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COMPRESSIBLE FLOWS WITH NON-VANISHING SURFACE TENSION AND DENSITY-DEPENDENT VISCOSITY

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Abstract: We consider a free boundary problem of the one-dimensional, compressible Navier-Stokes equations for an isothermal flow coupled with an Allen-Cahn equation modeling a phase transition of a medium. The medium is connecting to a vacuum state with a jump in the density. Here we consider density-dependent viscosity $\mu = \rho^\theta$ and a density-dependent transition layer. We prove that there exists an unique, weak solution globally in time, provided that $\theta < 1/2$.

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1. Introduction

We investigate the evolution of an one-dimensional isothermal viscous material flow which undergoes a phase transition. As preferred application we have biomaterials in mind. In the model, we assume that both the viscosity coefficient as well as the thickness of the phase transition layer depends on the mass density. First, the material flow is governed by the Navier-Stokes equations which can be written, in Eulerian coordinates, as

$$\rho_{|t} + (\rho v)_{|x} = 0\,, \tag{1}$$

$$(\rho v)_{|t} + (\rho v^2 + p)_{|x} = (\mu(\rho) v_{|x})_{|x}\,, \tag{2}$$

where $0 < x < \xi(t)$, $t > 0$. The unknown functions ρ and v denotes the density

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and velocity, respectively. $x = 0$ is the fixed boundary and $x = \xi(t)$ the free boundary. Further the pressure is given by

$$p \equiv p(\rho, \Phi) = \rho^2 \mathrm{dcp}\, \Phi + a\rho^3 + \frac{\tilde{\delta}_1^2}{2} \gamma \rho^{(\gamma-1)+(\beta-1)} \big| \Phi_{|x} \big|^2$$

with $\mathrm{dcp} := \mathrm{cp}_1 - \mathrm{cp}_2 \geq 0$ and $a \in \mathbb{R}_+$ arbitrary. The scalar function Φ represents the phase function and is used to indicate the fluid phases. Further, there exist scalar constants $\gamma \in \mathbb{R}$ and $\beta > 1$, $\beta \in \mathbb{R}$. $\mu \equiv \mu(\rho) := \rho^\theta$, $0 < \theta < 1/2$ is called the viscosity coefficient. In the conventional case, the viscosity is assumed to be constant. Here we analyze the density dependent state, but not the dependence in Φ.

The Navier-Stokes equations are coupled with a phase field equation of Allen-Cahn type describing the development of a phase change from the regarded material. It is well-known, if we apply phase-field methods on multiphase Navier-Stokes equations an additional surface tension arise, see Jacqmin [9] and Blesgen [3]. We assume, that the evolution of Φ, as the gradient flow equation, is mainly driven by a given Helmholtz free energy density $f \equiv f(\rho, \Phi, \nabla\Phi)$. Then, this surface tension has the form $\nabla\Phi \otimes f_{|\nabla\Phi}$. Further, from general thermodynamic law for the effective pressure p_f, resulting from f, it applies the form $-p_f = f - \rho f_{|\rho}$. All up, the additional pressure tensor which follows from the given energy density f has the form $p_{tr} = p_f + \nabla\Phi \otimes f_{|\nabla\Phi}$.

In this paper, we introduce the free energy density

$$f(\rho, \Phi, \nabla\Phi) := \rho W(\Phi) + \rho\big[\Phi f_1(\rho) + (1-\Phi) f_2(\rho)\big] + \rho\Big[\frac{\delta_1^2(\rho)}{2}\big|\nabla\Phi\big|^2 + \frac{\delta_2^4(\rho)}{4}\big|\nabla\Phi\big|^4\Big] + f_m(\rho)\,,$$

whereby ρ acts as parameter and the summands are in different way weighted by ρ. f behaves in ρ as convex function.

Here, the first summand stands for the double-well functional with logarithmic part

$$W(\Phi) = c_1\big(\Phi \ln \Phi + (1-\Phi)\ln(1-\Phi)\big) + c_2 \Phi(1-\Phi)\,,$$

where $c_2 > c_1$. It has exactly 2 minima, which display the two pure phases $\Phi \approx 1$ and $\Phi \approx 0$ of the medium.

The second summand represents the convex combination of chemical potentialities f_1, f_2 of the phases 1, 2. We emanate from the simplest, linear case

$$f_1(\rho) = cp_1 \rho \qquad \text{and} \qquad f_2(\rho) = cp_2 \rho.$$

cp_1 denotes the possibility rate of phase 1 to transform in phase 2. And cp_2

denotes the possibility of phase 2 to transform in phase 1. Accordingly, this summand models the deviation from thermodynamical equilibrium.

