$

$$+ C \cdot C_\delta \left(\int_{B(x_j;2\varepsilon)} |x|^{-bp^*} |u|^{p^*} \psi dx \right)^{\frac{p}{p^*}}. \quad (3.11)$$

Taking $\delta \to 0$ and $\varepsilon \to 0$ in (3.11), we have $\mu_j \leq \beta\nu_j$, i.e., $S_{(a,b)}\nu_j^{\frac{p}{p^*}} \leq \beta\nu_j$. Hence

$$\nu_j \geq \left(\frac{S_{(a,b)}}{\beta} \right)^{\frac{N}{dp}} \qquad \text{or} \qquad \nu_j = 0. \qquad \square$$

As a consequence of Lemma 3.3, the set J of Lemma 3.1 is finite. By this fact we may show

Lemma 3.4. *Suppose f satisfies $(A1)$. Let $\{u_n\} \subset D_a^{1,p}(\Omega)$ be a bounded sequence satisfying $\Phi'_\beta(u_n) \to 0$ in $(D_a^{1,p}(\Omega))'$ as $n \to \infty$. Then, up to a sequence*

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla|^{p-2}\nabla u \text{ weakly in } (L^{p'}(\Omega, |x|^{-ap}))^N,$$

where $p' = \frac{p}{p-1}$.

Now we have the $(PS)_c$ condition.

Proposition 3.5. *Under the assumptions (A0), (A1)-(A2), given $M > 0$, then there exists $\beta^* > 0$ such that $\Phi_\beta(u)$ satisfies the $(PS)_c$ condition in $D_a^{1,p}(\Omega)$ at every level $c < M$ provided $\beta \in (0, \beta^*)$.*

Proof. Given $M > 0$, let

$$\beta^* = \min \left\{ S_{(a,b)}, \left[\frac{S_{(a,b)}^{\frac{N}{dp}}}{((M+A)Nd^{-1})^{\frac{1}{\alpha}}} \right]^{\frac{1}{\frac{N}{dp} - \frac{1}{\alpha}}} \right\}, \quad (3.12)$$

where $A = c_1|\Omega| + c_2 \left(\int_\Omega |x|^{\frac{\sigma bp^*}{p^*-\sigma}} dx \right)^\alpha \triangleq c_1|\Omega| + c_2Q, \alpha = (p^* - \sigma)/p^*$, and c_1, c_2, σ are constants in $(A2)$. It follows from (A0) that $Q = \left(\int_\Omega |x|^{\frac{\sigma bp^*}{p^*-\sigma}} dx \right)^\alpha$ is well defined.

Consider $0 < \beta < \beta^*$, by (3.12) and $\sigma < p$, we get

$$1 < \left(\frac{S_{(a,b)}}{\beta} \right)^{\frac{N}{dp}}, \quad (3.13)$$

and

$$\left(\frac{(M+A)N}{d\beta} \right)^{\frac{1}{\alpha}} < \left(\frac{S_{(a,b)}}{\beta} \right)^{\frac{N}{dp}}. \quad (3.14)$$

Now given $c < M$, let $\{u_n\} \subset D_a^{1,p}(\Omega)$ be such that $\Phi_\beta(u_n) \to c$, and

$\Phi'_\beta(u_n) \to 0$ in $(D_a^{1,p}(\Omega))'$ as $n \to \infty$. We must show the existence of a subsequence of $\{u_n\}$ which converges strongly in $D_a^{1,p}(\Omega)$.

