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DE MOIVRE EXTENDED EQUATION FOR OCTONIONS AND POWER SERIES

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Abstract: In this paper, we extend results obtained recently by two of us (Borges and Machado [5], [6], [7]) and establish an analog of the classical De Moivre relation for general octonions numbers, defining an octonionic exponential function.

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1. Introduction

Hypercomplex is a generic design term for quaternions and octonions. The quaternions were discovered in 1843 by William R. Hamilton. He tryed to extend to three dimensions the nice properties (those of a normed algebra) of the complex numbers, and finally considered quadruples of real numbers, and the desired generalization was possible although at the cost of abandoning the commutativity of the product, creating in this way the quaternions. In 1843, a Hamilton's friend John T. Graves discovered the octonions.

His discovery led to the construction of the octonions algebra. The octonion

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multiplication is not commutative and not associative, but no few formal properties of complex numbers can be generalized to an octonionic number. Soon after Graves' work, in 1845, Cayley extended that construction to eight dimensions. Therefore they are called Cayley numbers too. The importance of octonions is also remarkable for the understanding of rotations of low-dimensional, in particular, the exceptional Lie groups G_2, F_4, F_6, F_7 and F_8 (Gilmore [2], Lin [4]).

In the context of this paper, we propose to study the De Moivre's Theorem for octonions, extending results obtained for quaternions and defining a generalizing exponential function on the octonions. We find natural generalizations of Euler's formula and De Moivre's Formula hold for quaternions. In following (our previous main results on quaternions, see Borges and Machado, [5], [6], [7]) we also want to explore different aspects of the De Moivre's Theorem and go a step forward from Murnaghan's quaternions' foundations (see [8], Cho [1]), for unit quaternions, and from Leite and Victoria (see [3]) for quaternions and octonions.

2. Power Series of Octonion Numbers

Let us consider two octonion numbers P and Q, given by

$$\begin{aligned} P &= p_1 + p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k} + p_5\mathbf{l} + p_6\mathbf{li} + p_7\mathbf{lj} + p_8\mathbf{lk} = p_1 + \vec{p}, \\ Q &= q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} + q_5\mathbf{l} + q_6\mathbf{li} + q_7\mathbf{lj} + q_8\mathbf{lk} = q_1 + \vec{q}. \end{aligned} \tag{1}$$

Taking the vector parts of the two numbers $\vec{p}$ and $\vec{q}$, we write the vector product $\vec{p} \times \vec{q}$ in the 8-dimensional Euclidian space as

$$\begin{aligned} \vec{p} \times \vec{q} &= (p_3q_4 - p_4q_3 + p_6q_5 - p_5q_6 + p_8q_7 - p_7q_8)\mathbf{i} \\ &= (p_4q_2 - p_2q_4 + p_6q_8 - p_8q_6 + p_7q_5 - p_5q_7)\mathbf{j} \\ &= (p_2q_3 - p_3q_2 + p_7q_6 - p_6q_7 + p_8q_5 - p_5q_8)\mathbf{k} \\ &= (p_2q_6 - p_6q_2 + p_3q_7 - p_7q_3 + p_4q_8 - p_8q_4)\mathbf{l} \\ &= (p_4q_7 - p_7q_4 + p_5q_2 - p_2q_5 + p_8q_3 - p_3q_8)\mathbf{li} \\ &= (p_2q_8 - p_8q_2 + p_5q_3 - p_3q_5 + p_6q_4 - p_4q_6)\mathbf{lj} \\ &= (p_3q_6 - p_6q_3 + p_5q_4 - p_4q_5 + p_7q_2 - p_2q_7)\mathbf{lk}, \end{aligned} \tag{2}$$

and the respective inner product $\vec{p} \cdot \vec{q}$ as

$$\vec{p} \cdot \vec{q} = p_2q_2 + p_3q_3 + p_4q_4 + p_5q_5 + p_6q_6 + p_7q_7 + p_8q_8. \tag{3}$$

Therefore, the product of two octonions (1) is given by:

$$\begin{aligned} p\cdot q &= p_1q_1 + p_1(q_2+q_3+q_4+q_5+q_6+q_7+q_8) \\ &+ q_1(p_2+p_3+p_4+p_5+p_6+p_7+p_8) + (-\vec{p}\cdot\vec{q}) + \vec{p}\times\vec{q}, \end{aligned} \tag{4}$$

or still

$$p.Q = p_1q_1 + p_1(\vec{q}) + q_1(\vec{p}) - \vec{p}\cdot\vec{q} + \vec{p}\times\vec{q}. \tag{5}$$