In our model, we want to set up a broader way to control the transition layer formation for the scalar function Φ. We define two gradient penalty coefficients δ_1 and δ_2, which regulate the thickness of the transition layer and vary with the evolution of ρ. That is, $\delta_{1,2}$ depend on ρ. We set $\delta_1 \equiv \delta_1(\rho) := \tilde{\delta}_1 \rho^{\frac{\gamma-2}{2}}$ and $\delta_2 \equiv \delta_2(\rho) := \tilde{\delta}_2 \rho^{-1}$, where $\tilde{\delta}_1, \tilde{\delta}_2 > 0$. On the one hand, for small ρ the gradient penalty coefficient is increased and a thicker interface will minimize the total free energy for the system. That means, regions residing nearly in vacuum states, need a large transition layer. This behavior is preferentially forced by $\delta_2(\rho)$. On the other hand, for large ρ, $\delta_2(\rho)$ wants to become faster zero than this is the case for $\delta_1(\rho)$. Therefore, the behavior of the transition layer is mainly driven by $\delta_1(\rho)$.

$f_m(\rho)$ stands for the molecular forces between the particles of the material. Here we define

$$f_m(\rho) = \frac{a}{2}\rho^3 - \mathrm{cp}_2 \rho^2 \,.$$

The last term annihilates itself in f, so that f remains convex in ρ.

The Allen-Cahn equation constitutes a gradient flow equation for Φ. With the specified Helmholtz energy f, we enforce the ansatz for the non-inert system

$$\rho(\Phi_{|t} + v \cdot \Phi_{|x}) \;=\; -\frac{1}{\eta}\frac{\delta f}{\delta \Phi} \,.$$

Accordingly,

$$\begin{aligned}\eta\rho\left(\Phi_{|t} + v \cdot \Phi_{|x}\right) &= -c_1\, \rho\, \ln\Big(\frac{\Phi}{1-\Phi}\Big) - c_2\, \rho(1-2\Phi) \\ &\quad - \left(\mathrm{cp}_1 - \mathrm{cp}_2\right)\rho^2 + \left(\delta_1^2(\rho)\rho\Phi_{|x}\right)_{|x} + \left(\delta_2^4(\rho)\rho\left|\Phi_{|x}\right|^2\Phi_{|x}\right)_{|x}, \quad (3)\end{aligned}$$

where $t > 0$, $0 < x < \xi(t)$ and $\eta > 0$ is the relaxation time. In the case $\eta = \mathcal{O}(\tilde{\delta}_2)$, the sharp interface model is assured.

It is clear, that in biomaterials physical laws work, especially thermodynamic laws. In accordance with the thermodynamics and mentioned above, we calculate for the pressure acting on each material particle and only resulting from the form the free energy density f

$$\begin{aligned}p_{tr} &= p_f + \nabla\Phi \otimes f_{|\nabla\Phi} = -f + \rho f_{|\rho} + \nabla\Phi \otimes f_{|\nabla\Phi} \\ &= -f + \rho\Big[c_1\big(\Phi \ln\Phi + (1-\Phi)\ln(1-\Phi)\big) + c_2\Phi(1-\Phi)\Big] \\ &\quad + 2\rho^2\left[\Phi\mathrm{cp}_1 + (1-\Phi)\mathrm{cp}_2\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tilde{\delta}_1^2}{2}(\gamma-1)\rho^{\gamma-1}\big|\Phi_{|x}\big|^2-\frac{3\tilde{\delta}_2^4}{4}\rho^{-3}\big|\Phi_{|x}\big|^4+a\frac{3}{2}\rho^3-\mathrm{cp}_2 2\rho^2\\
&+\ \tilde{\delta}_1^2\rho^{\gamma-1}\big|\Phi_{|x}\big|^2+\tilde{\delta}_2^4\rho^{-3}\big|\Phi_{|x}\big|^4\\
=\ &\rho^2\mathrm{dcp}\,\Phi+a\rho^3+\frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma-1}\big|\Phi_{|x}\big|^2 .
\end{aligned}
$$

If we set p_{tr} equal to p in the impulse conversation (2), then the model would fulfill the entropy principle for the regarded closed system (1), (2) and (3). But here we assume, that living material contains a certain amount of energy in itself. And we presume, that this amount is emitted in the course of the phase change. Therefore, we set up that the surface tension, implied by f, works in a more stronger sense as it would be expected in other materials. Hence, we scale the surface tension $\nabla\Phi\otimes f_{|\nabla\Phi}$ by the power of ρ with $\rho^{\beta-1}$.