First, we show the boundedness of $(PS)_c$ sequence. In fact, for n large enough, by $(A2)$ and Hölder inequality, we have

$$\begin{aligned} c+1+||u_n|| &\geq \Phi_\beta(u_n) - \frac{1}{p}\langle \Phi'_\beta(u_n), u_n\rangle \\ &\geq \frac{d\beta}{N}\int_\Omega |x|^{-bp^*}|u_n|^{p^*}dx - \int_\Omega \left(F(x,u_n) - \frac{1}{p}f(x,u_n)u_n\right)dx \\ &\geq \frac{d\beta}{N}||u_n||^{p^*}_{p^*,bp^*} - c_1|\Omega| - c_2Q||u_n||^{p^*(1-\alpha)}_{p^*,bp^*}. \end{aligned} \tag{3.15}$$

Furthermore, by Young's inequality, we have

$$||u_n||^{p^*(1-\alpha)}_{p^*,bp^*} \leq \delta||u_n||^{p^*}_{p^*,bp^*} + C_\delta \tag{3.16}$$

with $\delta = \frac{d\beta}{2Nc_2Q}, C_\delta = \alpha\left(\frac{1-\alpha}{\delta}\right)^{\frac{1-\alpha}{\alpha}}$. Then combining with (3.15) and (3.16), we get

$$||u_n||^{p^*}_{p^*,bp^*} \leq C + C||u_n||. \tag{3.17}$$

Now, by (3.2), (3.17) and $\Phi_\beta(u_n) \to c$ as $n \to \infty$, and Hölder inequality we have

$$\begin{aligned} \tfrac{1}{p}||u_n||^p &= \Phi_\beta(u_n) + \tfrac{\beta}{p^*}||u_n||^{p^*}_{p^*,bp^*} + \tfrac{\lambda}{p}\textstyle\int_\Omega |x|^{-p(a+1)+c}|u_n|^p dx + \int_\Omega F(x,u_n)dx \\ &\leq 1+|c|+C+C||u_n|| + |\lambda/p|\textstyle\int_\Omega |x|^{-p(a+1)+c}|u_n|^p dx \\ &\quad +C + C\textstyle\int_\Omega |x|^{-bp^*}|u_n|^{p^*}dx \\ &\leq C+C||u_n|| + C\textstyle\int_\Omega |x|^{-bp}|u_n|^p|x|^{-p[(a+1)-b]+c}dx \\ &\leq C+C||u_n|| + C\left(\int_\Omega |x|^{\frac{p^*(-dp+c)}{p^*-p}}\right)^{\frac{p^*-p}{p^*}}\left(\int_\Omega |x|^{-bp^*}|u_n|^{p^*}dx\right)^{\frac{p}{p^*}} \\ &\leq C+C||u_n|| + C||u_n||^p_{p^*,bp^*} \qquad (\text{by} \quad c > dp) \\ &\leq C+C||u_n|| + C(C+C||u_n||)^{\frac{p}{p^*}}. \end{aligned}$$

Therefore, $\{u_n\}$ is bounded in $D_a^{1,p}(\Omega)$. We may assume that there is a $u \in D_a^{1,p}(\Omega)$ such that $\{u_n\}$ satisfies (3.3), (3.5), (3.6) and, from Lemma 3.1 and Lemma 3.4 that

$$|x|^{-bp^*}|u_n|^{p^*} \rightharpoonup \nu = |x|^{-bp^*}|u|^{p^*} + \Sigma_{j\in J}\nu^{(j)}\delta_{x^{(j)}}, \tag{3.18}$$

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \text{ weakly in } (L^{p'}(\Omega,|x|^{-ap}))^N, \tag{3.19}$$

where $p' = \frac{p}{p-1}, \nu$ is a nonnegative bounded measure in $\bar{\Omega}, J$ is a finite set, $\{x_j; j \in J\}$ is a family of points in $\bar{\Omega}$, and $\{\nu^{(j)} : j \in J\}$ is a family of positive numbers.

