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FAMILIES OF DENSITY TYPE TOPOLOGIES

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Abstract: Originally defined by Haupt and Pauc in 1952, the density topology has since been well-studied. Consequently a Baire category analogue of the density topology was developed by Ciesielski and Larson (1993) and Wilczynski (1984). This paper presents another variation on the density topology. Given an arbitrary homeomorphism f with f and f^{-1} satisfying property N, we define an f-density operator and a corresponding f-density topology, T_f. In this paper, T_f is proven to be a topology and general properties of the topology are developed. Classes of functions, where $T_f = T_d$ are identified and it is shown that there is a function f so that $T_f \neq T_d$.

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1. Introduction

We denote the density topology as $T_d = \{A \in M : A \subset \Phi(A)\}$, where $\Phi(A) = \{x \in \mathbb{R} : \lim_{h\to 0} \frac{m(A\cap[x-h,x+h])}{2h} = 1\}$ and m stands for Lebesgue measure and M is the family of measurable sets. We use T_0 to refer to the natural topology, with A' for the limit points of A in T_0 and $\overline{A}$ for the closure of A in T_0. The compliment of a set A is A^c. We say $x \in \mathbb{R}$ is a dispersion point of $A \subset \mathbb{R}$ if $x \in \Phi(A^c)$.

Given an arbitrary function $f : \mathbb{R} \to \mathbb{R}$, a homeomorphism with f and

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f^{-1} satisfying property N, we define x to be an f-density point of $A \in M$ if and only if $f(x)$ is a density point of $f(A)$. We then let $\Phi_f(A) = \{x \in \mathbb{R} : x \text{ is an } f\text{-density point of } A\}$.

2. Properties of f-Density Operator

We determine if Φ_f has the same properties as Φ. We note that $A \sim B$ if and only if $m(A \Delta B) = 0$.

Theorem 1. *$\Phi_f(A) \sim A$.*

Proof. Let A be a measurable set. Since f is continuous and satisfies property N, we know that $f(A)$ is also measurable. Notice that $f(\Phi_f(A)) = f(\{x \in \mathbb{R} : f(x) \in \Phi(f(A))\}) = \Phi(f(A))$. Using the fact that f is bijective, we now see that

$$\begin{aligned} f(\Phi_f(A)\Delta A) &= f(\Phi_f(A) - A \cup A - \Phi_f(A)) \\ &= f(\Phi_f(A) - A) \cup f(A - \Phi_f(A)) \\ &= f(\Phi_f(A) \cap A^c) \cup f(A \cap \Phi_f(A)^c) \\ &\subset (f(\Phi_f(A)) \cap f(A)^c) \cup (f(A) \cap f(\Phi_f(A))^c) \\ &= (\Phi(f(A)) \cap f(A)^c) \cup (f(A) \cap \Phi(f(A))^c) \\ &= (\Phi(f(A)) - f(A)) \cup (f(A) - \Phi(f(A))) \\ &= \Phi(f(A))\Delta f(A). \end{aligned}$$

Now as $f(A)$ is measurable, $\Phi(f(A))$ is measurable. By the Lebesgue Density Theorem, $m(\Phi(f(A))\Delta f(A)) = 0$.

Since $f(\Phi_f(A)\Delta A) \subset \Phi(f(A))\Delta f(A)$, $m(f(\Phi_f(A)\Delta A)) = 0$. Since f^{-1} satisfies property N, $m(\Phi_f(A)\Delta A) = 0$. ☐

Theorem 2. *If $A \sim B$, then $\Phi_f(A) = \Phi_f(B)$.*

Proof. Let A and B be measurable sets with $A \sim B$.

Let $x \in \Phi_f(A)$. We must show that $x \in \Phi_f(B)$. We know

$$\lim_{h \to 0} \frac{m(f(A) \cap [f(x) - h, f(x) + h])}{2h} = 1.$$

Since f has property N, and since $A - B$ and $B - A$ are null sets, $m(f(A-B)) = 0 = m(f(B - A))$. Thus, we can conclude

$$\begin{aligned} &m(f(A) \cap [f(x) - h, f(x) + h]) \\ &= m((f(A) \cup f(B - A)) \cap [f(x) - h, f(x) + h]) \,. \end{aligned}$$

And since f is a function,

$$= m(f(A \cup (B - A)) \cap [f(x) - h, f(x) + h]) .$$

Since $A \cup (B - A) = A \cup B = B \cup (A - B)$, we have that

$$\begin{aligned} &= m(f(B \cup (A - B)) \cap [f(x) - h, f(x) + h]) \\ &= m((f(B) \cup f(A - B)) \cap [f(x) - h, f(x) + h]) \\ &= m(f(B) \cap [f(x) - h, f(x) + h]) . \end{aligned}$$

