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A MATHEMATICAL MODEL FOR THE CONTROL OF THE TRANSMISSION OF GENETIC DISEASES USING PURE FRACTIONS

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Abstract: We give a new form of representing the numbers in the unit interval $[0,1]$ and we call these numbers pure fractions. The theory and properties of pure fractions are studied and used to create a mathematical model for the control of the transmission of genetically inherited diseases.

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1. Introduction

The fundamental theory of mathematics is the theory of numbers. Number theory, in the past, has concentrated essentially on the properties of integers. However, in recent times, the theories and properties of other classes of numbers have proved worthy of more formal investigation. For instance see Stark [11], Niven [10], and Hardy and Wright [7] for information on continued fractions, Farey fractions, and irrational numbers. Among the class of numbers whose theory attracts much attention are the numbers in the unit interval $[0,1]$, which we call *pure fractions*.

The numbers in the unit interval $[0,1]$ play a significant role in mathematics. Every real number x has an important relationship to these fractions in the sense

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that every such x is expressible in the form $x = \lfloor x \rfloor + r$, where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$ and $r \in [0,1)$. The numbers in $[0,1]$ even became the basis of a new mathematical theory called fuzzy mathematics during the discovery of fuzzy sets Zadeh [13]. The theory of pure fractions originated from work on applications of fuzzy sets Eke [3]. This theory has also generated a number of applications in the study of cryptography Eke and Okrah [4].

In this paper we formalize some properties of pure fractions and give a biologically related application. We shall create a new form of representation for a given pure fraction. The motivation for the new form is the nature of the numbers $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$. which are numbers expressible in the form $\frac{n-1}{n}$ for an integer $n \geq 1$. We call such numbers *co-harmonic fractions*. An important characterization of a pure fraction s is that s can be represented in the form

$$s = \frac{r + \alpha - 1}{\alpha},$$

where $r \in [0,1]$ and α is a positive real number.

We start in Section 2 with introducing the representation and reducibility of pure fractions. Other preliminary properties of pure fractions are also given in this section. In Section 3, we introduce a biological application where we use pure fractions to create a mathematical model for the control of the transmission of genetic diseases from parents to their offspring. We conclude by giving a brief discussion in Section 4.

2. Representation and Reducibility of Pure Fractions

The concept of reducibility partitions the interval $[0,1)$ into two subintervals $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$ consisting, respectively, of *irreducible* and *reducible* pure fractions. This will be shown later in this section. However before getting into the details of reducibility, we introduce a form for representing pure fractions.

Let $\mathbb{R}^+$ denote the set of positive real numbers. Then, for any pure fraction r and $\alpha \in \mathbb{R}^+$, we define $r^{[\alpha]}$ by

$$r^{[\alpha]} := \frac{r + \alpha - 1}{\alpha}. \tag{1}$$

It will be shown by Proposition 2.1 that every pure fraction s can be *represented* in the form (1). However, we emphasize here that not every expression given by (1) is a pure fraction. For instance $\left(\frac{2}{3}\right)^{[1/4]} \notin [0,1]$ and is not a pure fraction. If $r^{[\alpha]}$ is not a pure fraction, then we call $r^{[\alpha]}$ an *indeterminate* in the theory of pure fractions. Thus, $\left(\frac{2}{3}\right)^{[1/4]}$ is an example of an indeterminate.

A given pure fraction may have more than one representation of the form given by (1). For instance, $s = \frac{2}{3}$ has two representations since $0^{[3]} = \frac{2}{3} = \left(\frac{1}{3}\right)^{[2]}$. When $s \neq 1$ has a representation (1), we call α the index of s relative to r, denoted by $i(s, r)$. Thus from (1)

$$\alpha = i(s, r) = \frac{1 - r}{1 - s}.$$

Proposition 2.1. *Let $r, s, t \in [0, 1)$ and $\alpha, \beta \in \mathbb{R}^+$, then we have the following properties:*

1. $s = r^{[\alpha]}$ if and only if $\alpha = i(s, r)$.

2. $r^{[1]} = r$ and $1^{[\alpha]} = 1$ (Identity).

3. $r^{[\alpha]} < r^{[\beta]}$ whenever $\alpha < \beta$, and $i(s, r) < i(t, r)$ whenever $s < t$ (Monotonicity).