We claim that $\int_\Omega d\nu < (S_{(a,b)}/\beta)^{\frac{N}{dp}}$. In fact, if $\int_\Omega d\nu \leq 1$, this follows by

(3.13). Otherwise, taking $n \to \infty$ in (3.15), we have

$$\frac{d\beta}{N}\int_{\Omega} d\nu \le c + c_1|\Omega| + c_2 Q\left(\int_{\Omega} d\nu\right)^{1-\alpha} \le (M + c_1|\Omega| + c_2 Q)\left(\int_{\Omega} d\nu\right)^{1-\alpha}.$$

Therefore, by (3.14), the claim is proved. As a consequence os this fact and Lemma 3.3 we conclude that $\nu_j = 0$ for all $j \in J$. Therefore, by (3.18),

$$\int_{\Omega} |x|^{-bp^*}|u_n|^{p^*}dx \to \int_{\Omega} |x|^{-bp^*}|u|^{p^*}dx, \tag{3.20}$$

Now from $\langle \Phi'_\beta(u_n), u_n\rangle = o(1)$ and $\langle \Phi'_\beta(u_n), u\rangle = o(1)$, taking $n \to \infty$, we obtain, using (3.20), (3.3), (3.19), (3.6) and (3.5),

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega}|x|^{-ap}|\nabla u_n|^p dx &= \beta\int_{\Omega}|x|^{-bp^*}|u|^{p^*}dx + \lambda\int_{\Omega}|x|^{-(a+1)p+c}|u|^p dx\\ &\quad + \int_{\Omega} f(x,u)u dx,\end{aligned}$$

$$\begin{aligned}\int_{\Omega}|x|^{-ap}|\nabla u|^p dx &= \beta\int_{\Omega}|x|^{-bp^*}|u|^{p^*}dx + \lambda\int_{\Omega}|x|^{-(a+1)p+c}|u|^p dx\\ &\quad + \int_{\Omega} f(x,u)u dx.\end{aligned}$$

Hence, since $D_a^{1,p}(\Omega)$ is uniformly convex, we have that $u_n \to u$ strongly in $D_a^{1,p}(\Omega)$. This completes the proof. □

4. Proof of the Main Results

Proof of Theorem 1.1. Since λ is not an eigenvalue of (1.3) and $\lambda < \lambda_1$ has been considered in [1], we assume that $\lambda \in (\lambda_l, \lambda_{l+1})$, for some $l \in \mathbb{N}$. Then there is an $A_0 \in \Gamma_l$ such that $\Psi(u) \le \lambda$ on A_0, where Γ_l is given in (2.1). Hence, by $F(x,t) \ge 0$ for $\forall (x,t) \in \Omega \times \mathbb{R}$,

$$\Phi_0(tu) \le \frac{t^p}{p}\left(1 - \frac{\lambda}{\Psi(u)}\right) \le 0, \quad \text{for all } u \in A_0 \text{ and } t \ge 0. \tag{4.1}$$

Take $\varphi_0 \in C(CA_0, S)$ with $\varphi_0|_{A_0} = id$, where $CA_0 = (A_0 \times [0,1])/(A_0 \times \{1\})$ is the cone over A_0. By $(A3)$ we have

$$\Phi_0(tu) \to -\infty \quad \text{as } t \to \infty$$

uniformly for u on compact set $\varphi_0(CA_0)$, so $\Phi_0 \le 0$ on $R\varphi_0(CA_0)$ when $R > 0$ is large enough. Moreover,

$$c_0 \triangleq \sup_{u\in\varphi_0(CA_0), t\ge 0} \Phi_0(tu) < \infty.$$

It follows from (A1)-(A2) and by Proposition 3.5, there exists a $\beta_1 > 0$ such that Φ_β satisfies the $(PS)_c$ conditions at all levels $c \leq c_0$ when $\beta \in (0, \beta_1)$.