$x=0$ is the fixed boundary

$$v(t,0)=0\,,$$

and $x=\xi(t)$ is the free boundary, i.e. the interface of the liquid or gaseous medium and the vacuum:

$$\frac{d\xi}{dt}(t)=v(t,\xi(t)) \qquad \text{and} \tag{4}$$

$$\Big(p-\mu v_{|x}\Big)(t,\xi(t))=0\,. \tag{5}$$

Further,

$$\Phi_{|x}(t,0)\ =\ \Phi_{|x}(t,\xi(t))\ =\ 0\,. \tag{6}$$

The aim of this article is to show the global existence of a weak solution and the uniqueness. Our main result is made precise in the next Section 2. Section 3 is devoted to the derivation of the fundamental energy inequality and a priori estimates. In the subsequent mathematical rigorous argumentation, two aspects are essential. First, it cannot appear regions for which the mass density is infinite, and second, regions where the mass is disappeared. These facts are proved in Section 4. The existence part of Theorem 2 is proved in Section 5 and Section 6. For that, we discretize the space variable and use the line method. The uniqueness is addressed in Section 7.

Remark. In the study we restrict ourselves on $0<\theta<1/2$. But this constant has not to be optimal.

2. The Weak Formulation of the Problem and Basic Assumptions

We rewrite the equations in the Lagrangian mass coordinates

$$z = \int_0^x \rho(t,x')dx' \,.$$

Assuming that is

$$\int_0^{\xi(t)} \rho(t,x')dx' = 1 \,.$$

The above problem is transformed into the ensuing fixed boundary problem;

$$\rho_{|t} + \rho^2 v_{|z} = 0 \,, \tag{7}$$

$$v_{|t} + p(\rho,\Phi)_{|z} = (\mu(\rho)\rho v_{|z})_{|z} \,, \tag{8}$$

$$\begin{aligned} \eta\Phi_{|t} = -c_1 \ln\left(\frac{\Phi}{1-\Phi}\right) - c_2(1-2\Phi) - \rho\,\mathrm{dcp} \\ + (\tilde{\delta}_1^2\rho^\gamma\Phi_{|z})_{|z} + (\tilde{\delta}_2^4|\Phi_{|z}|^2\Phi_{|z})_{|z} \end{aligned} \tag{9}$$

in $t > 0$ and $0 < z < 1$, where $p(\rho,\Phi) = \rho^2\,\mathrm{dcp}\,\Phi + a\rho^3 + \frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma+\beta}\left|\Phi_{|z}\right|^2$ with the boundary conditions

$$v(t,0) = 0 \,, \tag{10}$$

$$\big(p - \mu\rho v_{|z}\big)(t,1) = 0 \,, \tag{11}$$

$$\Phi_{|z}(t,0) = \Phi_{|z}(t,1) = 0 \tag{12}$$

and the initial data

$$(\rho,v,\Phi)(0,z) = (\rho_0,v_0,\Phi_0)(z) \,, \qquad 0 \le z \le 1 \,. \tag{13}$$

In this paper we consider the following assumptions:

(A.1) $\rho_0 \in C^{0,1}([0,1])$ and $\rho_0(z) \ge \underline{\rho}_0$, where $\underline{\rho}_0 > 0$ is a constant;

(A.2) $v_0 \in C^1([0,1])$ and $\frac{d}{dx}v_0 \in C^{0,1}([0,1])$;

(A.3) $\Phi_0 \in C^1([0,1])$ and $0 \le \Phi_0 \le 1$ and $\frac{d}{dx}\Phi_0 \in C^{0,1}([0,1])$;

(A.4) $0 < \theta < 1/2$.

Definition 1. A triple (ρ,v,Φ) is called a global weak solution for equations (7)-(13) if

$$\rho,\ v,\ \Phi\ \in L^\infty([0,T]\times[0,1]) \cap C^1([0,T];L^2(0,1)) \,, \tag{14}$$

$$\rho^{\theta+1}v_{|z} \in L^\infty([0,T]\times[0,1]) \cap C^0([0,T];L^2(0,1)) \,, \tag{15}$$

$$\rho^\gamma\Phi_{|z},\ \left|\Phi_{|z}\right|^2\Phi_{|z},\ \in L^\infty([0,T]\times[0,1]) \cap C^0([0,T];L^2(0,1)) \,, \tag{16}$$

$$\rho^{\gamma+\beta}\left|\Phi_{|z}\right|^2\ \in L^\infty([0,T]\times[0,1]) \cap C^0([0,T];L^2(0,1)) \,, \tag{17}$$