Thus, if

$$\lim_{h \to 0} \frac{m(f(A) \cap [f(x) - h, f(x) + h])}{2h} = 1,$$

then

$$\lim_{h \to 0} \frac{m(f(B) \cap [f(x) - h, f(x) + h])}{2h} = 1 .$$

So $x \in \Phi_f(B)$. It is analogous to show that if $x \in \Phi_f(B)$, then $x \in \Phi_f(A)$. □

Theorem 3. $\Phi_f(\emptyset) = \emptyset$ *and* $\Phi_f(\mathbb{R}) = \mathbb{R}$.

Proof. First, we prove the theorem for the real line. Now $x \in \Phi_f(\mathbb{R})$ if and only if $f(x) \in \Phi(f(\mathbb{R}))$. Since f is onto, $f(x) \in \Phi(f(\mathbb{R}))$ if and only if $f(x) \in \Phi(\mathbb{R}) = \mathbb{R}$. Lastly, $f(x) \in \mathbb{R}$ if and only if $x \in \mathbb{R}$. Thus $\Phi_f(\mathbb{R}) = \mathbb{R}$.

Now we prove the theorem for the empty set. Suppose there exists $x \in \Phi_f(\emptyset)$. Then $f(x) \in \Phi(f(\emptyset))$. Further $f(x) \in \Phi(\emptyset) = \emptyset$, which is impossible. So $\Phi_f(\emptyset) = \emptyset$. □

Theorem 4. $\Phi_f(A \cap B) = \Phi_f(A) \cap \Phi_f(B)$.

Proof. Now we know that $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$, for all measurable sets A, B. Notice that $\Phi(f(A \cap B)) = \Phi(f(A) \cap f(B))$. Also, $\Phi(f(A) \cap f(B)) = \Phi(f(A)) \cap \Phi(f(B))$, since $f(A)$ and $f(B)$ are measurable sets. Now $x \in \Phi_f(A \cap B)$ if and only if $f(x) \in \Phi(f(A \cap B)) = \Phi(f(A)) \cap \Phi(f(B))$ if and only if $x \in \Phi_f(A) \cap \Phi_f(B)$. □

From the previous theorem, we immediately have this last property of the operator Φ_f.

Theorem 5. *If* $A \subset B$, *then* $\Phi_f(A) \subset \Phi_f(B)$.

3. Counterexample

We will now see that there exists a function f and a set A such that $\Phi(A) \neq \Phi_f(A)$. Let $\rho_n = \frac{2^n}{3^n - 2^n}$.

$$f(x) = x - \frac{1}{2^n} + \frac{1}{3^n} \text{ on the interval } \left[\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{3^n}\right),$$
$$f(x) = \rho_n x - \rho_n \left(\frac{1}{2^{n-1}}\right) + \frac{1}{3^{n-1}},$$
$$\text{on the interval } \left[\frac{1}{2^n} + \frac{1}{3^n}, \frac{1}{2^{n-1}}\right).$$

This function is defined piece-wise for $n \in \mathbb{N}$. The function satisfies the Lipschitz condition because on half of the intervals the slope is 1, and on half of the intervals the slope starts at 2 and then decreases approaching zero because

$$\lim_{n\to\infty} \rho_n = 0.$$

We note that this function is also absolutely continuous, since it satisfies the Lipschitz condition. Thus, even given an absolutely continuous function f, it is possible for $\Phi(A) \neq \Phi_f(A)$ for some set A.

Example 6. *Given f as above, there exists A, where $0 \in \Phi(A)$ but $0 \notin \Phi_f(A)$.*

Proof. Let $a_n = \frac{1}{2^n}$ and $b_n = \frac{1}{2^n} + \frac{1}{3^n}$. Then define

$$A = \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup \bigcup_{n=1}^{\infty} (-a_n, -b_{n+1}).$$

We note that

$$A^c \cap (0, \infty) = \bigcup_{n=1}^{\infty} [a_{n+1}, b_{n+1}].$$

To prove that $0 \in \Phi(A)$, it shall suffice to prove that

$$\lim_{h\to 0^+} \sup \frac{m(A^c \cap [0,h])}{h} = 0.$$

Let $\epsilon > 0$. Since $\left(\frac{2}{3}\right)^n \to 0$, there exists n_0 such that

$$\left(\frac{2}{3}\right)^n < \frac{2}{3}\epsilon$$

for $n \geq n_0$. Note that for any $n \geq n_0$,

$$\frac{m(A^c \cap [0, b_n])}{m([0, b_n])} = \frac{\frac{3}{2} \cdot \frac{1}{3^n}}{\frac{1}{2^n} + \frac{1}{3^n}} \leq \frac{\frac{3}{2} \cdot \frac{1}{3^n}}{\frac{1}{2^n}} = \frac{3}{2}\left(\frac{2^n}{3^n}\right) < \epsilon.$$

Let h be an arbitrary number from $(0, b_{n_0})$. There is $n \geq n_0$ such that $h \in [b_{n+1}, b_n)$.