Replacing W as a subsequence of $W \oplus \mathbb{R}$ and CA_0 as a cone in $W \oplus \mathbb{R}$, with vertex at some point $\notin W$. Set

$$A_1 = \{tu: \ u \in A_0, t \in [0,1]\}, \qquad A = A_1 \cup CA_0$$

and

$$h(v) = \begin{cases} Rv, & v \in A_1, \\ R\varphi_0(v), & v \in CA_0. \end{cases}$$

Then $\Phi_\beta \leq \Phi_0 \leq 0$ on $h(A)$. By (A3) we get

$$\Phi_\beta(tu) = \frac{t^p}{p}\left(1 - \frac{\lambda}{\Psi(u)} + o(1)\right) \quad \text{as } t \to 0, u \in S. \tag{4.2}$$

Since $\lambda < \lambda_{l+1}$, the infimum of Φ_β on

$$B = \{u \in S_\rho: \ \Psi(u/\rho) \geq \lambda_{l+1}\}, \tag{4.3}$$

where $S_\rho = \{||u|| = \rho\}$, is positive if $0 < \rho < R$ is small enough. We complete the proof by showing that (A, h) links B with respect to $K = \{tz: \ z \in CA_0, t \in [0,1]\}$ and hence Φ_β has a critical point v.

Any $\gamma \in C(K, W)$ such that $\gamma|_A = h$ can be extended to an odd map $\tilde{\gamma}$ on $\tilde{K} = \{tz: \ z \in A_0 * Z_2, t \in [0,1]\}$, it suffices to show that $\tilde{\gamma}(\tilde{K}) \cap B \neq \emptyset$. Applying the piercing property to

$$S_0 = A_0 * Z_2, \ S_1 = \{||u|| \leq \rho\}, \ S_2 = \{||u|| \geq \rho\}, \varphi(v,t) = \tilde{\gamma}(tv),$$

yields

$$i(\tilde{\gamma}(\tilde{K}) \cap S_\rho) = i(\varphi(S \times [0,1]) \cap S_1 \cap S_2) \geq i(S_0) = i(A_0) + 1 \geq l + 1.$$

Therefore,

$$\max_{u \in \tilde{\gamma}(\tilde{K}) \cap S_\rho} \Psi(u/\rho) \geq \lambda_{l+1},$$

which means that $\tilde{\gamma}(\tilde{K}) \cap B \neq \emptyset$. □

Proof of Theorem 1.2. By (A4), we see that Φ_β is even. Denote by $\mathcal{A}$ the class of compact symmetric subsets of W and by Γ_β the group of odd homeomorphisms of γ of W such that $\gamma = id$ on $\Phi_\beta^{-1}((-\infty, 0])$. We fix a $\beta_* > 0$ and consider $\beta < \beta_*$. Let

$$i^*(A) := \min_{\gamma \in \Gamma_\beta} i(\gamma(A) \cap S_\rho), \quad A \in \mathcal{A} \tag{4.4}$$

be the pseudo-index of Benci [8] related to i, S_ρ and Γ_β.

We claim that for any $\beta \geq 0, \forall m \in \mathbb{N}$,

$$\mathcal{H}_m^\beta := \{A \in \mathcal{A}: \ i^*(A) \geq m\} \neq \emptyset.$$

In fact, take $A_0 \in \mathcal{H}_m^\beta, R > \rho$ large enough such that $\Phi_\beta \leq 0$ on RA_0 because of (A3), now let

$$A = \{tu : \ u \in RA_0, t \in [0,1]\}.$$

Then for any $\gamma \in \Gamma_\beta$ with $\gamma|_{RA_0} = id$ and $\gamma(0) = 0$, and applying the piercing property to

$$S_0 = RA_0, \ S_1 = \{||u|| \leq \rho\}, \ S_2 = \{||u|| \geq \rho\}, \varphi(u,t) = \gamma(tu)$$

yields

$$i(\gamma(A) \cap S_\rho) = i(\varphi(S_0 \times [0,1]) \cap S_1 \cap S_2) \geq i(S_0) = i(A_0) \geq m.$$