for an arbitrary T, and the following equations hold:

$$\rho_{|t} + \rho^2 v_{|z} = 0 \tag{18}$$

for a.e. $z \in (0,1)$ and for any $t \geq 0$, and

$$\int_0^1 \left[\varphi v_{|t} - \varphi_{|z}(p - \mu\rho v_{|z})\right] dz = 0 \tag{19}$$

for any test function $\varphi \in C_0^\infty((0,1])$ and for a.e. $t \in [0,T]$, and

$$\int_0^1 \Big[\varphi\Phi_{|t} + \varphi\Big\{c_1 \ln\Big(\frac{\Phi}{1-\Phi}\Big) + c_2(1-2\Phi) + \rho\,\mathrm{dcp}\Big\} + \varphi_{|z}\Big\{\tilde{\delta}_1^2\rho^\gamma\Phi_{|z} + \tilde{\delta}_2^4\big|\Phi_{|z}\big|^2\Phi_{|z}\Big\}\Big] dz = 0 \tag{20}$$

for any test function $\varphi \in C_0^\infty((0,1])$ and for a.e. $t \in [0,T]$.

Under assumptions (A.1)-(A.4), we will prove the existence of a global weak solution to the initial boundary value problem (7)-(13) in the sense of Definition 1. In what follows, we use C or $C(T)$ to label a generic constant, which depends only on the initial data, the time T.

The main theorem states

Main Theorem 2. *Under assumptions (A.1)-(A.4), there exist constants γ and β, so that the free boundary value problem has a weak global solution $(\rho(t,z), v(t,z), \Phi(t,z))$ with $\rho, v, \Phi \in L^\infty([0,T]\times[0,1]) \cap C^1([0,T];L^2(0,1))$ and ρ satisfies*

$$\frac{1}{C(T)} \;\leq\; \rho(t,z) \;\leq\; C \quad \text{a.e.} \quad (t,z) \in [0,T]\times[0,1]\,.$$

3. Estimates

The difficulty of solving equations (7)-(9) relies on the logarithmic nonlinearity. The method to handle, is to introduce a regularized system, in which $\ln\left(\frac{\Phi}{1-\Phi}\right)$ is replaced by polynomials. In this way, the ln-term is approximated by a function, which is extended on the whole $\mathbb{R}^1$. For large N the series diverges outside $[-1,1]$.

First, we consider following transformation

$$w = 1 - 2\Phi\,.$$

Because of

$$g_N(w) := 4c_1 \sum_{k=0}^{N} \frac{w^{2k+1}}{2k+1}\,,$$

system (7)–(13) can be written as

$$\rho_{|t} + \rho^2 v_{|z} = 0\,, \tag{21}$$

$$v_{|t} + p(\rho, w)_{|z} = (\mu(\rho)\rho v_{|z})_{|z}\,, \tag{22}$$

$$\eta\, w_{|t} = -g_N(w) + 2c_2 w + 2\rho\,\mathrm{dcp} + \tilde{\delta}_1^2(\rho^\gamma w_{|z})_{|z} + \frac{\tilde{\delta}_2^4}{4}\big(\big|w_{|z}\big|^2 w_{|z}\big)_{|z}\,, \tag{23}$$

in $t > 0$ and $0 < z < 1$, where $p(\rho, w) = \rho^2 \mathrm{dcp}\,\frac{1}{2}(1-w) + \rho^3 a + \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2$ with the boundary conditions

$$v(t,0) = 0\,, \tag{24}$$

$$\big(p - \mu\rho v_{|z}\big)(t,1) = 0\,, \tag{25}$$

$$w_{|z}(t,0) = w_{|z}(t,1) = 0 \tag{26}$$

and the initial condition

$$(\rho, v, w)(0,z) = (\rho_0, v_0, 1 - 2\Phi_0)(z)\,, \qquad 0 \le z \le 1\,. \tag{27}$$

First, we will show that the solution satisfies a priori estimates.