If $n \in [b_{n+1}, a_n)$, then

$$\frac{m(A^c \cap [0,h])}{m([0,h])} \leq \frac{m(A^c \cap [0, b_{n+1}])}{b_{n+1}} < \epsilon$$

because $\frac{a}{b} \leq \frac{a}{b-x}$ for $b \geq x \geq 0$.

If $h \in [a_n, b_n)$, then

$$\frac{m(A^c \cap [0,h])}{h} \leq \frac{m(A^c \cap [0, b_n])}{b_n} < \epsilon$$

because $\frac{a}{b} \leq \frac{a+x}{b+x}$ for $x \geq 0$ and $b \geq a > 0$.

Thus, 0 is a dispersion point of A^c and $0 \in \Phi(A)$.

It is not hard to show that $0 \notin \Phi_f(A)$ because

$$\frac{m((f(b_{n+1}), f(a_n)))}{m((f(a_{n+1}), f(b_{n+1})))} = 1$$

for all $n \in \mathbb{N}$. □

We note that homeomorphisms with property N which satisfy the Lipschitz condition do not necessarily have $\Phi(A) = \Phi_f(A)$ for all measurable A.

4. A Topology Based on *f*-Density

Theorem 7. *Let S denote the family of all measurable sets. The set $T_f = \{A \in S : A \subset \Phi_f(A)\}$ is a topology.*

Proof. Now $\mathbb{R} = \Phi_f(\mathbb{R})$ and $\emptyset = \Phi_f(\emptyset)$, so $\mathbb{R}, \emptyset \in T_f$. Let $A, B \in T_f$, so $A \subset \Phi_f(A)$ and $B \subset \Phi_f(B)$. Now $\Phi_f(A \cap B) = \Phi_f(A) \cap \Phi_f(B) \supset A \cap B$. Thus $A \cap B \in T_f$.

It remains to show that T_f is closed under arbitrary union. Let $F = \{A_\alpha : A_\alpha \subset \Phi_f(A_\alpha)\}$ for $\alpha \in \Gamma$. Let $A = \bigcup_{\alpha \in \Gamma} A_\alpha$. Since f is a homeomorphism, for all $\alpha \in \Gamma$ we have $f(A_\alpha) \subset f(\Phi_f(A_\alpha))$. Since $f(\Phi_f(A_\alpha)) = \Phi(f(A_\alpha))$, we then know $f(A_\alpha) \in T_d$ for all $\alpha \in \Gamma$. Thus $f(A) = \bigcup_{\alpha \in \Gamma} f(A_\alpha) \in T_d$, and so is measurable. Since f is a homeomorphism satisfying property N, we know A is measurable. Also, since $f(A) \in T_d$, $f(A) \subset \Phi(f(A))$. We then have $A \subset f^{-1}(\Phi(f(A))) = \Phi_f(A)$, since for all B measurable, $f^{-1}(\Phi(f(B))) = \Phi_f(B)$.

Thus $A \in T_f$. □

We may also give the following equivalent definition of T_f.

Theorem 8. *Set* $T_2 = \{\Phi_f(A)\backslash N : A \in S,\ m(N) = 0\}$. *Then* $T_f = T_2$.

Proof. Let $U \in T_f$. Thus $U \subset \Phi_f(U)$ and U is measurable. Thus $U = \Phi_f(U)\backslash(\Phi_f(U)\backslash U)$, where $m(\Phi_f(U)\backslash U) = 0$. So $U \in T_2$.

Let $U \in T_2$. So $U = \Phi_f(A)\backslash N$, where A is measurable and $m(N) = 0$. Since f is a homeomorphism that satisfies property N, $\Phi_f(A)$ is measurable and thus $\Phi_f(A)\backslash N$ is measurable. Notice that since $\Phi_f(A)\backslash N \sim A$, we have $\Phi_f(\Phi_f(A)\backslash N) = \Phi_f(A)$. Thus $\Phi_f(A)\backslash N \subset \Phi_f(A) = \Phi_f(\Phi_f(A)\backslash N)$. So $U \in T_f$. □

The following lemma highlights the connection between T_f and Φ_f.

