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SOME CLASSES OF GENERALIZED COMPLETELY REGULAR SPACES

Bhamini M.P. Nayar
Department of Mathematics
Morgan State University
1700, E. Cold Spring Lane, Baltimore, MD 21251, USA
e-mail: Bhamini.Nayar@morgan.edu

Abstract: Studying spaces defined by functions into real number system, we study four classes of spaces, namely the classes of SR-spaces, SER-spaces, SSR-spaces and $\alpha-CR$-spaces. A space is said to be an SR-space (respectively, SER-space, SSR-space. $\alpha - CR - space$), if for each closed set $F \subseteq X$ and a point $x \in X - F$ there is a semi-continuous (respectively, upper semi-continuous and quasi lower semi-continuous, lower semi-continuous and quasi upper semi-continuous, upper semi-continuous and lower α-continuous) function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$. Each of these classes of spaces is characterized by characteristic functions obtained from different generalizations of continuous functions and has properties which are analogous to the properties of completely regular spaces. Hence each of these types of spaces is considered as a Generalized Completely Regular Space.

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1. Introduction

Investigating topological spaces through functions is a classical approach used by many researchers. In that attempt one often seeks weaker or generalized forms of continuity as tools. Some of the notions used to obtain such generalized concepts are θ-closure operator [19], $u - closure$ operator [8], regularly

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open sets, regularly closed sets, semi-open sets of Levine [10] , and α-sets of Njastad [15]. Very often combinations of the above concepts are used towards this end. Recently, in [8], using θ-closure and u-closure operators, several weak continuity forms and graph conditions were introduced. These operators were then used to give alternative proofs of some classical results and/or new characterizations of some well-known classes of spaces. They were used also to improve certain existing results. The $u - closure$ operator was utilized to isolate a "second category type" property of topological spaces. Dickman and Krystock [4], have used the above mentioned concepts, to study several new and established results. In [1], using some of the above mentioned concepts, several classes of functions were introduced. In [8] an extensive list of articles in this category is given. In the present article we shall study four classes of spaces, each of which is characterized by characteristic functions obtained from some of the generalized forms of continuity. We do not assume any separation axiom, unless stated.

When one looks at spaces defined by functions into the real number system, the concept of complete regularity comes foremost. A space X is *completely regular* if each closed subset and a point not in that set can be separated by continuous real valued functions. Hence if a space has the property that the characteristic function of each of its closed subsets is continuous, then that space is completely regular. We shall use analogous results of this implication to complete regularity to study the following four classes of spaces. We call these classes of spaces as the classes of SR, SER, SSR and $\alpha - CR$ spaces (in the sequel, it will become clear why these names are chosen). Three of the above mentioned classes of spaces, the classes of SR, SER, and SSR spaces contain the class of regular spaces. The class of $\alpha - CR$ spaces is contained in the class of regular spaces and contains the class of completely regular spaces. Construction of examples to distinguish the class of $\alpha - CR$ spaces from the class of regular spaces and the class of completely regular spaces falls in the study of construction of spaces which are regular but not completely regular and will be done separately. In [6], [13], [17], and [18] one could see study of spaces of this nature.

In [9] Kohli gave a unified treatment of different separation axioms in the model of the definition of complete regularity. For unifying purpose, a P-set is defined and a space X is *completely P-regular* if for each closed P-set and a point outside the set, there is a real valued continuous function on X, which has zero at the point and the value one on the set. For different separation axioms, the nature of the P-set varies.

Studying the class of spaces in which every closed set is θ-semi-closed, Ganster [5] studied the class of strongly s-regular spaces. It was identified as a class of spaces which is contained in the class of s-regular spaces of Maheshwary and Prasad [11] and contains the class of regular spaces. It was also shown that strong s-regularity is independent of almost regularity of Singal and Arya [16] and of semi-regularity.

Recall that a real-valued function f on a space X is *upper semi-continuous*, denoted as $u.s.c.$(respectively, *lower semi-continuous*, denoted as $l.s.c.$), if the set $\{x|f(x) < a\}$ (respectively, $\{x|f(x) > b\}$ is open, where a and b are any two real numbers. If the set $\{x|f(x) < a\}$ is semi-open (respectively, an α-set), we say that f is *quasi upper semi-continuous*, denoted as $q - u.s.c.$(respectively, *upper α-continuous*, denoted as $u.\alpha - c.$) and if $\{x|f(x) > b\}$ is semi-open (respectively, an α-set), we say that f is *quasi lower semi-continuous*, denoted as $q - l.s.c.$ (respectively, *lower α-continuous*, denoted as $l.\alpha - c.$), a and b being any two real numbers. A set $A \subseteq X$ is *semi-open* [10] if there is an open set $U \subseteq X$ such that $U \subseteq A \subseteq clU$, where clU represents the closure of U. A function $f : X \to Y$ is *semi-continuous* [10] if the inverse image of every open set is semi-open. If a function $f : X \to [0, 1]$ is u.s.c. and q-l.s.c. or l.s.c. and q-u.s.c., then f is semi-continuous, but a semi-continuous function need not be u.s.c. and q-l.s.c. or l.s.c. and q-u.s.c. as can be easily seen. A set $A \subseteq X$ is an *α-set* [15] if $A \subseteq int(cl(intA))$ where $intA$ represents the interior of A. A function $f : X \to Y$ is *α-continuous* [1] if the inverse image of every open set is an α-set. If a function is u.s.c. and l.α-c. or l.s.c. and u.α-c., then it is α-continuous but the converse need not be true. A space X is an *SR-space* (respectively, *SER-space*, *SSR-space*, *$\alpha - CR$-space*), if for each closed set $F \subseteq X$ and a point $x \in X - F$, there is a semi-continuous (respectively, u.s.c and q-l.s.c, l.s.c and q-u.s.c., u.s.c. and l.α-c.) function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$. We shall call $\alpha - CR$-space as *α-completely regular* space and show that the classes of SR, SER and SSR spaces are respectively equivalent to the classes of s-regular spaces of Maheshwary and Prasad [11], semi-regular spaces, and the class of strongly s-regular spaces of Ganster [5] and hence the respective names SR, SER, and SSR spaces. It is shown that an α-subset of an α-completely regular space is α-completely regular and product of α-completely regular spaces is α-completely regular. Every completely regular space is α-completely regular and an α-completely regular space is regular. For spaces in each of the above classes, we give characterizations using real valued functions. Since these four classes of spaces are defined and characterized in terms of functions into the real number system, in a way similar to the definition of completely regular spaces, we call this a study of some Classes of Generalized

Completely Regular Spaces.