4. $\left(r^{[\alpha]}\right)^{[\beta]} = r^{[\alpha\beta]} = r^{[\beta\alpha]}$ (Power rule).

5. $i(s, r) = i(r, s)^{-1}$ (Inversion).

Proof. Properties 1, 2, 4, and 5 are immediate, so it remains to prove 3. Suppose for $r^{[\alpha]}$ and $r^{[\beta]}$ we have $\alpha < \beta$, where $r \in [0, 1)$. Then

$$\beta(r - 1) < \alpha(r - 1).$$

From this we have

$$\beta r + \alpha\beta - \beta < \alpha r + \alpha\beta - \alpha \Rightarrow \beta(r + \alpha - 1) < \alpha(r + \beta - 1)$$
$$\Rightarrow \frac{r + \alpha - 1}{\alpha} < \frac{r + \beta - 1}{\beta} \Rightarrow r^{[\alpha]} < r^{[\beta]}.$$

Now suppose for $i(s, r)$ and $i(t, r)$ we have $s < t$, where $s, t \in [0, 1)$. Then

$$-t < -s \Rightarrow 1 - t < 1 - s \Rightarrow \frac{1}{1 - s} < \frac{1}{1 - t}$$
$$\Rightarrow \frac{1 - r}{1 - s} < \frac{1 - r}{1 - t} \Rightarrow i(s, r) < i(t, r). \quad \square$$

Definition 2.1. When $s = r^{[n]}$ and n is a positive integer, we call s a power of r, and r a root of s.

Proposition 2.2. *Let $r, s, t \in [0, 1)$ and the relation "r is a root of s" be denoted by "$r \prec s$". Then "$r \prec s$" is a partial ordering on $[0, 1)$.*

Proof. To prove "$r \prec s$" is a partial ordering we have the following.

(1) $r \prec r$ since $r = r^{[1]}$. This shows reflexivity.

(2) Suppose $r \prec s$ and $s \prec r$. Then $s = r^{[n]}$ and $r = s^{[m]}$ for some positive

integers n and m. Then substituting for r and by Proposition 2.1(4)

$$s = r^{[n]} = \left(s^{[m]}\right)^{[n]} = s^{[mn]}.$$

Then $mn = 1$ since $s^{[1]} = s = s^{[mn]}$. This implies $m = 1$ and $n = 1$ and shows that $s = r$. Thus we have antisymmetry.

(3) Suppose $r \prec s$ and $s \prec t$. Then $s = r^{[n]}$ and $t = s^{[m]} = r^{[nm]}$. Hence $r \prec t$ and we have transitivity. $\square$

Proposition 2.3. *Let $s \in [0,1)$. Then, the following conditions are equivalent for a positive integer n.*

1. *$s = r^{[n]}$ for some $r \in [0,1)$.*
2. *$i(s,0) \geq n$.*
3. *$1 > s \geq \frac{n-1}{n}$.*

Proof. Since (2) $\Rightarrow$ (3) is immediate we now prove the other cases.

(1) $\Rightarrow$ (2): Consider $s = r^{[n]}$. Then by Proposition 2.1 (1), $n = i(s,r)$. Then since $r, s \in [0,1)$

$$\begin{aligned} n &= i(s,r) = \frac{1-r}{1-s} \\ &\leq \frac{1-0}{1-s} = i(s,0). \end{aligned}$$

(3) $\Rightarrow$ (1): Assume $1 > s \geq \frac{n-1}{n}$. Then by $0^{[n]} = \frac{n-1}{n}$ and parts (2) and (4) of Proposition 2.1 we get

$$1^{[n]} > s \geq 0^{[n]} \text{ and } 1 > s^{\left[\frac{1}{n}\right]} \geq 0^{[1]}.$$

Thus, we let $r = s^{\left[\frac{1}{n}\right]}$ and hence $r^{[n]} = s$ for some positive integer n, where $r \in [0,1)$. $\square$

Definition 2.2. A pure fraction s is called reducible if s satisfies any of the equivalent conditions of Proposition 2.3 for some integer $n \geq 2$. Otherwise, s is called irreducible.

As a consequence of Proposition 2.3, a pure fraction s is irreducible if and only if $s = 1$ or $s < 1/2$. Thus, we have a partition of $[0,1)$ into $I_1 = [0,1/2)$ and $I_2 = [1/2,1)$ consisting, respectively, of irreducible and reducible pure fractions.