Lemma 9. *If* $T_f = T_g$, *then* $\Phi_f(A) = \Phi_g(A)$ *for all measurable* A.

Proof. Suppose $T_f = T_g$. Let A be a measurable set. Now $\Phi_f(A) \in T_f = T_g$, so $\Phi_f(A) \subset \Phi_g(\Phi_f(A)) = \Phi_g(A)$. Also $\Phi_g(A) \in T_g = T_f$, so $\Phi_g(A) \subset \Phi_f(\Phi_g(A)) = \Phi_f(A)$. Thus $\Phi_f(A) = \Phi_g(A)$. □

Thus since in our counterexample above we found a function f and a set A, where $\Phi(A) \neq \Phi_f(A)$, we also know that $T_d \neq T_f$ from the lemma.

5. Limit Points in T_f

We now present a theorem describing the limit points of T_f.

Theorem 10. *A point* x *is a* T_f*-limit point of* A *if and only if* $f(x)$ *is a* T_d*-limit point of* $f(A)$.

Proof. Let x be a T_f-limit point of A. Thus for all $O \in T_f$ with $x \in O$ we have $O\backslash\{x\} \cap A \neq \emptyset$. Let $V \in T_d$ with $f(x) \in V$. Since f is a homeomorphism with property N, we have U, where $U = f^{-1}(V)$, with U measurable and $x \in U$. Further $U = f^{-1}(V) \subset f^{-1}(\Phi(V))$.

We shall prove that $f^{-1}(\Phi(V)) = \Phi_f(U)$. Let $z \in f^{-1}(\Phi(V))$, so $f(z) \in \Phi(V) = \Phi(f(U))$, so $z \in \Phi_f(U)$. Let $z \in \Phi_f(U)$, so $f(z) \in \Phi(f(U)) = \Phi(V)$, so $z \in f^{-1}(\Phi(V))$.

Therefore we have $U \subset \Phi_f(U)$ and so $U \in T_f$. Since x is a T_f-limit point of A, we know $U\backslash\{x\} \cap A \neq \emptyset$. Thus there exists $y \in U\backslash\{x\} \cap A$. So $f(y) \in f(U\backslash\{x\}\cap A) = f(U\backslash\{x\})\cap f(A) = (f(U)\cap\{f(x)\}^c)\cap f(A) = V\backslash\{f(x)\}\cap f(A)$, since f is bijective. Since $f(y) \in V\backslash\{f(x)\} \cap f(A)$, $V\backslash\{f(x)\} \cap f(A) \neq \emptyset$.

Therefore $f(x)$ is a T_d-limit point of $f(A)$.

Similarly, let $f(x)$ be a T_d-limit point of $f(A)$. So for all $O \in T_d$ with $f(x) \in O$ we have $O \backslash \{f(x)\} \cap f(A) \neq \emptyset$. Let $U \in T_f$ with $x \in U$. So $f(x) \in f(U)$, and since f is a homeomorphism with property N, $f(U)$ is measurable. Also $f(U) \subset f(\Phi_f(U)) = \Phi(f(U))$, so $f(U) \in T_d$, giving us that $f(U) \backslash \{f(x)\} \cap f(A) \neq \emptyset$. Thus there exists $y \in f(U) \backslash \{f(x)\} \cap f(A)$. Since f is bijective, we have $f^{-1}(y) \in U \backslash \{x\} \cap A$, which means that $U \backslash \{x\} \cap A \neq \emptyset$ and thus x is a T_f-limit point of A. □

Corollary 11. *A point x is a T_f-limit point of A if and only if*

$$\lim_{h \to 0^+} \sup \frac{m^*(f(A) \cap [f(x) - h, f(x) + h])}{2h} > 0.$$

Proof. This follows from the concept of a T_d-limit point [6]. □

6. Interior

Theorem 12. *For a set A, measurable kernel B of the set A, and f-density topology T_f,*

$$Int_{T_f} A = A \cap \Phi_f(B)\,.$$

Proof. We prove using the definition of

$$T_f = \{\Phi_f(M) \backslash N : M \text{ is a measurable set and } N \text{ is a nullset}\}.$$

Let $x \in Int_{T_f}(A)$. So $x \in A$ and we must show $x \in \Phi_f(B)$. Since $x \in Int_{T_f}(A)$, there exists an open set $O^* \in T_f$ such that $x \in O^* \subset A$. Since O^* is open, there exists a measurable set O and nullset N such that $O^* = \Phi_f(O) - N$. So $x \in \Phi_f(O)$. We know $m(O \backslash B) = 0$, because if not, B would not be the kernel of A. So $m(O \cap B) = m(O)$, and $\Phi_f(O) = \Phi_f(O \cap B) \subset \Phi_f(B)$. Thus, $x \in \Phi_f(B)$.