Lemma 3. *Let (ρ, v, w) be the solution of (21)-(23). Then we have*

$$\int_0^1 \frac{1}{4}\Big\{\eta\big|w_{|t}\big|^2 + \frac{d}{dt}\Big[4c_1\sum_{k=0}^{N}\frac{w^{2k+2}}{(2k+2)(2k+1)}\Big] \tag{28}$$

$$+ c_2\frac{d}{dt}\big[(1-w)(1+w)\big] + \frac{d}{dt}\Big[\frac{\tilde{\delta}_1^2}{2}\rho^\gamma\big|w_{|z}\big|^2\Big] + \frac{d}{dt}\Big[\frac{\tilde{\delta}_2^4}{4^2}\big|w_{|z}\big|^4\Big]\Big\}dz$$

$$+ \int_0^1 \frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma+1}\big|w_{|z}\big|^2 v_{|z}\,dz\,d\tau + \int_0^1 \rho\,\mathrm{dcp}\,\frac{d}{dt}\Big[\frac{1}{2}\Big(1-w\Big)\Big]dz \;=\; 0\,.$$

Proof. After multiplying equation (23) with $w_{|t}$, we integrate from 0 to 1. This yields

$$\int_0^1\Big\{\eta\big|w_{|t}\big|^2 + g_N(w)w_{|t} - c_2\frac{d}{dt}\big(w^2\big) - \tilde{\delta}_1^2(\rho^\gamma w_{|z})_{|z}w_{|t}$$

$$- \frac{\tilde{\delta}_2^4}{4}\big(\big|w_{|z}\big|^2 w_{|z}\big)_{|z}w_{|t}\Big\}dz \;=\; \int_0^1 2\rho\,\mathrm{dcp}\,w_{|t}\,dz\,.$$

We calculate further the fourth summand on the right-hand side. It holds, after exploiting the boundary conditions (26),

$$-\int_0^1 \tilde{\delta}_1^2(\rho^\gamma w_{|z})_{|z}w_{|t}\,dz = \int_0^1 \tilde{\delta}_1^2\rho^\gamma w_{|z}w_{|tz}\,dz$$

$$= \frac{d}{dt}\Big[\int_0^1 \frac{\tilde{\delta}_1^2}{2}\rho^\gamma |w_{|z}|^2 dz\Big] - \int_0^1 \frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma-1}\rho_{|t}|w_{|z}|^2 dz$$
$$= \frac{d}{dt}\Big[\int_0^1 \frac{\tilde{\delta}_1^2}{2}\rho^\gamma |w_{|z}|^2 dz\Big] + \int_0^1 \frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma+1}|w_{|z}|^2 v_{|z} dz\,,$$

and

$$-\int_0^1 \frac{\tilde{\delta}_2^4}{4}\big(|w_{|z}|^2 w_{|z}\big)_{|z} w_{|t} dz = \frac{d}{dt}\Big[\int_0^1 \frac{\tilde{\delta}_2^4}{4^2}|w_{|z}|^4 dz\Big].$$

Further, it is

$$g_N(w)w_{|t} = \frac{d}{dt}\Big[4c_1\sum_{k=0}^{N}\frac{w^{2k+2}}{(2k+2)(2k+1)}\Big].$$

By inserting this in above equation, we derive that

$$\int_0^1 \Big\{\eta|w_{|t}|^2 + \frac{d}{dt}\Big[4c_1\sum_{k=0}^{N}\frac{w^{2k+2}}{(2k+2)(2k+1)}\Big] + c_2\frac{d}{dt}\big[(1-w)(1+w)\big] + \frac{d}{dt}\Big[\frac{\tilde{\delta}_1^2}{2}\rho^\gamma|w_{|z}|^2\Big] + \frac{d}{dt}\Big[\frac{\tilde{\delta}_2^4}{4^2}|w_{|z}|^4\Big]\Big\}dz + \int_0^1 \frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma+1}|w_{|z}|^2 v_{|z} dz \;=\; \int_0^1 2\rho\,\mathrm{dcp}\,w_{|t} dz\,. \quad \square$$

Lemma 4. *It holds*

$$\int_0^1 \frac{1}{2}\frac{d}{dt}|v|^2 dz - \int_0^1 \Big(\rho^2\mathrm{dcp}\,\frac{1}{2}(1-w)v_{|z} + a\rho^3 v_{|z}\Big)dz - \int_0^1 \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}|w_{|z}|^2 v_{|z} dz + \int_0^1 \rho^{1+\theta}|v_{|z}|^2 dz \;=\; 0\,. \tag{29}$$

Proof. We multiply equation (22) with v and integrate. By using the boundary conditions (24), (25), this equation follows immediately. $\square$

Lemma 5. *It holds*

$$\int_0^1 \rho_{|t}\,\mathrm{dcp}\,\frac{1}{2}(1-w)\,dz + \int_0^1 \frac{d}{dt}\Big(\frac{1}{2}a\rho^2\Big)dz + \int_0^1 \Big(\rho^2\mathrm{dcp}\,\frac{1}{2}(1-w)v_{|z} + a\rho^3 v_{|z}\Big)dz = 0\,. \tag{30}$$