This article is organized into two sections. In Section 3 we shall give characterizations and several related properties of these spaces. Examples are provided in Section 4. Below we shall give some additional preliminary concepts which are used in the paper.

2. Preliminary Concepts

We shall denote the family of all open subsets of X, containing a set A, by $\Sigma(A)$, ($\Sigma(\{x\})$, if $A = \{x\}$). Recall that a set $A \subseteq X$ is said to be *semi-open* [10] if there is an open set $U \subseteq X$ such that $U \subseteq A \subseteq clU$ and the complement of a semi-open set is said to be *semi-closed.* We shall denote all semi-open subsets of X containing A by $SO(A)$ ($SO(\{x\})$, if $A = \{x\}$). *θ-closure* [19] (respectively, *θ-semi-closure* [7]) of a set A, denoted as $cl_\theta A$ (respectively, $scl_\theta A$), is $\{x \in X : A \cap clV \neq \emptyset$ is satisfied for each $V \in \Sigma(\{x\})\}$ (respectively, $\{x \in X : A \cap clV \neq \emptyset$ is satisfied for each $V \in SO(\{x\})$). A set B is θ-closed, if $B = cl_\theta B$ and B is θ-semi-closed if $B = scl_\theta B$. A set A is *regularly open*, (respectively, *regularly closed*) if $A = int(cl(A))$ or equivalently, interior of a closed set (respectively, $A = cl(intA))$ or equivalently, closure of an open set).

It may be noted that every regularly open set is open and every open set is semi-open. Also it is known that the reverse statements are not true in general. The closure of a semi-open set is regularly closed and a regularly closed set is semi-open as well as closed. Therefore, for a topological space $(X,\ T)$, the family of all regularly closed subsets is same as the family of the closures of all semi-open subsets of $(X,\ T)$. A set A is regularly closed if and only if it is closed and semi-open. Hence the family of all regularly closed subsets of a space is the family of all closed semi-open subsets. In light of this, we can see that $scl_\theta A = \{x \in X : V \cap A \neq \emptyset$, for any $V \in RC(\{x\})\}$, where $RC(\{x\})$ is the family of regularly closed sets containing $\{x\}$. Thus, it is evident that a regularly open set is θ-semi-closed. A regularly open set is open as well as semi-closed. Also, interior of a semi-closed set is regularly open. For, if A is semi-closed, there is a closed set C such that $intC \subseteq A \subseteq C$. Now, $intC \subseteq int(cl(A)) \subseteq (int(clC)) = intC$. Thus $intC = int(cl(A))$, which is the interior of a closed set. A set A is regularly open if and only if it is open and semi-closed. Thus the family of regularly open sets is the family of those sets which are the interior of a semi-closed set. In other words, the family of all regularly open sets of a space is the family of open semi-closed subsets of the space.

Recall that a function $f : X \to Y$ is *semi-continuous* [10] (respectively, *irresolute* [3]), if the inverse image of every open set (respectively, semi-open set) is semi-open and $f : X \to Y$ is said to be $\alpha - continuous$[1] if the inverse image of every open set is an α-set. A space X is said to be *s-regular* [11] if for each closed set $F \subseteq X$ and $x \in X - F$ there exists a pair of disjoint semi-open sets U and V such that $x \in U$ and $F \subseteq V$. A space $(X,\ T)$ is *strongly s-regular* [5] if for any closed set $A \subseteq X$ and any point $x \in X - A$, there is an $F \in RC(X,\ T)$ with $x \in F$ and $F \cap A = \emptyset$, where $RC(X,\ T)$ denotes the family of regularly closed subsets of the space $(X,\ T)$. Recall that a space $(X,\ T)$ is *semi-regular* if for each closed set $A \subseteq X$ and $x \in X - A$, there is an $F \in RO(X,\ T)$ such that $x \in F$ and $F \cap A = \emptyset$, where $RO(X,\ T)$ represents the family of regularly open subsets of $(X,\ T)$. A space $(X,\ T)$ is *almost regular* [16] if for each $F \in RC(X,\ T)$ and $x \in X - F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

3. Characterizations and Related Properties

In this section we shall study four classes of spaces, three of theses classes of spaces contain the class of regular spaces. These three classes of spaces, namely, the classes of SR, SER, and SSR-spaces are characterized using generalizations of continuous functions which are characteristic functions of closed subsets. We treat them in a fashion similar to the properties of completely regular spaces. These classes of spaces are dealt with in Subsection 3.1 whereas the class of $\alpha - CR$ is treated in Subsection 3.2. These characterizations will show that the class of SR-spaces is equivalent to the class of s-regular spaces of Maheshwary and Prasad [11], SER-spaces are equivalent to the well-known semi-regular spaces and the class of SSR spaces is equivalent to the class of strongly s-regular spaces of Ganster [5].

3.1. SR-Spaces, SER-Spaces and SSR-Spaces

Definitions 3.1.1. A space $(X,\ T)$ is said to be an *SR-space* (respectively, an *SER-space*, *SSR-space*) if for each closed set $F \subseteq X$ and a point $x \in X - F$, there is a semi-continuous (respectively, an u.s.c. and q-l.s.c., l.s.c. and q-u.s.c.) function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.

From the above definitions, it is clear that an SSR-space as well as an SER-space is an SR-space. Examples in Section 4 will show that the reverse implications need not be true in general.

Definition 3.1.2. (see [11]) A space $(X,\ T)$ is said to be *s-regular* if for each closed set $F \subseteq X$ and a point $x \in X - F$, there exist disjoint semi-open sets U and V contained in X such that $x \in U$ and $F \subseteq V$.

Definition 3.1.3. A space$X$ is *semi-regular* if for each $x \in X$ and an open set U containing x, there is a regularly open set V such that $x \in V \subseteq U$.

Definition 3.1.4. (see [5]) A space $(X,\ T)$ is *strongly s-regular* if for any closed set F and a point $x \in X - F$, there is a $G \in RC(X,\ T)$ such that $x \in G$ and $G \cap F = \phi$.

We shall use the following preliminary results in the sequel and Theorem 3.1.5 is stated without proof.