Let $x \in A \cap \Phi_f(B)$. We examine the set $\Phi_f(B) - (\Phi_f(\{x\} \cup B) \Delta (\{x\} \cup B))$. We know that $x \in \Phi_f(B) - (\Phi_f(\{x\} \cup B) \Delta (\{x\} \cup B))$ because $x \in \Phi_f(B)$ and $\Phi_f(B) = \Phi_f(\{x\} \cup B)$, so x cannot be in the symmetric difference of $(\Phi_f(\{x\} \cup B) \Delta (\{x\} \cup B))$. Also, since B is measurable, $\Phi_f(\{x\} \cup B)$ is measurable and we know that $(\Phi_f(\{x\} \cup B) \Delta (\{x\} \cup B))$ is a nullset. Thus, $\Phi_f(B) - (\Phi_f(\{x\} \cup B) \Delta (\{x\} \cup B))$ is in the f-density topology.

It remains to show that $\Phi_f(B) - (\Phi_f(\{x\} \cup B) \Delta (\{x\} \cup B)) \subset A$.

Let $x^* \in \Phi_f(B) \backslash A$. We must show that $x^* \in (\Phi_f(\{x\} \cup B) - (\{x\} \cup B)) \cup$

$((\{x\} \cup B) - \Phi_f(\{x\} \cup B))$. Since $x^* \in \Phi_f(B)\backslash A$, $x^* \in \Phi_f(\{x\} \cup B)$, and since $(\{x\} \cup B) \subset A$, $x^* \in (\Phi_f(\{x\} \cup B) - (\{x\} \cup B))$. Thus, $\Phi_f(B) - (\Phi_f(\{x\} \cup B)\Delta(\{x\} \cup B)) \subset A$, and we are finished. □

7. Closure

Lemma 13. *For all measurable sets A, $\Phi(A) \subset \Phi(A)'$ and $\Phi_f(A) \subset \Phi_f(A)'$.*

Proof. First we show $\Phi(A) \subset \Phi(A)'$. Let A be a measurable set. Suppose there exists $x \in \Phi(A)\backslash\Phi(A)'$. So there exists $U \in T_0$ with $x \in U$ and $U\backslash\{x\} \cap \Phi(A) = \emptyset$. Thus,

$$\lim_{h \longrightarrow 0} \frac{m(A \cap [x-h, x+h])}{2h} = \lim_{h \longrightarrow 0} \frac{m(\Phi(A) \cap [x-h, x+h])}{2h} = 0\,,$$

since $A \sim \Phi(A)$. Thus $x \notin \Phi(A)$, which is a contradiction.

Now we show $\Phi_f(A) \subset \Phi_f(A)'$. Suppose now there exists $x \in \Phi_f(A)\backslash\Phi_f(A)'$. So there exists $U \in T_0$ with $x \in U$ and $U\backslash\{x\} \cap \Phi_f(A) = \emptyset$. So $f(U)\backslash\{f(x)\} \cap \Phi(f(A)) = \emptyset$ with $f(U)$ open and $f(x) \in f(U)$. Thus we have

$$\lim_{h \longrightarrow 0} \frac{m(\Phi(f(A)) \cap [f(x)-h, f(x)+h])}{2h} = 0,$$

so $\lim_{h \longrightarrow 0} \frac{m(f(A) \cap [f(x)-h, f(x)+h])}{2h} = 0$ since $\Phi(f(A)) \sim f(A)$. So $f(x) \notin \Phi(f(A))$. But since $x \in \Phi_f(A)$, $f(x) \in \Phi(f(A))$, so we have a contradiction. □

Theorem 14. *For a homeomorphism f with property N and measurable set B,*

$$\overline{\Phi(B)} = \overline{\Phi_f(B)}\,.$$

Proof. Let B be a measurable set. Since $\Phi(B) \subset (\Phi(B))'$, $\Phi_f(B) \subset (\Phi_f(B))'$, and $\overline{B} = B \cup B'$, it will suffice to prove that $\Phi(B)' = (\Phi_f(B))'$.