Proof. Take equation (21) and multiply with $\Big[\mathrm{dcp}\,\frac{1}{2}(1-w) + a\rho\Big]$, then we obtain by summation the assertion. $\square$

Lemma 6. *It holds*

$$\tilde{\delta}_1^4\rho^{2\gamma}\big|w_{|z}\big|^2(t,z)+\frac{1}{2}\tilde{\delta}_1^2\tilde{\delta}_2^4\rho^{\gamma}\big|w_{|z}\big|^4(t,z)+\frac{\tilde{\delta}_2^8}{4^2}\big|w_{|z}\big|^6(t,z)$$
$$\leq\ 4\int_0^1\eta^2\big|w_{|t}\big|^2(t,.)dz'+C\int_0^1\big|g_N(w)\big|^2(t,.)dz'+16\,\mathrm{dcp}^2\int_0^1\rho^2(t,.)dz'\,.$$

Proof. We integrate equation (23) from 0 to z:

$$\tilde{\delta}_1^2\rho^{\gamma}w_{|z}(t,z)+\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^2w_{|z}(t,z)+\int_0^z\eta\,w_{|t}(t,.)dz'$$
$$=\tilde{\delta}_1^2\rho^{\gamma}w_{|z}(t,0)+\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^2w_{|z}(t,0)$$
$$+\int_0^z g_N(w)(t,.)dz'-2c_2\int_0^z w(t,.)dz'-2\,\mathrm{dcp}\int_0^z\rho(t,.)dz'\,.$$

Then by squaring

$$\Big(\tilde{\delta}_1^2\rho^{\gamma}w_{|z}(t,z)+\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^2w_{|z}(t,z)\Big)^2$$
$$=\Big(\tilde{\delta}_1^2\rho^{\gamma}w_{|z}(t,0)+\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^2w_{|z}(t,0)+\int_0^z\eta\,w_{|t}(t,.)dz'$$
$$+\int_0^z g_N(w)(t,.)dz'-2c_2\int_0^z w(t,.)dz'-2\,\mathrm{dcp}\int_0^z\rho(t,.)dz'\Big)^2$$
$$\leq 2^2\Big(\int_0^1\eta^2\big|w_{|t}\big|^2(t,.)dz'+\int_0^1\big|g_N(w)\big|^2(t,.)dz'$$
$$+4c_2^2\int_0^1 w^2(t,.)dz'+4\mathrm{dcp}^2\int_0^1\rho^2(t,.)dz'\Big)\,.\quad\square$$

Lemma 7. *It holds*

$$\int_0^1\big|g_N(w)\big|^2dz+\int_0^1\tilde{\delta}_1^2\rho^{\gamma}\big|w_{|z}\big|^2g_N'(w)dz+\int_0^1\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^4g_N'(w)dz$$
$$\leq\ C+C\int_0^1\eta^2\big|w_{|t}\big|^2dz+C\int_0^1\rho^2dz\,.$$

Proof. We multiply equation (23) with $g_N(w)$ and integrate from 0 to 1. Then

$$\int_0^1\eta\,w_{|t}g_N(w)dz+\int_0^1\big|g_N(w)\big|^2dz$$
$$=\ 2c_2\int_0^1 w\,g_N(w)dz+2\mathrm{dcp}\int_0^1\rho\,g_N(w)dz$$

$$- \int_0^1 \tilde{\delta}_1^2 \rho^\gamma w_{|z} g_N'(w) w_{|z} dz - \int_0^1 \frac{\tilde{\delta}_2^4}{4} \left|w_{|z}\right|^2 w_{|z} g_N'(w) w_{|z} dz \,.$$

Then, with Young's inequality and ε_1 fixed and small

$$\frac{1}{2}\int_0^1 \left|g_N(w)\right|^2 dz + \int_0^1 \tilde{\delta}_1^2 \rho^\gamma \left|w_{|z}\right|^2 g_N'(w) dz + \int_0^1 \frac{\tilde{\delta}_2^4}{4} \left|w_{|z}\right|^4 g_N'(w) dz + \int_0^1 \frac{2}{3} w^4 dz$$
$$\leq \frac{c_2}{\varepsilon_1}\int_0^1 w^2 dz + \frac{1}{2\varepsilon_1}\int_0^1 \eta^2 \left|w_{|t}\right|^2 dz + \frac{\mathrm{dcp}}{\varepsilon_1}\int_0^1 \rho^2 dz \,.$$