Proposition 2.4. *Let $s \in [0,1)$. Then, we have the following properties.*

1. There exists $r \in [0,1)$ and a positive integer n such that $s = r^{[n]}$ and $\frac{1}{n+1} > r$. In particular, r is irreducible.

2. If $s = r^{[n]}$ for an irreducible $r \in [0,1)$ and a positive integer n, then

$$\frac{i\,(s,0)}{2} < n.$$

In particular, the smallest such positive integer n is given by

$$n_0\,(s) = \left\lfloor \frac{i\,(s,0)}{2} \right\rfloor + 1. \tag{2}$$

Proof. (1). Let $n = \lfloor i\,(s,0) \rfloor$ be the greatest integer $\leq i\,(s,0)$. Then

$$n \leq i\,(s,0) < n+1.$$

Hence by parts (3) and (4) of Proposition 2.1, respectively, we get

$$0^{[n]} \leq s < 0^{[n+1]} \text{ and } 0 \leq s^{\left[\frac{1}{n}\right]} < 0^{\left[\frac{n+1}{n}\right]}.$$

Since $0^{\left[\frac{n+1}{n}\right]} = \frac{1}{n+1}$, we let $r = s^{\left[\frac{1}{n}\right]} \in [0,1)$. Then, clearly $s = r^{[n]}$ and $\frac{1}{n+1} > r$. Moreover, $\frac{1}{2} \geq \frac{1}{n+1} > r$. Therefore r is irreducible.

(2). Suppose $s = r^{[n]}$ for some irreducible $r \in [0,1)$. Then $r < \frac{1}{2} = 0^{[2]}$ and $s = r^{[n]} < 0^{[2n]}$. Consequently by Proposition 2.1 (3), $i\,(s,0) < 2n$. This shows

$$\frac{i\,(s,0)}{2} < n \text{ and } n \geq n_0\,(s) = \left\lfloor \left(\frac{i\,(s,0)}{2} \right) \right\rfloor + 1.$$

We have shown that the set

$$S = \left\{ n \in \mathbb{N} \mid r^{[n]} = s \text{ for some irreducible pure fraction } r \right\}$$

has $n_0\,(s) = \left\lfloor \left(\frac{i(s,0)}{2} \right) \right\rfloor + 1$ as a lower bound. We must now show that $n_0\,(s)$ is itself an element of S. Let $\beta = \frac{i(s,0)}{2}$ and $m = \frac{i(s,0)}{2} + 1$. We know that

$$0 < \beta < n_0\,(s) \leq m.$$

Hence $\frac{1}{m} \leq \frac{1}{n_0(s)} < \frac{1}{\beta}$. Consequently by Proposition 2.1 (3)

$$s^{[1/m]} \leq s^{[1/n_0(s)]} < s^{[1/\beta]}.$$

It is easy to show that $s^{[1/\beta]} = 1/2$ and $s^{[1/m]} \geq 0$. Thus

$$0 \leq s^{[1/n_0(s)]} < 1/2.$$

This shows $n_0\,(s)$ is indeed an element of S. □

We need the following definitions before stating Corollary 2.5 of the above proposition.

Definition 2.3. The degree of a pure fraction $s \neq 1$, denoted by $\deg(s)$, is the highest positive integer n such that $s \geq \frac{n-1}{n}$. The pure fraction $s = 1$

has no degree.

Definition 2.4. Let $s \in [0,1)$. By the principal root of the pure fraction s we shall mean the irreducible root r of s such that $i(s,r)$ is minimal. We denote the principal root of s by $r_0(s)$ and $i(s,r_0(s))$ by $n_0(s)$.

Corollary 2.5. *A pure fraction $s \neq 1$ of degree n has exactly n roots, and at least one of the roots is irreducible.*

Proof. The roots of s are $s^{\left[\frac{1}{k}\right]}$, $k = 1, \ldots, n$. The uniqueness of $s^{\left[\frac{1}{k}\right]}$ as the kth root of s can be easily verified. Thus, the only property left to prove is irreducibility which we get by Proposition 2.4. Hence at least one of the n roots of s is irreducible. $\square$

Example 2.1. If $n > 0$ and $s = \frac{n-1}{n}$, then by Proposition 2.4

$$n_0(s) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } r_0(s) = \frac{\left\lfloor \frac{n-1}{2} \right\rfloor}{n}.$$

Proposition 2.6. *The following conditions are equivalent for any two pure fractions s and t.*

1. $s \leq t$,
2. $s^{[1/n_0(t)]}$ *is either irreducible or indeterminate,*
3. $n_0(s) \leq n_0(t)$,
4. $t^{[1/n_0(s)]}$ *is reducible if* $n_0(s) \neq n_0(t)$.