Theorem 3.1.5. *If $f : X \to Y$ is semi-continuous and if $g : Y \to Z$ is continuous, then $g \circ f$ is semi-continuous.*

Theorem 3.1.6. *Let $g : X \to Y$ be continuous and $h : X \to Z$ be semi-continuous. Then $f : X \to Y \times Z$ defined by $f(x) = (g(x), h(x))$ is semi-continuous.*

Proof. Let $A \times B$ be a basic open subset of $Y \times Z$. Then $f^{-1}(A \times B) = g^{-1}(A) \cap h^{-1}(B)$. Since g is continuous and h is semi-continuous, $g^{-1}(A)$ is open and $h^{-1}(B)$ is semi-open. Therefore, $f^{-1}(A \times B)$ is semi-open and hence f is semi-continuous. □

The following lemma can be proved as a consequence of Theorems 3.1.5 and 3.1.6.

Lemma 3.1.7. *Let g and f, respectively, be continuous and semi-continuous real-valued functions on X. Then each of the following is true.*

(a) $g + f$ is semi-continuous.

(b) gf is semi-continuous,

(c) $\frac{g}{f}$ is semi-continuous, if $0 \notin f(X)$.

(d) $\frac{f}{g}$ is semi-continuous, if $0 \notin g(X)$.

(e) Max$\{f, g\}$ is semi-continuous.

(f) Min$\{f, g\}$ is semi-continuous.

(g) f^k is semi-continuous for each positive integer k.

Proof. We prove (a) and (c). $h : R \times R \to R$ defined as $h(x, y) = x + y$ is continuous. Hence the proof of (a) follows from Theorems 3.1.5 and 3.1.6. The proof of (c) follows from the facts that $h(x) = \frac{1}{x}$ is continuous, that f is semi-continuous and (b). □

Definitions 3.1.8. A subset $Z \subseteq X$ is said to be an *s-zero set* if there is a real-valued semi-continuous function f on X such that $Z = \{x | f(x) = 0\}$. The complement of an s-zero set is said to be an *s-cozero set.*

Clearly, every s-zero set is semi-closed and an s-cozero set is semi-open. In the following theorem we see that for an SR-space every open set is a union of family of s-cozero sets.

Theorem 3.1.9. *An open subset of an SR-space is a union of s-cozero sets.*

Proof. Let U be a non-empty open subset of an SR-space X. Let $x \in U$. Then there is a semi-continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X - U) = 1$. Define the function $g = 1 - f$. In view of Lemma 3.1.7, g is semi-continuous, $g(x) = 1$ and $g(X - U) = 0$. Let C be the s-cozero set of g. Then $x \in C$ and $C \cap (X - U) = \phi$. Hence $x \in C \subseteq U$. □

In the following Theorems 3.1.10, 3.1.11 and 3.1.12, we shall show that the concepts of being an SR-space, SER-space and SSR-space are respectively equivalent to the concepts of being an s-regular space, semi-regular space and strongly s-regular space. Moreover, the above mentioned theorems give characterizations of SR-space, SER-space and SSR-space, respectively, analogous to the Urysohn's Lemma in the case of normal spaces.

Theorem 3.1.10. *The following are equivalent for a space X.*

(a) X is an SR-space.

(b) X is s-regular

(c) For each closed set F and a point $x \in X - F$, there is an irresolute function $g : X \to [0, 1]$ such that $g(x) = 1$ and $g(F) = 0$.

Proof. $(a) \Rightarrow (b)$. Let $F \subseteq X$ be a closed set and $x \in X - F$. Let $g : X \to [0, 1]$ be a semi-continuous function such that $g(x) = 0$ and $g(F) = 1$. Then, for an a such that $0 < a < 1$, $g^{-1}(a, 1]$ and $g^{-1}[0, a)$ are the required semi-open sets.

$(b) \Rightarrow (c)$. Since X is s-regular, for each closed set $F \subseteq X$ and $x \in X - F$, there is a semi-open set $V \subseteq X$ such that $x \in V \subseteq sclV \subseteq X - F$. Since $sclV$ is both semi-open and semi-closed, the characteristic function of $sclV$, χ_{sclV} is irresolute and satisfies the conditions of (c).

$(c) \Rightarrow (a)$. Follows easily from the fact that an irresolute function g is semi-continuous and in view of Lemma 3.1.7, $1 - g$ is semi-continuous. Also, $1 - g$ satisfies the conditions of (a). □

Theorem 3.1.11. *The following are equivalent for a space X:*

(a) X is an SER-space.

(b) Each closed subset of X is an intersection of regularly closed subsets.

(c) Each open subset of X is a union of regularly open sets.

(d) X is semi-regular.

(e) For each closed set $F \subseteq X$ and a point $x \in X - F$ there is a l.s.c. and q-u.s.c. function $f : X \to [0,1]$ such that $f(F) = 0$ and $f(x) = 1$.

Proof. $(a) \Rightarrow (b)$. Let $F \subseteq X$ be closed and $x \in X - F$. Therefore, there is an u.s.c. and q-l.s.c. function $f : X \to [0,1]$, such that $f(F) = 1$ and $f(x) = 0$. Since f is u.s.c. and q-l.s.c., $f^{-1}[0, \frac{1}{2})$ is an open set and $f^{-1}(\frac{1}{2}, 1]$ is a semi-open set containing x and F respectively. Also, $f^{-1}[0, \frac{1}{2}) \bigcap f^{-1}(\frac{1}{2}, 1] = \emptyset$. For each $x \in X - F$, such a choice of f is possible. Therefore, each $x \in X - F$, $x \notin cl f^{-1}(\frac{1}{2}, 1]$, closure of a semi-open set. Thus each closed set F is contained in a regularly closed set and $x \in X - F$ is not in that regularly closed set. Therefore, each closed set is intersection of regularly closed sets.

$(b) \Rightarrow (c)$. Follows easily.

$(c) \Rightarrow (d)$. Clear from the definition of semi-regular spaces.

$(d) \Rightarrow (e)$. X is semi-regular and let F be a closed subset of X. If $x \in X - F$, there is a regularly open set A such that $x \in A \subseteq X - F$. A being regularly open, it is open and semi-closed. Hence the characteristic function of A, χ_A, is l.s.c. and q-u.s.c.. Moreover, $F \subseteq X - A = \chi_A^{-1}[0, \frac{1}{2})$. Also, $\chi_A(x) = 1$ and $\chi_A(F) = 0$. Thus there is a l.s.c. and q-u.s.c. function $g = \chi_A$ such that $g(x) = 1$ and $g(F) = 0$. □

Theorem 3.1.12. *The following are equivalent for a space X:*

(a) X is an SSR-space.

(b) Each closed subset of X is an intersection of semi-closed neighborhoods.

(c) Each closed set is an intersection of regularly open sets.

(d) X is strongly s-regular.