Because of

$$\frac{c_2}{\varepsilon_1} w^2 - \frac{2}{3} w^4 \;\leq\; \frac{3}{8}\frac{c_2^2}{\varepsilon_1^2} \,,$$

it follows the assertion. □

Theorem 8. *It holds following inequality*

$$\int_0^1 \frac{1}{2} v^2 \, dz + \int_0^1 \left[c_1 \sum_{k=0}^N \frac{w^{2k+2}}{(2k+2)(2k+1)} \right] dz$$
$$+ c_2 \int_0^1 \frac{1}{4}\big[(1-w)(1+w)\big] dz + \int_0^1 \frac{1}{4}\frac{\tilde{\delta}_1^2}{2} \rho^\gamma \left|w_{|z}\right|^2 dz$$
$$+ \int_0^1 \frac{1}{4}\frac{\tilde{\delta}_2^4}{4^2} \left|w_{|z}\right|^4 dz + \int_0^1 \left(\rho \,\mathrm{dcp}\, \frac{1}{2}(1-w) + \rho^2 \frac{1}{2} a\right) dz$$
$$+ C \int_0^t \int_0^1 \frac{1}{4}\, \eta \left|w_{|t}\right|^2 dz d\tau + C \int_0^t \int_0^1 \rho^{\theta+1} \left|v_{|z}\right|^2 dz d\tau$$
$$\leq\; C^*(\rho_0, v_0, 1 - 2\Phi_0) \;+\; C \,.$$

Proof. By adding all equations (30), (29) and (28). After that we integrate over $[0, t]$, then we get the desired inequality by taking assumptions (A.1)-(A.3) about the initial functions into account. It holds following inequality

$$\int_0^1 \frac{1}{2} v^2 \, dz + \int_0^1 \left[c_1 \sum_{k=0}^N \frac{w^{2k+2}}{(2k+2)(2k+1)} \right] dz$$
$$+ c_2 \int_0^1 \frac{1}{4}\big[(1-w)(1+w)\big] dz + \int_0^1 \frac{1}{4}\frac{\tilde{\delta}_1^2}{2} \rho^\gamma \left|w_{|z}\right|^2 dz$$
$$+ \int_0^1 \frac{1}{4}\frac{\tilde{\delta}_2^4}{4^2} \left|w_{|z}\right|^4 dz + \int_0^1 \left(\rho \,\mathrm{dcp}\, \frac{1}{2}(1-w) + \frac{1}{2} a \rho^2\right) dz$$
$$+ \int_0^t \int_0^1 \frac{1}{4}\, \eta \left|w_{|t}\right|^2 dz d\tau + \int_0^t \int_0^1 \rho^{\theta+1} \left|v_{|z}\right|^2 dz d\tau$$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{2} v_0^2 \, dz + \int_0^1 \left[c_1 \sum_{k=0}^{N} \frac{w_0^{2k+2}}{(2k+2)(2k+1)} \right] dz \\
&\quad + c_2 \int_0^1 \frac{1}{4} \big[(1-w_0)(1+w_0) \big] dz + \int_0^1 \frac{1}{4} \frac{\tilde{\delta}_1^2}{2} \rho_0^{\gamma} \big| w_{0|z} \big|^2 dz \\
&\quad + \int_0^1 \frac{1}{4} \frac{\tilde{\delta}_2^4}{4^2} \big| w_{0|z} \big|^4 dz + \int_0^1 \Big(\rho_0 \, \mathrm{dcp} \, \frac{1}{2} (1 - w_0) + \frac{1}{2} a \rho_0^2 \Big) dz \\
&\quad + \int_0^t \int_0^1 \frac{\tilde{\delta}_1^2}{2} \gamma \, \big(\frac{1}{4} \rho^{\gamma+\beta} - \rho^{\gamma+1} \big) \big| w_{|z} \big|^2 v_{|z} \, dz d\tau \\
&\le C + \int_0^t \int_0^1 \frac{\tilde{\delta}_1^2}{2} \gamma \big(\frac{1}{4} \rho^{\gamma+\beta} + \rho^{\gamma+1} \big) \big| w_{|z} \big|^2 \big| v_{|z} \big| dz d\tau \\
&\le C + \frac{1}{2\varepsilon_1} \int_0^t \int_0^1 \Big(\frac{\tilde{\delta}_1^2}{2} \gamma \Big)^2 \rho^{2\gamma-\theta+1} \big| w_{|z} \big|^4 dz d\tau \\
&\quad + \frac{\varepsilon_1}{2} \int_0^t \int_0^1 \rho^{1+\theta} \big| v_{|z} \big|^2 dz d\tau + \frac{1}{2\varepsilon_1} \int_0^t \int_0^1 \Big(\frac{\tilde{\delta}_1^2}{8} \gamma \Big)^2 \rho^{2\gamma+2\beta-\theta-1} \big| w_{|z} \big|^4 dz d\tau \\
&\quad + \frac{\varepsilon_1}{2} \int_0^t \int_0^1 \rho^{1+\theta} \big| v_{|z} \big|^2 dz d\tau \\
&\le C + \frac{1}{2\varepsilon_1} \int_0^t \int_0^1 \Big(\frac{\tilde{\delta}_1^2}{2} \gamma \Big)^2 \rho^{2\gamma-\theta+1} \big| w_{|z} \big|^4 dz d\tau \\
&\quad + \frac{1}{6\varepsilon_1} \int_0^t \int_0^1 \rho^{(2\gamma+2\beta-\theta-1)\cdot 3} dz d\tau + \frac{1}{3\varepsilon_1} \Big(\frac{\tilde{\delta}_1^2}{8} \gamma \Big)^3 \int_0^t \int_0^1 \big| w_{|z} \big|^6 dz d\tau \\
&\quad + \varepsilon_1 \int_0^t \int_0^1 \rho^{1+\theta} \big| v_{|z} \big|^2 dz d\tau \, .
\end{aligned}
$$