Proof. (1) $\Rightarrow$ (2): Consider $s \leq t$. Then by Proposition 2.1 (3) and Definition 2.4,

$$s^{[1/n_0(t)]} \leq t^{[1/n_0(t)]} = r_0(t).$$

Hence, $s^{[1/n_0(t)]} < 1/2$ since $r_0(t)$ is irreducible. Consequently $s^{[1/n_0(t)]}$ is either irreducible or indeterminate.

(2) $\Rightarrow$ (3): If $s^{[1/n_0(t)]}$ is either irreducible or indeterminate, then since

$$n_0(t) = i\left(s, s^{[1/n_0(t)]}\right)$$

it must be the case that

$$n_0(t) \geq n_0(s) = i(s, r_0(s)).$$

(3) $\Rightarrow$ (4): If

$$n_0(s) \leq n_0(t) \leq \deg(t),$$

$t^{[1/n_0(s)]}$ is well defined since $t^{[1/n_0(s)]}$ is a root of t. If $t^{[1/n_0(s)]}$ is irreducible, then since

$$i\left(t, t^{[1/n_0(s)]}\right) = n_0(s) < n_0(t) = i(t, r_0(t))$$

we contradict the definition of $r_0(t)$. Therefore, $t^{[1/n_0(s)]}$ must be reducible.

(4) $\Rightarrow$ (1): If $n_0(s) \neq n_0(t)$, then

$$t^{[1/n_0(s)]} > s^{[1/n_0(s)]} = r_0(s)$$

since $r_0(s)$ is irreducible but $t^{[1/n_0(s)]}$ is reducible. Hence by Proposition 2.1 (3), $t > s$. □

For any integer $n \geq 1$, we denote the co-harmonic pure fraction $\frac{n-1}{n}$ by c_n.

Corollary 2.7. *If* $\deg(s) = n$, *then* $n_0(s) = \lfloor \frac{n}{2} \rfloor + 1 = n_0(c_n)$.

Proof. By Example 2.1,

$$n_0(c_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n_0(c_{n+1}) = \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

Now by Proposition 2

$$n_0(c_n) \leq n_0(s) \leq n_0(c_{n+1}).$$

This leads to the following inequality

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq n_0(s) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

Now considering the cases when n is even, and when n is odd gives the result. □

3. Pure Fractions and Genetic Diseases

In this section we create a mathematical model for the control of the transmission of genetic diseases using pure fractions. The concepts of reducibility and representation of pure fractions have important roles in the development of the model.

Consider a hypothetical genetic disease D and a gene G responsible for the disease D. As done in fuzzy logic for truth values Nguyen and Walker [9], we imagine every number in the unit interval $[0, 1]$ as representing a possible genotype for G. That is, each pure fraction $s \in [0, 1]$ measures a corresponding degree of possession and transmission of the gene G by an individual denoted by X. We note here that the pure fraction s can also be thought of as the probability of possession and transmission of the gene G by the individual denoted by X.

A person with a "high incidence" of the gene G is defined as someone who has a genotype falling in the subinterval $I_2 = [1/2, 1)$. A person with such a genotype may suffer from discrimination when selecting a spouse. However, we make the assumption that such an individual would like to have a spouse so that their descendant of some future generation would be more fortunate and would have a genotype of G that does not fall in subinterval I_2. Such an individual is a descendant with a "low incidence" of G and has a genotype that falls in the subinterval $I_1 = [0, 1/2)$. Based on what is stated above, we will devise a mathematical model, where an individual with a high incidence of the gene G can be matched with the appropriate spouse so that a future descendant can have a low incidence of the gene G.

3.1. The Problem

Consider a genetic disease D and a gene G responsible for the disease D. Let H denote the set of humans and $X \in H$ an individual with parents denoted as X' and Y'. Let $S : H \to [0, 1]$ denote a function such that the following axioms hold.