(e) For each closed set F and a point $x \in X - F$, there is an u.s.c. and q-l.s.c. function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(F) = 0$.

Proof. $(a) \Rightarrow (b)$. Suppose that X is an SSR-space and let F be a closed subset of X. For each $x \in X - F$, there is a l.s.c. and q-u.s.c. function $f : X \to [0,1]$ such that $f(F) = 1$ and $f(x) = 0$. Then $F \subseteq f^{-1}(\frac{1}{2}, 1]$ and $x \in f^{-1}[0, \frac{1}{2})$, where $f^{-1}(\frac{1}{2}, 1]$ is an open set and $f^{-1}[0, \frac{1}{2})$ is a semi-open set. Since $f^{-1}[0, \frac{1}{2}) \bigcap f^{-1}(\frac{1}{2}, 1] = \emptyset$, $x \notin scl f^{-1}(\frac{1}{2}, 1]$. For each $x \in X - F$, such a choice of f is possible. Therefore, F is an intersection of semi-closed neighborhoods.

$(b) \Rightarrow (c)$. Let $G \subseteq X$ be a closed set. In view of (b), $G = \bigcap_F W$, where F is a family of semi-closed neighborhoods of G. Since W is semi-closed, there is a closed set C such that $intC \subseteq W \subseteq C$. Also, $clW \subseteq C$ and hence, $intC \subseteq int(clW) \subseteq intC \subseteq W$. Furthermore, since each W is a semi-closed neighborhood of G, $G \subseteq intW \subseteq int(clW) \subseteq W$. Thus $G \subseteq \bigcap_F int(clW) \subseteq \bigcap_F W = G$. Hence, $G = \bigcap_F int(clW)$, intersection of regularly open sets.

$(c) \Rightarrow (d)$. Follows from Theorem 1 of [5].

$(d) \Rightarrow (e)$. X is strongly s-regular and let F be a closed subset of X and $x \in X - F$. In view of Theorem 1 of [5], there is a $U \in RC(x)$ such that $x \in U \subseteq X - F$. Then χ_U, the characteristic function of U, is an u.s.c. and q-l.s.c. function such that $\chi_U(F) = 0$ and $\chi_U(x) = 1$.

$(e) \Rightarrow (a)$. Follows from the fact that, if $f : X \rightarrow [0, 1]$ is u.s.c. and q-l.s.c., then $g = 1 - f$ is l.s.c. and q-u.s.c. □

Corollary 3.1.13. *If a space X is regular, then each open subset of X is a union of*

(i) regularly open sets;

(ii) regularly closed sets;

(iii) θ-semi-closed sets.

Proof. Follows from Theorems 3.1.11 and 3.1.12 and from the facts that a regular space is semi-regular as well as strongly s-regular and a regularly open set is θ-semi-closed. □

Corollary 3.1.14. *If X is regular, then X is an SR-space, SER-space as well as an SSR-space.*

Proof. Follows from Theorems 3.1.10, 3.1.11 and 3.1.12 and also from the facts that a regular space is s-regular, semi-regular and strongly s-regular. □

Definitions 3.1.15. A subset $Z \subseteq X$ is said to be a *lower s-zero set* (respectively, an *upper s-zero set*) if $Z = \{x : g(x) = 0\}$ for some l.s.c. and q-u.s.c. (respectively, u.s.c. and q-l.s.c.) function $g : X \rightarrow [0, 1]$. The complement of a lower s-zero set (respectively, upper s-zero set) is called a *lower s-cozero set* (respectively, *upper s-cozero set*). □

Theorem 3.1.16. *If X is an SER-space (respectively, an SSR-space), the family of lower s-cozero sets (respectively, upper s-cozero sets) form a base for the topology.*

Proof. Suppose X is an SER-space and let $x \in X$ and let V be an open set containing x. Then $X - V$ is a closed set and $x \notin X - V$. Hence, in view of Theorem 3.1.11(e), there is a l.s.c. and q-u.s.c. function $g : X \to [0, 1]$, such that $g(x) = 1$ and $g(X - V) = 0$. Let Z_g be the zero set of g. Then $x \in X - Z_g \subseteq V$. Also $X - Z_g = X - g^{-1}\{0\} = g^{-1}(0, 1]$, an open set since g is a l.s.c. function. The case when X is an SSR-space can be similarly handled.

3.2. α-Completely Regular Spaces (α-CR-space)

Definition 3.2.1. A space X is said to be *α-Completely Regular (denoted as α-CR-space)* if for each closed set F and a point $x \in X - F$ there is an u.s.c. and l.α-c. function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.

Since every continuous function is both l.α-c. and u.s.c., every completely regular space is an α-CR-space. In the following Theorem we shall show that every α-CR-space is regular.

Theorem 3.2.2. *An α-CR-space is regular.*

Proof. Let X be an α-CR-space and F be a closed subset of X. Suppose that $x \in X - F$. Then there is a l.α.c. and u.s.c. function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$. Then $V = f^{-1}(\frac{1}{2}, 1]$ is an α-set and $U = f^{-1}[0, \frac{1}{2})$ is an open set. Also $U \cap V = \emptyset$, $F \subseteq V$ and $x \in U$. Since V is an α-set, $V \subseteq int(cl(intV))$. We shall show that $U \cap (int(cl(intV)) = \emptyset$. Suppose that $U \cap (int(cl(intV)) \neq \emptyset$. Then $U \cap (cl(intV)) \neq \emptyset$. Since U is open, $U \cap (cl(intV)) \subseteq cl(U \cap (intV))$ and hence $cl(U \cap (intV)) \neq \emptyset$. Therefore, $U \cap (intV) \neq \emptyset$ and hence $U \cap V \neq \emptyset$, a contradiction. Also $F \subseteq int(cl(intV))$, an open set. Thus we have two disjoint open sets, containing x and F respectively and hence X is regular. □

Corollary 3.2.3. *An α-CR-space is an SR, SER as well as an SSR-space.*

Proof. Immediate, since a regular space is SR, SER as well as SSR, in view of Theorems 3.1.10, 3.1.11 and 3.1.12. □

Corollary 3.2.4. *Each open subset of an α-CR-space can be written as a union of*

(i) regularly open sets;

(ii) regularly closed sets;

(iii) θ-semi-closed sets.