Let $x \in \Phi(B)'$. We must prove that $x \in (\Phi_f(B))'$. Since $x \in \Phi(B)'$, there exists a sequence $\{x_n\} \in \Phi(B)$ such that $\{x_n\} \to x$ with $\{x_n\} \neq x$ for all n. Let $\epsilon > 0$. There exists an n such that $|x_n - x| < \epsilon$. Since $x_n \in \Phi(B)$,

$$\lim_{h \to 0} \frac{m(B \cap (x_n - h, x_n + h)}{2h} = 1.$$

Let $\epsilon_2 > 0$. So for some $h > 0$,

$$\frac{m(B \cap (x_n - h, x_n + h))}{2h} > 1 - \epsilon_2\,.$$

Because f has property N, it must be that

$$m(f(B \cap (x_n - h, x_n + h)) > 0.$$

Since f is a homeomorphism, $m(f(B) \cap f(x_n - h, x_n + h)) > 0$. Since for any measurable A, $\Phi(A) \sim A$, we know that

$$m(\Phi(f(B) \cap f(x_n - h, x_n + h))) > 0$$

and therefore is non-empty. So there exists a $x^* \in (x_n - h, x_n + h)$ such that $x^* \in \Phi_f(B)$.

Lastly, since x_n is closer to x than ϵ, and x^* is closer to x_n than h, we know

$$|x^* - x| < \epsilon + h.$$

We can constrict ϵ and ϵ_2 (ϵ_2 restricts h), thereby inductively defining a sequence of $\{x_n^*\} \in \Phi_f(B)$ so that $\{x_n^*\} \to x$. □

However, we can see that the closure of a set A in T_d, $\overline{A}^{T_d}$, is not necessarily the same as the closure of the set in T_f, $\overline{A}^{T_f}$, in the following informal discussion.

We may take a continuous f such that for $x \in (0, \frac{1}{2}]$, the intervals $[\frac{1}{2^{n+1}} + \frac{1}{3^{n+1}}, \frac{1}{2^n}]$ map onto $[\frac{1}{2^n} - \frac{1}{3^n}, \frac{1}{2^n}]$ and the intervals $[\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}}]$ map onto $[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{3^n}]$ in an increasing, one-to-one manner. For $x \in [-\frac{1}{2}, 0)$, we set $f(x) = -f(-x)$. Then we set $f(x) = x$ for $x \in (-\infty, -\frac{1}{2}) \cup \{0\} \cup (\frac{1}{2}, \infty)$.

Denote $A = \bigcup_{n=1}^{\infty}[\frac{1}{2^{n+1}} + \frac{1}{3^{n+1}}, \frac{1}{2^n}]$ and $B = \bigcup_{n=1}^{\infty}[\frac{1}{2^n} - \frac{1}{3^n}, \frac{1}{2^n}]$. Such a function will map $A \cup -A$ onto $B \cup -B$. We thus have $0 \in \Phi(A) \subset \overline{A}^{T_d}$. However, it can be shown that $f(0) = 0 \in \Phi(f(A)^c)$ since 0 is a dispersion point of $f(A)$, so $0 \notin \overline{A}^{T_f}$.

8. Conditions for $T_d = T_f$

We now examine the conditions under which $T_d = T_f$.

Theorem 15. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism where f and f^{-1} satisfy property N. Given $x \in \mathbb{R}$, if there exists $U \in T_0$ with $x \in U$ and there exists $B, C \in \mathbb{R}^+$, where for all $y, z \in U, B \leq \frac{|f(z)-f(y)|}{|z-y|} \leq C$, then for all measurable sets A, $x \in \Phi(A)$ if and only if $x \in \Phi_f(A)$.*

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism where f and f^{-1} satisfy property N and let $x \in \mathbb{R}$. Assume we have $U \in T_0$ with $x \in U$, where there exists $B, C \in \mathbb{R}^+$ so that for all $y, z \in U$, we have $B \leq \frac{|f(z)-f(y)|}{|z-y|} \leq C$. Let A be a measurable set.

Let $x \in \Phi(A)$. So $\lim_{h \longrightarrow 0} \frac{m(A^c \cap [x-h, x+h])}{2h} = 0$. Now there exists $\delta > 0$

where $(x-\delta, x+\delta) \subset U$. So for $h < \delta$, $[x-h, x+h] \subset U$, and thus for all $h < \delta$, $m(f(A^c \cap [x-h, x+h])) \leq Cm(A^c \cap [x-h, x+h])$. We then know:

$$0 = \lim_{h \longrightarrow 0} \frac{m(A^c \cap [x-h, x+h])}{2h} \geq \frac{1}{C} \lim_{h \longrightarrow 0} \frac{m(f(A^c \cap [x-h, x+h]))}{2h} = \frac{1}{C} \lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x-h), f(x+h)])}{2h}.$$

Again, for $h < \delta$, since $[x-h, x+h] \subset U$, we have $|f(x) - f(x-h)| \geq B|x-x+h|$, so $f(x) - Bh \geq f(x-h)$. Similarly, for $h < \delta$, $f(x+h) \geq f(x) + Bh$. Thus for $h < \delta$, $[f(x) - Bh, f(x) + Bh] \subset [f(x-h), f(x+h)]$. Thus

$$0 \geq \lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x-h), f(x+h)])}{2h} \geq \lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x) - Bh, f(x) + Bh])}{2h}.$$

Hence $\lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x) - Bh, f(x) + Bh])}{2Bh} \leq 0$, so $f(x) \in \Phi(f(A))$, giving us $x \in \Phi_f(A)$.