1. The "genotype" $s = S(X)$ satisfies:

i) $s = 0$ if and only if X is completely free of the gene G.

ii) $s \in (0, 1/2]$ if and only if X has a "low incidence" of gene G.

That is, X is a healthy carrier of G.

iii) $s \in (1/2, 1]$ if and only if X has a "high incidence" of gene G.

That is, X is in danger of D.

2. If s is in the range of S and r is a root of s (i.e., $s = r^{[n]}$ for some positive integer n), then r is also in the range of S.

3. $S(X) = \left(S\left(X'\right) + S\left(Y'\right)\right)/2$. That is, the genetic structure of an individual is proportional to that of his (her) parents X' and Y'.

4. If both parents X' and Y' are in danger of the disease D, then an individual Y will not accept X for a martial partner if

$$S(Y) < \min\left(S\left(X'\right)^{[1/2]}, S\left(Y'\right)^{[1/2]}\right).$$

Using the function $S : H \to [0, 1]$ devise a scheme for "marital connections" such that an individual X with genotype $S(X) \in (1/2, 1]$ can be "tolerated for marriage" by an individual Y only if they can have a descendant Z such that $S(Z) \in [0, 1/2]$.

Hereafter, we will use the small letter x to denote the genotype $S(X)$ of an individual X. To solve the "marital connection" problem, we will consider the sequences $\{X_n\}$ and $\{Y_n\}$, where n is a positive integer and X_n and Y_n

are individuals, and their corresponding sequences $\{x_n\}$ and $\{y_n\}$ of genotypes. Moreover, Y_n is thought of as the spouse of X_n given that

$$y_{n+1} \nless \min\left(x_n^{[1/2]}, y_n^{[1/2]}\right).$$

We note that this condition follows from Axiom 4 above and is shown below in the proof of Theorem 3.1.

Before solving the problem, we need the following definition and theorem.

Definition 3.1. Two sequences $\{x_n\}$ and $\{y_n\}$ of pure fractions are said to constitute a Genetic Remediation Scheme (GRS) if:

1. x_1 is reducible.
2. $y_k = x_k^{[1/2]}, k = 1, 2, \ldots$
3. $x_{k+1} = \frac{1}{2}(x_k + y_k)$
4. The iteration stops when x_N is irreducible for some positive integer N.

Theorem 3.1. *The sequences $\{x_n\}$ and $\{y_n\}$ of pure fractions that constitute a GRS are decreasing and finite. In addition, for each k such that y_k is reducible we have $y_{k+1} \geq y_k^{[1/2]}$.*

Proof. Since the sequences constitute a GRS we have

$$y_k = x_k^{[1/2]} = 2x_k - 1 < 2x_k - x_k = x_k$$

implies

$$x_{k+1} = \frac{1}{2}(x_k + y_k) < x_k.$$

Then by Proposition 2.1

$$y_{k+1} = x_{k+1}^{[1/2]} < x_k^{[1/2]} = y_k.$$

This proves the sequences are decreasing. Moreover we have $y_k < x_k$ implies $y_k < \frac{1}{2}(x_k + y_k) = x_{k+1}$. Hence $y_k^{[1/2]} < x_{k+1}^{[1/2]}$ whenever y_k is reducible.

Now to prove the sequences are finite, suppose there exists no positive integer N such that x_N is irreducible. Then $x_n \geq 1/2$ for all n. Since $\{x_n\}$ is decreasing and bounded, $\{x_n\}$ converges to some pure fraction s. Similarly, $\{y_n\}$ converges to some pure fraction t. Consequently,

$$s = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}(x_n + y_n) = \frac{1}{2}(s + t).$$

Thus, $s = t$. Then

$$s = \lim_{n\to\infty} y_n = \lim_{n\to\infty} (2x_n - 1) = 2s - 1$$

implies $s = 1$. But $s \leq x_1 < 1$. Therefore, we have a contradiction and this completes the proof. □