Proof. Follows from the above corollary and Corollary 3.1.13. □

Theorem 3.2.5. *A space X is an α-CR-space if and only if for each closed set $F \subseteq X$ and $x \in X - F$, there is an u.α-c. and l.s.c. function $f :\to [0, 1]$ such that $f(x) = 1$ and $f(F) = 0$.*

Proof. Follows from the fact that if $g : X \to [0,1]$ is l.α-c. and u.s.c., then $1 - g = f : X \to [0,1]$ is u.α-c. and l.s.c. □

Definition 3.2.6. A subset $Z_g \subseteq X$ is said to be an *upper α-zero set* if $Z_g = \{x \in X : g(x) = 0\}$ for some u.α-c. and l.s.c. function $g : X \to [0,1]$. The complement of an upper α-zero set is called an *upper α-cozero set.*

Clearly, if Z_g is an upper α-zero set, then $X - Z_g = g^{-1}(0,1]$ and hence is open. Therefore, each upper α-cozero set is open.

Theorem 3.2.7. *If X is α-CR, then the family of upper α-cozero sets forms a base for the topology on X.*

Proof. Let X be an α-CR-space and $x \in X$. Suppose that U is an open set containing x. Then $X - U$ is a closed subset and $x \notin X - U$. Then in view of Theorem 3.2.5, there is an u.α.c. and l.s.c. function $g : X \to [0,1]$ such that $g(X - U) = 0$ and $g(x) = 1$. Clearly, $x \in X - Z_g$ and $X - U \subseteq Z_g$. Thus $x \in X - Z_g \subseteq U$. Therefore, the family of upper α-cozero sets form a base for the topology on X. □

Theorem 3.2.8. *An α- subset of an α-CR-space is α-CR.*

Proof. Suppose that $Y \subseteq X$ be an α- set and X is α-CR. Let F be a closed subset of Y and $y \in Y - F$. There is a closed subset K of X such that $F = X \cap K$ and $y \in X - K$. Hence there is a l.α-c. and u.s.c. function $f : X \to [0,1]$ such that $f(y) = 0$ and $f(K) = 1$. Let $h = f|Y$. Then h is a l.α-c. and u.s.c. function on Y, since Y is an α-set. Thus $h(F) = 1$ and $h(y) = 0$. Hence Y is an α-CR-space. □

Corollary 3.2.8. *An open subspace of an α-CR-space is α-CR.*

To study the above properties in a product space, we shall need the following properties of functions on the product space.

Theorem 3.2.9. *Let $\{X_\alpha | \alpha \in \Sigma\}$ be a family of spaces. Let $X = \Pi_\Sigma X_\alpha$ and let Γ be a finite non-empty subset of Σ. For each $\alpha \in \Gamma$, let $g_\alpha : X_\alpha \to Y_\alpha$ be semi-continuous (respectively, α-continuous). Then $g : X \to \Pi_\Gamma Y_\alpha$ defined by $g(x) = (g_\alpha(P_\alpha(x)))$ is semi-continuous (respectively, α-continuous), where P_α is the projection from X to X_α.*

Proof. Let $\Pi_\Gamma V_\alpha$ be a basic open subset of $\Pi_\Gamma Y_\alpha$. Then $g^{-1}(\Pi_\Gamma V_\alpha)= \bigcap_{\alpha \in \Gamma} P_\alpha^{-1}(g_\alpha^{-1} V_\alpha)$. Since for each $\alpha \in \Gamma, g_\alpha : X_\alpha \to Y_\alpha$ is semi-continuous (respectively, α-continuous), $g^{-1}(\Pi_\Gamma V_\alpha)$ is a semi-open (respectively, an α-set) in X. Therefore g is semi-continuous (respectively, α-continuous). □

Theorem 3.2.10. *Let $\{X_\alpha | \alpha \in \Sigma\}$ be a family of topological spaces and Γ be a non-empty and finite subfamily of Σ. For each $\alpha \in \Gamma$, let $g_\alpha : X_\alpha \to R$ be* *semi-continuous (respectively, α-continuous). Let $X = \Pi_\Sigma X_\alpha$. Then for each of the following cases, $\lambda : X \to R$, as defined, is semi-continuous (respectively, α-continuous).*

(a) $\lambda(x) = \Sigma_{\alpha\in\Gamma} g_\alpha \circ P_\alpha(x)$.

(b) $\lambda(x) = \Pi_{\alpha\in\Gamma} g_\alpha \circ P_\alpha(x)$.

(c) $\lambda(x) = \max\{g_\alpha \circ P_\alpha(x)|\alpha \in \Gamma\}$.

(d) $\lambda(x) = \min\{g_\alpha \circ P_\alpha(x)|\alpha \in \Gamma\}$.

We shall give the proof of (c). The proofs of (a), (b) and (d) can be done similarly.

Proof of (c). Let $Y_\alpha = R$ for each $\alpha \in \Gamma$. Then $h : \Pi Y_\alpha :\to R$, defined as $h(x) = \max\{P_\alpha(x)|\alpha\ in \Gamma\}$ is continuous. If g is defined as in Theorem 3.2.9 above, then for each $x \in X, (h \circ g)(x) = h((g_\alpha \circ P_\alpha(x))) = \max\{g_\alpha \circ P_\alpha(x)|\alpha \in \Gamma\} = \lambda(x)$. In view of Theorem 3.2.9, g is semi-continuous (respectively, α-continuous). Hence λ is semi-continuous (respectively, α-continuous.) □

Theorem 3.2.11. *Let $\{X_\alpha|\alpha \in \Sigma\}$ be a collection of spaces with the property that if $F_\alpha \subseteq X_\alpha$ is closed and $x_\alpha \in X_\alpha - F_\alpha$. Then there is a semi-continuous (respectively, α-continuous) function $g_\alpha : X_\alpha \to [0, 1]$ such that $g_\alpha(x_\alpha) = 0$ and $g_\alpha(F_\alpha) = 1$. Let $X = \Pi_\Sigma X_\alpha$. Then for each closed set $F \subseteq X$ and $x \in X - F$, there is a semi-continuous (respectively, α-continuous) function $g : X \to [0, 1]$ such that $g(x) = 0$ and $g(F) = 1$.*