On the other hand, let $x \in \Phi_f(A)$. Then

$$0 = \lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x) - Ch, f(x) + Ch])}{2Ch}.$$

As above we have $\delta > 0$, so that $(x-\delta, x+\delta) \subset U$. So for $h < \delta$, $[x-h, x+h] \subset U$. Thus we have $f(x) - Ch \leq f(x-h)$ and $f(x) + Ch \geq f(x+h)$. So for $h < \delta$, $[f(x-h), f(x+h)] \subset [f(x) - Ch, f(x) + Ch]$. So,

$$0 = \lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x) - Ch, f(x) + Ch])}{2Ch} \geq \lim_{h \longrightarrow 0} \frac{m(f(A)^c \cap [f(x-h), f(x+h)])}{2Ch} = \frac{1}{C} \lim_{h \longrightarrow 0} \frac{m(f(A^c \cap [x-h, x+h])}{2h}.$$

Since $[x-h, x+h] \subset U$ for $h < \delta$ we have $f^{-1}(f(A^c \cap [x-h, x+h])) \leq \frac{1}{B} f(A^c \cap [x-h, x+h])$. So,

$$0 \geq \lim_{h \longrightarrow 0} \frac{m(f(A^c \cap [x-h, x+h]))}{2h} \geq B \lim_{h \longrightarrow 0} \frac{m(f^{-1}(f(A^c \cap [x-h, x+h])))}{2h} = B \lim_{h \longrightarrow 0} \frac{m(A^c \cap [x-h, x+h])}{2h}.$$

Thus $0 = \lim_{h \longrightarrow 0} \frac{m(A^c \cap [x-h, x+h])}{2h}$, so $x \in \Phi(A)$. □

From this theorem, we get the following corollary, which follows immediately from the previous theorem.

Corollary 16. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism, where f and f^{-1} satisfy property N. If for all $x \in \mathbb{R}$, there exists $U \in T_0$ with $x \in U$ and there exists $B, C \in \mathbb{R}^+$, where for all $y, z \in U$, $B \leq \frac{|f(z)-f(y)|}{|z-y|} \leq C$, then $T_f = T_d$.*

This condition from the theorem above is not the necessary condition for $T_d = T_f$, as the following discussion will show.

9. Density Continuous

We note a helpful connection between f-density topologies and density continuous functions.

Theorem 17. *Given f_1, f_2 homeomorphisms with f_1, ${f_1}^{-1}$, f_2, ${f_2}^{-1}$ satisfying property N, we see that $T_{f_1} = T_{f_2}$ if and only if $f_1 \circ {f_2}^{-1}$ and $f_2 \circ {f_1}^{-1}$ are density continuous.*

Proof. Assume $T_{f_1} = T_{f_2}$. We show that $f_2 \circ {f_1}^{-1}$ is density continuous. Let $U \in T_d$. Now ${f_2}^{-1}(U) \in T_{f_2}$ since $f_2 \circ {f_2}^{-1}(U) \subset \Phi(f_2 \circ {f_2}^{-1}(U))$. So ${f_2}^{-1}(U) \in T_{f_1}$. Thus $f_1 \circ {f_2}^{-1}(U) \subset \Phi(f_1 \circ {f_2}^{-1}(U))$ and so $f_1 \circ {f_2}^{-1}(U) \in T_d$. Since the preimage of U by $f_2 \circ {f_1}^{-1}$ is open in T_d, $f_2 \circ {f_1}^{-1}$ is density continuous. Similar work shows that $f_1 \circ {f_2}^{-1}$ is also density continuous.

We now assume that $f_1 \circ {f_2}^{-1}$ and $f_2 \circ {f_1}^{-1}$ are density continuous. Let $U \in T_{f_1}$. Thus $f_1(U) \in T_d$. Since $f_1 \circ {f_2}^{-1}$ is density continuous, $f_2 \circ {f_1}^{-1} \circ f_1(U) \in T_d$, which is to say, $f_2(U) \in T_d$. Consequently, $U \in T_{f_2}$, giving us that $T_{f_1} \subset T_{f_2}$. The fact that $T_{f_2} \subset T_{f_1}$ follows similarly, □

We immediately have several corollaries.