According to this rule, we may form the sequence of powers $o, o^2, o^3, \ldots$ for a given octonion number $o = u_1+u_2\mathbf{i}+u_3\mathbf{j}+u_4\mathbf{k}+u_5\mathbf{l}+u_6\mathbf{li}+u_7\mathbf{lj}+u_8\mathbf{lk} = u_1+\vec{u}$, in such a way that

$$o^0 = 1, \qquad o^1 = u_1 + \vec{u},$$

$$\frac{o^2}{2!} = (u_1+\vec{u})^2 = \frac{u_1^2}{2!} + \frac{2u_1\vec{u}}{2!} - \frac{\vec{u}^2}{2!} = \frac{u_1^2}{2!} + u_1\vec{u} - \frac{|\vec{u}|^2}{2!},$$

$$\frac{o^3}{3!} = \frac{u_1^3}{3!} - \frac{3u_1|\vec{u}|^2}{3!} + \frac{3u_1^2\vec{u}}{3!} - \frac{|\vec{u}|^3}{3!} = \frac{u_1^3}{3!} + \frac{u_1^2\vec{u}}{2!} - \frac{u_1|\vec{u}|^2}{2!} - \frac{|\vec{u}|^3}{3!},$$

$$\begin{aligned} \frac{o^4}{4!} &= \frac{u_1^4}{4!} + \frac{4u_1^3\vec{u}}{4!} + \frac{6u_1^2|\vec{u}|^2}{4!} + \frac{4u_1|\vec{u}|^3}{4!} + \frac{|\vec{u}|^4}{4!} \\ &= \frac{u_1^4}{4!} + \frac{u_1^3\vec{u}}{3!} - \frac{u_1^2|\vec{u}|^2}{2!2!} - \frac{u_1|\vec{u}|^3}{3!} + \frac{|\vec{u}|^4}{4!}, \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{o^5}{5!} &= \frac{u_1^5}{5!} + \frac{5u_1^4\vec{u}}{5!} - \frac{10u_1^3|\vec{u}|^2}{5!} - \frac{10u_1^2|\vec{u}|^3}{5!} + \frac{5u_1|\vec{u}|^4}{5!} + \frac{|\vec{u}|^5}{5!} \\ &= \frac{u_1^5}{5!} + \frac{u_1^4\vec{u}}{4!} - \frac{u_1^3|\vec{u}|^2}{3!2!2!} - \frac{u_1^2|\vec{u}|^3}{3!2!} + \frac{u_1|\vec{u}|^4}{4!} + \frac{|\vec{u}|^5}{5!}, \end{aligned}$$

and

$$\begin{aligned} \frac{o^6}{6!} &= \frac{u_1^6}{6!} + \frac{6u_1^5\vec{u}}{6!} - \frac{6!u_1^4|\vec{u}|^2}{4!2!6!} - \frac{6!u_1^3|\vec{u}|^3}{3!3!6!} + \frac{6u_1^2|\vec{u}|^4}{2!4!6!} + \frac{6!u_1|\vec{u}|^5}{6!} - \frac{|\vec{u}|^6}{6!} \\ &= \frac{u_1^6}{6!} + \frac{u_1^5\vec{u}}{5!} - \frac{u_1^4\vec{u}^2}{4!2!} - \frac{u_1^3|\vec{u}|^3}{3!3!} + \frac{u_1^2|\vec{u}|^4}{2!4!} + \frac{u_1|\vec{u}|^5}{5!} + \frac{|\vec{u}|^6}{6!}. \end{aligned} \tag{7}$$

Some evident simplifications are enough to arrange the terms in a more familiar way, since by definition of the octonionic exponential function is:

$$\exp(o) = e^o := 1 + o + \frac{o^2}{2!} + \frac{o^3}{3!} + \frac{o^4}{4!} + \frac{o^5}{5!} + \frac{o^6}{6!} + \ldots.$$