Proof. Let $\{X_\alpha|\alpha \in \Sigma\}$ be a collection of spaces satisfying the hypothesis of the theorem and let $X = \Pi_\Sigma X_\alpha$. Let $F \subseteq X$ be a closed subset and $x \in X - F$. There is a non-empty finite $\Gamma \subseteq \Sigma$ and for each $\alpha \in \Gamma$, an open subset V_α of X_α such that $x \in \bigcap_\Gamma P_\alpha^{-1}(V_\alpha) \subseteq X - F$. For each $\alpha \in \Gamma, P_\alpha(x) \in V_\alpha$ and $X_\alpha - V_\alpha$ is a closed subset of X_α. So there is a semi-continuous (respectively, α-continuous) function $g_\alpha : X_\alpha \to [0, 1]$ such that $g_\alpha(P_\alpha(x)) = 0$ and $g_\alpha(X_\alpha - V_\alpha) = 1$. Let $\lambda : X \to [0, 1]$ be defined as $\lambda(y) = \max\{g_\alpha \circ P_\alpha(y)|\alpha \in \Gamma\}$. Then λ is semi-continuous (respectively, α -continuous), in view of Theorem 3.2.10. Moreover, $\lambda(x) = \max\{g_\alpha \circ P_\alpha(x)|\alpha \in \Gamma\} = 0$. Suppose $y \in F$. Then, since $y \notin X - F$, there is a $\mu \in \Gamma$ such that $y \notin P_\mu^{-1}(V_\mu)$, since $y \notin \bigcap_\Gamma P_\alpha^{-1}(V_\alpha)$. For such a $\mu, P_\mu(y) \notin V_\mu$. Hence $g_\mu \circ P_\mu(y) = 1$ and $\lambda(y) = 1$. □

Corollary 3.2.12. *Product of SR-spaces is an SR-space.*

Proof. Immediate, in view of Theorem 3.2.11. □

Theorem 3.2.13. *Let $\{X_\alpha|\alpha \in \Sigma\}$ be a family of spaces and let Γ be a non-empty finite subset of Σ. For each $\alpha \in \Gamma$, let $g_\alpha : X_\alpha \to R$ be a function and let $X = \Pi_\Sigma X_\alpha$. Then the following statements are true:*

If g_α is q-l.s.c. (respectively, q-u.s.c.,l.α-c, u.α-c.) for each $\alpha \in \Gamma$, then in each of the following cases, $\lambda : X \to R$, as defined, is q-l.s.c. (respectively, q-u.s.c.,l-α-c., u-α-c.).

(a) $\lambda(x) = \max\{g_\alpha \circ P_\alpha(x) | \alpha \in \Gamma\}$.

(b) $\lambda(x) = \min\{g_\alpha \circ P_\alpha(x) | \alpha \in \Gamma\}$.

We shall first prove (a) and (b) when g_α is q-l.s.c. for each $\alpha \in \Gamma$.

Proof. (a) Suppose that g_α is q-l.s.c. for each $\alpha \in \Gamma$, and suppose also that $x \in X$ and $\lambda(x) > a$, where a is any real number. Then there is a $\mu \in \Gamma$ such that $g_\mu \circ P_\mu(x) > a$. For such a μ, there is a semi-open subset A_μ of X_μ such that $g_\mu(A_\mu) \subseteq (a, \infty)$ and $P_\mu(x) \in A_\mu$. Then $x \in P_\mu^{-1}(A_\mu)$ and $P_\mu^{-1}(A_\mu)$ is a semi-open subset of X. Let $z \in P_\mu^{-1}(A_\mu)$. Then $P_\mu(z) \in A_\mu$ and $g_\mu(P_\mu(z)) \in (a, \infty)$ since $g_\mu(A_\mu) \subseteq (a, \infty)$. Hence $\lambda(z) = \max\{g_\alpha \circ P_\alpha(z) | \alpha \in \Gamma\} > a$. Thus we have a semi-open subset $A = P_\mu^{-1}(A_\mu)$ of X satisfying $x \in A$ and $\lambda(A) \subseteq (a, \infty)$. Thus λ is q-l.s.c..

(b) Suppose that g_α is q-l.s.c. for each $\alpha \in \Gamma$, let $x \in X$ and $\lambda(x) > a$. Then $g_\alpha \circ P_\alpha(x) > a$ for all $\alpha \in \Gamma$. So, for each $\alpha \in \Gamma$ there is a semi-open subset $A_\alpha \subseteq X_\alpha$ such that $P_\alpha(x) \in A_\alpha$ and $g_\alpha(A_\alpha) \subseteq (a, \infty)$. Furthermore, $\cap_\Gamma P_\alpha^{-1}(A_\alpha)$ is a semi-open subset of X and $x \in \cap_\Gamma P_\alpha^{-1}(A\alpha)$. Now let $z \in \cap_\Gamma P_\alpha^{-1}(A_\alpha)$. Then $P_\alpha(z) \in A_\alpha$ for each $\alpha \in \Gamma$ and $g_\alpha \circ P_\alpha(z) > a$. Thus λ is q-l.s.c..

Proof of (a) and (b) when g_α is q-u.s.c. or $l.\alpha - c.$ or $u.\alpha - c.$ for each $\alpha \in \Gamma$ follows from the above result and from the following: (1) The function λ is q-u.s.c. if and only if $-\lambda$ is q-l.s.c. and $\max D = -min(-D)$, where D is a nonempty finite collection of reals and $-D = \{-x | x \in D\}$; (2) If $A_\mu \subseteq X_\mu$ is an α-set, then $P_\mu^{-1}(A_\mu)$ is an α-subset of X; (3) If for each $\alpha \in \Gamma, A_\alpha$ is an α-subset of X_α, then $\cap_\Gamma P_\alpha^{-1}(A_\alpha)$ is an α-subset of X; and (4) λ is $u.\alpha - c.$ if and only if $-\lambda$ is $l.\alpha - c.$.

Thus the proof is complete. □

Theorem 1.2.14. *Product of $\alpha - CR$ spaces is $\alpha - CR$.*

Proof. Let $\{X_\alpha | \alpha \in \Sigma\}$ be a family of $\alpha - CR$ spaces and let $X = \Pi_\Sigma X_\alpha$. Let $x \in X$ and $F \subseteq X$ be closed and $x \in X - F$. Then there is a nonempty finite subset Γ of Σ such that for each $\alpha \in \Gamma$, there is an open subset V_α of X_α and $x \in \cap_\Gamma P_\alpha^{-1}(V_\alpha) \subseteq X - F$. Then $P_\alpha(x) \in V_\alpha$ for each $\alpha \in \Gamma$ and X_α is $\alpha - CR$. Hence there is a l.α-c. and u.s.c. function $g_\alpha : X_\alpha \to [0, 1]$ such that $g_\alpha(P_\alpha(x)) = 0$ and $g_\alpha(X_\alpha - V_\alpha) = 1$. Let $\lambda(p) = \max\{g_\alpha \circ P_\alpha(p) | \alpha \in \Gamma\}$. Then λ is l.α-c. and u.s.c., in view of Theorem 3.2.13. Moreover, $\lambda(x) = \max\{g_\alpha \circ P_\alpha(x) | \alpha \in \Gamma\} = 0$. Suppose that $y \in F$. Then $y \notin \cap_\Gamma P_\alpha^{-1}(V_\alpha)$. Hence there is a $\mu \in \Gamma$ such that $y \notin P_\mu^{-1}(V_\mu)$. For such a μ, $g_\mu(P_\mu(y)) = 1$ and so $\lambda(y) = 1$. Therefore, $\lambda(F) = 1$. Thus X is $\alpha - CR$. □