Corollary 18. *Given f a homeomorphism with f, f^{-1} satisfying property N, we see that $T_f = T_d$ if and only if f and f^{-1} are density continuous.*

It is known that all functions of the form x^α for $\alpha \in \mathbb{R}^+$ are density continuous [1]. Since the inverses of these functions are also of the form x^α, we have the following corollary.

Corollary 19. *Given f a homeomorphism with f, f^{-1} satisfying property N and $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R}^+$, we know $T_f = T_d$.*

We thus see that Corollary 2 is not the necessary condition for $T_f = T_d$, since the function $f(x) = x^3$ does not satisfy the conditions of the corollary and yet $T_f = T_d$ for this function.

We conclude by showing that Theorem 12 is best possible. That is, we will show that there exists a function f, where f is not density continuous but f^{-1} is density continuous.

Example 20. *Let*

$$f(x) = \begin{cases} \frac{1}{n!} \text{ when } x = \frac{1}{n!} \text{ for all } n \in \mathbb{N}\,, \\ \frac{1}{n!} - \frac{n}{10^n(n+1)!} \text{ when } x = \frac{1}{n!} - \frac{n}{10(n+1)!} \text{ for all } n \in \mathbb{N}\,. \end{cases}$$

Define $f(x)$ to be linear elsewhere. The function f is not density continuous, but f^{-1} is density continuous.

Proof. Clearly, f is a homeomorphism with f, f^{-1} satisfying property N. Let $a_n = \frac{1}{(n+1)!}$, $b_n = \frac{1}{n!} - \frac{n}{10(n+1)!}$. Set $A = (-\infty, 0) \cup \bigcup_{n=1}^{\infty}(a_n, b_n)$ and note $f(A) = (-\infty, 0) \cup \bigcup_{n=1}^{\infty}(f(a_n), f(b_n))$. It can be shown that $0 \in \Phi(f(A))$ but $0 \notin \Phi(A)$ since the relative measure of $(f(b_n), f(a_{n-1}))$ in $(f(a_n), f(a_{n-1}))$ goes to zero, but the relative measure of (b_n, a_{n-1}) in (a_n, a_{n-1}) does not. So $f(A) \in T_d$ but $A \notin T_d$. Thus f is not density continuous. We will now show that f^{-1} is density continuous. Let $E \subset \mathbb{R}$. By Theorem 11, it suffices to show that if $0 \in \Phi(E)$ then $0 \in \Phi(f(E))$. Assume $0 \in \Phi(E)$ and let $\epsilon > 0$. Since $f((-\infty, 0)) = (-\infty, 0)$, we only need show that $\frac{m(f(E)\cap[0,h])}{h} > 1 - \epsilon$ for some h. Since $0 \in \Phi(E)$, there exists $\delta > 0$ so that when $h < \delta$, we have $\frac{m(E\cap[0,h])}{h} > 1-\epsilon$. So for some $b_n < h < \delta$, $m(E \cap [0, b_n]) > b_n - b_n\epsilon$ and so $m(E \cap [a_n, b_n]) > b_n - b_n\epsilon - a_n$. Now $m(f(E) \cap [f(a_n), f(b_n)]) > \frac{f(b_n)-f(a_n)}{b_n-a_n} m(E \cap [a_n, b_n])$. So,

$$\frac{m(f(E) \cap [f(a_n), f(b_n)])}{f(b_n)} > (1 - \epsilon)\frac{(f(b_n) - f(a_n))b_n}{(b_n - a_n)f(b_n)} - \frac{(f(b_n) - f(a_n))a_n}{(b_n - a_n)f(b_n)}.$$

Now

$$\frac{f(b_n) - f(a_n)}{b_n - a_n}\frac{b_n}{f(b_n)} = \frac{10^n(9n) + 10^{n+1} - 9n}{10^n(9n) - 9n + 10^n(9)} \longrightarrow 1\,, \quad \text{as } n \longrightarrow \infty.$$

Also

$$\frac{f(b_n) - f(a_n)}{b_n - a_n}\frac{a_n}{f(b_n)} = \frac{10^{n+1} - 10}{10^n(9n) + (9)10^n - 9n} \longrightarrow 0\,, \quad \text{as } n \longrightarrow \infty.$$

Therefore

$$\frac{m(f(E) \cap [0, f(b_n)])}{f(b_n)} > \frac{m(f(E) \cap [f(a_n), f(b_n)])}{f(b_n)} \longrightarrow 1 - \epsilon.$$

Thus $0 \in \Phi(f(E))$ and so f^{-1} is density continuous. Thus for the function f as above, $T_d \subsetneq T_f$. □

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