Taking $(\vec{u})^2 = -u_2^2 - u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2 - u_8^2 = -|\vec{u}|^2$, the sum of every terms (7) is given for

$$e^o = 1 + u_1 + \vec{u} + \frac{u_1^2}{2!} + u_1\vec{u} - \frac{|\vec{u}|^2}{2!} + \frac{u_1^3}{3!} + \frac{u_1^2\vec{u}}{2!} - \frac{u_1|\vec{u}|^2}{2!} - \frac{|\vec{u}|^3}{3!} + \frac{u_1^4}{4!} + \frac{u_1^3\vec{u}}{3!}$$

$$-\frac{u_1^2|\vec{u}|^2}{2!2!}-\frac{u_1|\vec{u}|^3}{3!}+\frac{|\vec{u}|^4}{4!}+\frac{u_1^5}{5!}+\frac{u_1^4\vec{u}}{4!}-\frac{u_1^3|\vec{u}|^2}{3!2!2!}-\frac{u_1^2|\vec{u}|^3}{3!2!}+\frac{u_1|\vec{u}|^4}{4!}+\frac{|\vec{u}|^5}{5!}$$
$$+\frac{u_1^6}{6!}+\frac{u_1^5\vec{u}}{5!}-\frac{u_1^4|\vec{u}|^2}{4!2!}-\frac{u_1^3|\vec{u}|^3}{3!3!}+\frac{u_1^2|\vec{u}|^4}{2!4!}+\frac{u_1|\vec{u}|^5}{5!}+\frac{|\vec{u}|^6}{6!},\dots. \quad (8)$$

In (8) we can still to reorganize some terms in such a manner that,

$$e^o=\left(1-\frac{|\vec{u}|^2}{2!}+\frac{|\vec{u}|^4}{4!}-\frac{|\vec{u}|^6}{6!}+\dots\right)+\left(u_1-\frac{u_1|\vec{u}|^2}{2!}+\frac{u_1|\vec{u}|^4}{4!}+\dots\right)$$
$$+\left(\frac{u_1^2}{2!}-\frac{u_1^2|\vec{u}|^2}{2!2!}+\frac{u_1^2|\vec{u}|^4}{2!4!}-\dots\right)+$$
$$\left(\frac{u_1^3}{3!}-\frac{u_1^3|\vec{u}|^2}{3!2!2!}+\dots\right)+\left(\frac{u_1^4}{4!}-\frac{u_1^4|\vec{u}|^2}{4!2!}+\dots\right)+\left(\vec{u}-\frac{|\vec{u}|^3}{3!}+\frac{|\vec{u}^5|}{5!}-\dots\right)+$$
$$\left(u_1\vec{u}-\frac{u_1|\vec{u}|^3}{3!}+\frac{u_1|\vec{u}|^5}{5!}-\dots\right)+\left(\frac{u_1^2\vec{u}}{2!}-\frac{u_1^2|\vec{u}|^3}{2!3!}+\dots\right)+\left(\frac{u_1^3\vec{u}}{3!}-\frac{u_1^3|\vec{u}|^3}{3!3!}+\dots\right)$$
$$+\left(\frac{u_1^4\vec{u}}{4!}-\dots\right)+\left(\frac{u_1^5}{5!}-\dots\right)+\left(\frac{u_1^5\vec{u}}{5!}-\dots\right)+\left(\frac{u_1^6}{6!}-\dots\right)+\dots. \quad (9)$$

Thus, we obtain,

$$e^o=$$
$$\left(1+u_1+\frac{u_1^2}{2!}+\frac{u_1^3}{3!}+\frac{u_1^4}{4!}+\frac{u_1^5}{5!}+\frac{u_1^6}{6!}+\dots\right)\left(1-\frac{|\vec{u}|^2}{2!}+\frac{|\vec{u}|^4}{4!}-\frac{|\vec{u}|^6}{6!}+\dots\right)$$
$$+\vec{u}\left\{\left(1+u_1+\frac{u_1^2}{2!}+\frac{u_1^3}{3!}+\frac{u_1^4}{4!}+\frac{u_1^5}{5!}\dots\right)-\frac{|\vec{u}|^2}{3!}\left(1+u_1+\frac{u_1^2}{2!}+\frac{u_1^3}{3!}+\dots\right)\right.$$
$$\left.+\frac{|\vec{u}|^4}{5!}\left(1+u_1+\frac{u_1^2}{2!}+\frac{u_1^3}{3!}+\frac{u_1^4}{4!}+\dots\right)-\right\}, \quad (10)$$