On the other hand,

$$c_0 := \inf_{A \in \mathcal{H}_{l+k}^0} \max_{u \in A} \Phi_0(u) < \infty.$$

It follows from (A1), (A2) and Proposition 3.5 that there is a $0 < \beta_k \leq \beta_*$ such that Φ_β satisfies the $(PS)_c$ conditions at all levels $c \leq c_0$ when $\beta \in (0, \beta_k)$. Denote

$$c_m := \inf_{A \in \mathcal{H}_{l+m}^\beta} \max_{u \in A} \Phi_\beta(u), \quad 1 \leq m \leq k.$$

It is easy to see that $c_1 \leq c_2 \leq \cdots \leq c_k \leq c_0$. Taking $A \in \mathcal{H}_{l+1}^\beta, \gamma = id$ in (4.4), we have $i(A \cap S_\rho) \geq l + 1$. So

$$\max_{u \in A \cap S_\rho} \Psi(u/\rho) \geq \lambda_{l+1},$$

and hence $A \cap B \neq \emptyset$. By (4.2), taking $\rho > 0$ small enough that $\inf \Phi_{\beta_*}(B) > 0$, where B is given in (4.3). We have

$$\max_{u \in A} \Phi_\beta(u) \geq \inf_{u \in B} \Phi_{\beta_*}(u) > 0$$

and therefore, $c_1 > 0$. Now, following the way of [8], we can obtain that Φ_β has at least k pairs of nontrivial critical points (see Benci [8]). This completes the proof. □

Remark 4.1. In [1], [2], the parameter λ is restricted to locate in $(0, \lambda_1)$ or (λ_1, λ_2). But in this paper we loose this restriction on λ by taking $\beta > 0$ small enough. By the way, we say that even if in the case $\lambda = 0$, Theorem 1.2 does not coincidence with the main results in [6].

Acknowledgements

The work is supported by NSFC 10571096, 10001019 and SRF for ROCS, SEM and NCET of the Ministry for Chinese Educations.

References

[1] V. Benci, On critical point theory for indefinite functionals in presence of symmetries, *Tran. Amer. Math. Soc.*, **271** (1982), 533-573.

[2] H. Brezis, E. Lieb, Arelation between pointwise convergence of functions and functionals, *Proc. Amer. Math. Soc.*, **88** (1983), 486-490.

[3] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, *Compositio Mathematica*, **53** (1984), 259-275.

[4] F. Catrina, Z.Q. Wang, On the Caffarelli-Kohn-Nirenber inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions, *Comm. Pure Appl. Math.*, **LIV** (2001), 229-258.

[5] G. Cerami, S. Solomini, M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, *J. Funct. Anal.*, **69** (1986), 289-306.

[6] K.S. Chou, D. Geng, On the critical dimension of a semilinear degenerate elliptic equation involving critical Sobolev-Hardy exponent, *Nonlinear Analysis*, **26** (1996), 1965-1984.

[7] E.R. Fadell, P.H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, *Invent. Math.*, **45** (1978), 139-174.

[8] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana*, **1**, No. 1 (1985), 145-201.

[9] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana*, **1**, No. 2 (1985), 45-121.

[10] K. Perera, A. Szulkin, p-Laplacian problems where the nonlinearity cross an eigenvalue, *Disc. Conti. Dyn. Syst.*, **13** (2005), 743-753.

[11] P.H. Rabinowitz, *Minimax Methods in Critical Point Theorey with Applications to Differential Equations*, CBMS Reg. Conf. Series Math., Volume 65, Amer. Math. Soc., Providence, RI (1986).

[12] E.A.B. Silva, M.S. Xavier, Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents, *Ann. I.H. Poincaré*, **20** (2003), 341-358.

[13] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin (2000).

[14] B. Xuan, Existence results for superlinear singular equation of Caffarelli-Kohn-Nirenberg type, *ArXiv:* math./AP/0404035 (2004).

[15] B. Xuan, The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights, *Nonlinear Analysis*, **62** (2005), 703-725.

[16] Z. Wei, X. Wu, A multiplicity result for quasilinear elliptic equations involving critical Sobolev exponents, *Nonlinear Analysis*, **18** (1992), 559-567.