4. Examples

Spaces which are minimal for different topological properties have been topics of investigation for many researchers and the relationship of such spaces with compact spaces have been a significant area of study. While it is known that a regular space is minimal Hausdorff if and only if it is compact, the following is a classical example of a non-compact minimal Hausdorff space. This example is given here to further show that the concept of regularity cannot be replaced either by the concept of an SR- space or by the concept of an SSR- space, as indicated by Remark 4.3. This is also an example of an SSR and an SR-space which is minimal Hausdorff but not compact and hence need not be regular. Therefore, this example shows that a space which is SR, SSR and SER need not be regular. In [5] Examples are provided to distinguish the classes of strongly s-regular, semi-regular and almost regular classes of spaces. However, in this section we shall provide examples to distinguish the classes of SR, SER and SSR spaces from the class of regular spaces.

Example 4.1. (Example of a Minimal Hausdorff, SR-Space which is not Regular) Let $X = \{0, 1\} \cup \{n + \frac{1}{k} | n, k \in N\}$, where N is the set of natural numbers. Let the topology on X be generated by the relative topology on $X - \{0, 1\}$ from the usual topology on the reals; sets of the form $\{0\} \cup \{n + \frac{1}{2k} | n \geq m : m, k \in N\}$ for some $m \in N$ and $\{1\} \cup \{n + \frac{1}{2k+1} | n \geq p : p, k \in N\}$ for some $p \in N$. To see that X is an SR-space, note that if F is a closed subset of X and $x \in X - F$, there is a set H of one of the following four(4) forms such that $x \in H$ and $F \subset X - H$. Each such H is both semi-open and semi-closed, so $\chi_{(X-H)}$ is semi-continuous in each case:

(1) $H = \{x\} \cup \{x + \frac{1}{n} | n \geq m\}, x \in N - \{0, 1\}$,

(2) $H = \{x\}, x \in X - (N \cup \{0\})$,

(3) $H = \{0\} \cup \{n + \frac{1}{2} | n \geq m\} : n, m \in N$, for some m,

(4) $H = \{1\} \cup \{n + \frac{1}{3} | n \geq p\} : n, p \in N$, for some p.

Remark 4.2. (The Space in Example 4.1 is an SER-Space) It is clear from the fact that X is minimal Hausdorff and hance H-closed and semi-regular. Therefore, in view of Theorem 3.1.11, X is an SER-space. However, to verify this independently, let F be a closed set and let $x \in X - F$. If $x \notin N \cup \{0\}$, then $\{x\}$ is open and closed and $F \subset X - \{x\}$. If $x \in \{0,1\}$, there is a basic open set H such that $H \cap F = \emptyset$. For such a basic open set H, $X - H$ is closed and semi-open. If $x \in N - \{0,1\}$, there is a basic open set H such that $x \in H$ and $H \cap F = \emptyset$, H is open and closed. For each of the above cases, $\chi_{(X-H)}$ is u.s.c. and q-l.s.c., which satisfies the required conditions of an SER-space.

Remark 4.3. (The Space in Example 4.1 is an SSR-Space) To see that X is an SSR-space, let F be a closed subset of X and let $x \in X-F$. If $x \notin N \cup \{0,1\}$, then $\{x\}$ is open and closed and $F \subset X-\{x\}$. If $x \in N-\{1\}$, then $F \cap \{x+\frac{1}{k} | k \geq m\} = \emptyset$, for some $m \in N$. For such an m, $H = \{x\} \cup \{x + \frac{1}{k} | k \geq m\}$ is open, closed and $F \subset X - H$. If $x = 0$, then there is an m such that $F \cap \{n + \frac{1}{2k} | n \geq m, k \in N\} = \emptyset$. For such an m let $H = \{0\} \cup \{n + \frac{1}{2} | n \geq m\}$. Then H is regularly closed and hence $X - H$ is open and semi-closed and $F \subset X - H$. The case when $x = 1$ can be handled in a similar fashion by considering $H = \{1\} \cup \{n + \frac{1}{3} | n \geq p\}$. For each such case $\chi_{(X-H)}$ is l.s.c. and q-u.s.c., satisfying the required conditions of an SSR-space.

In [2], [12] and [14], two generalizations of the concept of normal spaces were introduced and studied. A space X is *s-normal* [2] if for any two disjoint semi-closed subsets A and B of X, there exist disjoint semi-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$. It is well-known that a minimal Hausdorff normal space is compact. However, an s-normal minimal Hausdorff space need not be compact.

Remark 4.4. (The Space in Example 4.1 is an s-Normal Space) Suppose that A and B are two disjoint semi-closed subsets of X. We distinguish four (4) cases and show in these cases that there are disjoint semi-open subsets P and Q such that $A \subset P$ and $B \subset Q$. The other cases can be handled in a similar fashion.

Case 1. $\underline{\{0,1\} \cap (A \cup B) = \emptyset}$. For each $m \in A \cap N$, there is an infinite $N(m) \subset N$ such that $\{m + \frac{1}{k} | k \in N(m)\} \cap B = \emptyset$. Choose such an $N(m)$ and let $A(m) = \{m + \frac{1}{k} | k \in N(m)\}$. For each $m \in B \cap N$, some infinite $K(m) \subset N$ satisfies $\{m + \frac{1}{k} | k \in K(m)\} \cap A = \emptyset$. Choose such a $K(m)$, and let $B(m) = \{m + \frac{1}{k} | k \in K(m)\}$. Let $P = A \cup (\cup_{N \cap A} A(m))$ and $Q = B \cup (\cup_{B \cap N} B(m))$. Then $A \subset P, B \subset Q, P \cap Q = \emptyset$ and P and Q are semi-open.