3.2. A Solution to the Problem

Let X_1 be an individual with genotype $x_1 = S(X_1)$, where x_1 is reducible. Then Theorem 3.1 guarantees that x_1 generates a GRS with two finite sequences $\{x_1, \dots, x_N\}$ and $\{y_1, \dots, y_{N-1}\}$. Corresponding to these two sequences are the sequences $\{X_1, \dots, X_N\}$ and $\{Y_1, \dots, Y_{N-1}\}$ of humans where, for $k = 1, \dots, N-1$, Y_k is assumed to be the spouse of X_k and X_{k+1} is assumed to be an offspring of X_k and Y_k satisfying

$$S(X_{k+1}) = x_{k+1} = \frac{1}{2}(x_k + y_k).$$

Clearly, whenever x_k and y_k are both reducible we have that

$$y_{k+1} \geq y_k^{[1/2]} = \min\left(x_k^{[1/2]}, y_k^{[1/2]}\right).$$

This implies that Y_{k+1} can accept X_{k+1} for a spouse. The individual X_N is the required descendant of X_1 satisfying $S(X_N) \in [0, 1/2]$. Thus, this solves the problem.

Example 3.1. Let $x_1 = 13/15 > 1/2$ be the pure fraction associated with individual X_1. The iterative scheme and Theorem 3.1 are used to find x_N such that $x_N < 1/2$. The iterations are given in the following table.

$x_1 = 13/15,$	$y_1 = x_1^{[1/2]} = 2(13/15) - 1 = 11/15$
$x_2 = \frac{1}{2}(x_1 + y_1) = 12/15 > 1/2,$	$y_2 = x_2^{[1/2]} = 2(12/15) - 1 = 9/15$
$x_3 = \frac{1}{2}(x_2 + y_2) = 7/10 > 1/2,$	$y_3 = x_3^{[1/2]} = 2(7/10) - 1 = 2/5$
$x_4 = \frac{1}{2}(x_3 + y_3) = 11/20 > 1/2,$	$y_4 = x_4^{[1/2]} = 2(11/20) - 1 = 1/10$
$x_5 = \frac{1}{2}(x_4 + y_4) = 13/40 < 1/2,$	Iteration stops.

Hence, $N = 5$ and after the 5-th generation X_1 and Y_1 will have the descendant X_5 who has a "low incidence" of the gene G with genotype $x_5 = S(X_5) = 13/40 \in [0, 1/2]$.

4. Discussion

The motivation for the development of the mathematical model in Section 3 is the study of the genotypes for sickle-cell anemia which is a genetic disease that is prevalent among people of African descent. However, as remarked earlier, pure fractions representing genotypes are only probabilities of possession and transmission of the gene. The occurrence of sickle-cell anemia is predicated on an individual having a gene known as the S-gene and a genotype known as the SS-genotype. There are three types of genotypes encountered in studies of sickle-cell anemia. They are called AA, AS, and SS. The effect of genotype AA is that the carrier has no S-gene, no fear of ever developing sickle-cell anemia, and no fear of ever transmitting the S-gene to offspring. The carrier of the genotype AS is a heterozygote who cannot suffer from sickle-cell anemia but can transmit the S-gene to offspring. The carrier of the genotype SS is called a sickler. Such a person will surely develop sickle-cell anemia over time and will surely transmit the S-gene to offspring. The genotypes for sickle-cell anemia are represented by the pure fractions $0, 1/2$, and 1 corresponding to AA, AS, and SS respectively. See Cox and Schinazi [2], Feng et al [5], Feng et al [6] and Jones [8] for more on studies of the S-gene, and Allison [1] and Strasser [12] for an introduction to the sickle-cell disease.

Since our model is intended for a general genetic disease we have assumed, as done in fuzzy logic for truth values Nguyen and Walker [9], a continuum of genotypes represented by pure fractions. Thus, every pure fraction s is a possible genotype for the general genetic disease D.

For the general genetic disease D, the people in danger are those with genotypes $x \in (1/2, 1]$. The genotypes in $(1/2, 1)$ are all reducible pure fractions. The mathematical model created in this paper is for the purpose of controlling the transmission of genes with "high incidence" genotypes to arrive at genotypes $x \in [0, 1/2)$ with "low incidence" in order to produce heterozygous offspring over future generations. The mathematics of pure fractions was very convenient for this purpose.

The theory of pure fractions is quite new in the sense that the authors believe this work is the first study to formalize and standardize special structural properties of pure fractions. The theory shows potential for further research and applications. In particular the type of Remediation Scheme presented by Definition 3.1 and Theorem 3.1 can be modified to apply to various other areas of study.

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