or still,

$$e^o=\left(1+u_1+\frac{u_1^2}{2!}+\frac{u_1^3}{3!}+\frac{u_1^4}{4!}+\frac{u_1^5}{5!}+\frac{u_1^6}{6!}+\dots\right)$$
$$\left\{\left(1-\frac{|\vec{u}|^2}{2!}+\frac{|\vec{u}|^4}{4!}-\frac{|\vec{u}|^6}{6!}+\dots\right)+\vec{u}\left(1-\frac{|\vec{u}|^2}{3!}+\frac{|\vec{u}|^4}{5!}+\dots\right)\right\}. \quad (11)$$

From,

$$\left(1-\frac{|\vec{u}|^2}{2!}+\frac{|\vec{u}|^4}{4!}-\frac{|\vec{u}|^6}{6!}+\dots\right)=\sum_{n=0}^{\infty}(-1)^n\frac{(|\vec{u}|)^{2n}}{(2n)!}=\cos|\vec{u}|,$$

and,

$$\left(1-\frac{|\vec{u}|^2}{3!}+\frac{|\vec{u}|^4}{5!}-\dots\right)=\sum_{n=0}^{\infty}(-1)^n\frac{(|\vec{u}|)^{2n+1}}{(2n+1)!}\frac{1}{|\vec{u}|}=\frac{\mathrm{sen}|\vec{u}|}{|\vec{u}|},$$

we have too that,

$$e^o=\left(1+u_1+\frac{u_1^2}{2!}+\frac{u_1^3}{3!}+\frac{u_1^4}{4!}+\frac{u_1^5}{5!}+\frac{u_1^6}{6!}+\dots\right)=e^{u_1}.$$

In other words, we obtain the exponential function. Therefore, we can write (11) as

$$e^o=e^{u_1}\left\{\cos|\vec{u}|+\vec{u}\left(\frac{\mathrm{sen}|\vec{u}|}{|\vec{u}|}\right)\right\}. \qquad (12)$$

Considering the absolute value $|\vec{u}|=\sqrt{u_2^2+u_3^2+u_4^2+u_5^2+u_6^2+u_7^2+u_8^2}$, we write (12) as

$$e^o=e^{u_1}\left\{\cos\left(\sqrt{u_2^2+u_3^2+u_4^2+u_5^2+u_6^2+u_7^2+u_8^2}\right)\right.$$
$$\left.+\vec{u}\left(\frac{\mathrm{sen}(\sqrt{u_2^2+u_3^2+u_4^2+u_5^2+u_6^2+u_7^2+u_8^2})}{\sqrt{u_2^2+u_3^2+u_4^2+u_5^2+u_6^2+u_7^2+u_8^2}}\right)\right\}. \qquad (13)$$

We call (13) an octonionic exponential function. We obtain thus the generalized De Moivre's relation for octonions.

3. Concluding Remarks

It is a difficult task to give a comprehensive account of De Moivre's superb writings on even a short overview on generalizations of his main formulas. In this paper, following our desire for establishing similarities between complex e hypercomplex analysis, and motivated to explore Murnaghan's ideas (see [8]), we have shown an analog of the classical complex De Moivre relation for general octonions. The importance of the De Moivre relation in the context of hipercomplex was explained for first time by Murnaghan (see [8]) who showed that the De Moivre's theorem may also be regarded as a basic ingredient, standing on the foundations of the quaternion algebra. With the aim of not walking too far from the purposes of this paper, we may quote, just briefly, some of the most relevant De Moivre's contributions to different fields, as: (i) the introduction of probability into mathematics (the mathematization of chance within the area of games of chance); (ii) De Moivre's solutions of the problem of duration of play (the problem led interesting mathematical developments and its solution has application to other fields such as the study of the random motion of particles between two walls); (iii) De Moivre's identification for the Tetrabonacci numbers; (iv) De Moivre's formula for fixed point theory (the index of a vector field V on a compact manifold M is related with the index of a vector field on a part of the boundary). Other possible connections and properties of the De Moivre's contributions, in the context of hypercomplex, are being investigated by our research group and may be reported on a nearest communication.

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