Case 2. $\underline{0 \in A, 1 \notin B}$. There is an infinite set $N(0) \subset N$ and for each integer $m \in N(0)$ there is an integer $k(m)$ such that $\{m + \frac{1}{2k(m)} | m \in N(0)\} \cap B = \emptyset$. For each of the sets $A(m)$ and $B(m)$, as defined by the process in Case 1, let $C(m) = A(m) - \{m + \frac{1}{2k(m)}\}$ and $D(m) = B(m) - \{m + \frac{1}{2k(m)}\}$. Let $A(0) = \{m + \frac{1}{2k(m)} | m \in N(0)\}$. Let $P = A \cup A(0) \cup (\cup_{N \cap A} C(m))$, $Q = B \cup (\cup_{N \cap B} D(m))$.

Then P and Q are semi-open, disjoint and $A \subset P$ and $B \subset Q$.

Case 3. $\{0,1\} \subset A$. There are infinite sets $N(0) \subset N$ and $N(1) \subset N$ and for $m \in N(0)$, $n \in N(1)$, there are integers $k(m)$ and $k(n)$ such that $\{m + \frac{1}{2k(m)} | m \in N(0)\} \cap B = \emptyset$, $\{n + \frac{1}{2k(n)+1} | n \in N(1)\} \cap B = \emptyset$. For each of the sets $C(m)$ and $D(m)$, as defined as by the process in Case 2, let $E(m) = C(m) - \{m + \frac{1}{2k(m)+1}\}$ and $F(m) = D(m) - \{m + \frac{1}{2k(m)+1}\}$. Let $A(0) = \{m + \frac{1}{2k(m)} | m \in N(0)\}$ and $A(1) = \{m + \frac{1}{2k(m)+1}\ m \in N(1)\}$. Let $P = A \cup A(0) \cup A(1) \cup (\cup_{N \cap A} E(m))$ and $Q = B \cup (\cup_{N \cap B} F(m))$. Then P and Q are disjoint semi-open sets and $A \subset P$, $B \subset Q$.

Case 4. $0 \in A, 1 \in B$. There are infinite sets $N(0) \subset N$, $N(1) \subset N$ and for $m \in N(0)$, $n \in N(1)$, integers $k(m)$ and $k(n)$, respectively, such that $\{m + \frac{1}{2k(m)} | m \in N(0)\} \cap B = \emptyset$, $\{m + \frac{1}{2k(m)+1} | m \in N(1)\} \cap A = \emptyset$. Let $A(0)$, $E(m)$ and $F(m)$ be defined as by the process in Case 3, let $B(1) = \{m + \frac{1}{2k(m)+1} | m \in N(1)\}$, $P = A \cup A(0) \cup (\cup_{N \cap A} E(m))$, and $Q = B \cup B(1) \cup (\cup_{N \cap B} F(m))$. Then P and Q are disjoint, semi-open and $A \subset P$, $B \subset Q$.

Example 4.5. (Example of a Minimal Hausdorff, SER, SSR and s-Normal Space which is not $\alpha - CR$) As demonstrated above, the space in Example 4.1 is an SR, SER, SSR and s-normal space. We have seen that it is not regular. Therefore, in view of Theorem 3.2.2, the space is not $\alpha - CR$.

Example 4.6. (Example of an H-Closed, SR-Space which is not SER) Let X be the space in Example 4.1 and let $Y = X - (\{k + \frac{1}{2n+1} | k + \frac{1}{2n+1} \in X\} \cup \{1\})$ with the subspace topology from X. Y is known to be H-closed. To see that Y is an SR-space, let $x \in Y$, $F \subset Y$ be closed and let $x \in Y - F$. If $x = 0$, then there is an $m \in N$ such that $F \cap \{n + \frac{1}{2k} | n \geq m, k \in N\} = \emptyset$. Let $H = \{0\} \cup \{n + \frac{1}{2} | n \geq m\}$. Then H is semi-open, semi-closed and $H \subset Y - F$. If $x \in N - \{0\}$, then there is an $m \in N$ such that $F \cap \{x + \frac{1}{2k} | k \geq m\} = \emptyset$. Let $H = \{x\} \cup \{x + \frac{1}{2k} | k \geq m\}$. Then H is open, closed and $F \subset Y - H$. If $x \in Y - (N \cup \{0\})$, then $H = \{x\}$ is open, closed and $F \subset Y - H$. Thus Y, as a subspace, is SR. Y is not an SER-space, since N is a closed subset of Y such that $0 \in Y - N$. Any semi-open set W of Y satisfying the condition $N \subset W$ also satisfies the condition that $m + \frac{1}{2k} \in W$ for infinitely many $m, k \in N$. Thus, $0 \in cl(W)$. Hence in view of Theorem 3.1.11, Y is not SER.

Example 4.7. (Example of an SSR-Space which is not SER) The space Y in Example 4.6 is an SSR-space. As in the discussion about the space being an SR-space, note that the characteristic function χ_{Y-H} in each case is a l.s.c. and q-u.s.c. function, satisfying the required conditions for the space to be an SSR-space.

Example 4.8. (A Closed Subspace of an SER, and SSR-Apace need not be an SER-Space) The space Y in Example 4.6 is a closed subspace of the the SER and SSR-space X of the Example 4.1.

Example 4.9. (Example of an SER-Space which is not SSR) Let $X = [-1, 1]$ and let the topology $T = \{U \subseteq X|$ either $0 \notin U$ or $(-1, 1) \subseteq U\}$. Then X is not regular since $[0, 1]$ is a closed subset of X and if $-1 < a < 0$, then $a \notin [0, 1]$. But a is an element of any open set containing $[0, 1]$. To show that X is SER, let F be a closed subset of X and $a \in X - F$. If $0 \notin F$, then F is open as well as closed, so χ_F is continuous. If $0 \in F$, let $b \in X - \{-1, 0, a, 1\}$ and let $A = F \cup \{b\}$. Then χ_A is u.s.c. and q-l.s.c. Therefore, X is SER. To show that X is not SSR, suppose that $a \in X - \{-1, 0, 1\}$. Let $H = (X - \{a\}) \cup \{0\}$. Then $a \in X - H$ and any regularly closed set containing a intersects H. Therefore, X is not SSR.

Remark. It is well known that every pseudo metric space is completely regular and hence the topology generated by a pseudo metric is $\alpha - CR$. Therefore, for the study of those spaces which are $\alpha - CR$ but not completely regular, the study of generalized forms of a pseudo metric and topology generated by such functions is significant. Also, since every real valued function on a countable connected space is constant, a countable and connected space with more than one point is not completely regular. Therefore, the study of those spaces which are countable, connected and regular is important in the study of $\alpha - CR$-spaces as it is in the study of spaces which are regular, but not completely regular.

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