Now we choose γ and β in such a way, so that

$$
2\gamma - \theta + 1 = 0 \qquad \text{and} \qquad (2\gamma + 2\beta - \theta - 1) \cdot 3 \;=\; 2 \, .
$$

Then we have

$$
\gamma \;=\; \frac{\theta - 1}{2} \qquad \text{and} \qquad \beta \;=\; \frac{4}{3} \qquad \Rightarrow \qquad \gamma + \beta = \frac{3\theta + 5}{6} \, .
$$

With Lemma 6, it is valid

$$
\begin{aligned}
&\int_0^1 \frac{1}{2} v^2 \, dz + \int_0^1 \frac{1}{4} \left[4 c_1 \sum_{k=0}^{N} \frac{w^{2k+2}}{(2k+2)(2k+1)} \right] dz \\
&\quad + c_2 \int_0^1 \frac{1}{4} \big[(1-w)(1+w) \big] dz + \int_0^1 \frac{1}{4} \frac{\tilde{\delta}_1^2}{2} \rho^{\gamma} \big| w_{|z} \big|^2 dz \\
&\quad + \int_0^1 \frac{1}{4} \frac{\tilde{\delta}_2^4}{4^2} \big| w_{|z} \big|^4 dz + \int_0^1 \Big(\rho \, \mathrm{dcp} \, \frac{1}{2} (1 - w) + \rho^2 \frac{1}{2} a \Big) dz
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^t\int_0^1 \frac{1}{4}\eta\big|w_{|t}\big|^2 dzd\tau + \int_0^t\int_0^1 \rho^{\theta+1}\big|v_{|z}\big|^2 dzd\tau \\
&\le C + \frac{1}{2\varepsilon_1}\Big(\frac{\tilde\delta_1^2}{2}\gamma\Big)^2 \int_0^t\int_0^1 \big|w_{|z}\big|^4 dzd\tau + \frac{1}{6\varepsilon_1}\int_0^t\int_0^1 \rho^2 dzd\tau \\
&\quad + \varepsilon_1 \int_0^t\int_0^1 \rho^{1+\theta}\big|v_{|z}\big|^2 dzd\tau + \frac{1}{3\varepsilon_1}\Big(\frac{\tilde\delta_1^2}{2}\gamma\Big)^3 \int_0^t \frac{1}{\tilde\delta_2^8}\Big[C + 4\int_0^1 \eta^2\big|w_{|t}\big|^2 dz \\
&\qquad\qquad + C\int_0^1 \big|g_N(w)\big|^2 dz + C\int_0^1 \rho^2 dz\Big] d\tau\,,
\end{aligned}
$$

with Lemma 7, it is valid

$$
\begin{aligned}
&\le C + C\int_0^t\int_0^1 \big|w_{|z}\big|^4 dzd\tau + C\int_0^t\int_0^1 \rho^2 dzd\tau \\
&\quad + \varepsilon_1 \int_0^t\int_0^1 \rho^{1+\theta}\big|v_{|z}\big|^2 dzd\tau + \frac{\tilde\delta_1^6}{\tilde\delta_2^8} C \int_0^t\int_0^1 \eta^2\big|w_{|t}\big|^2 dzd\tau\,.
\end{aligned}
$$

Now, we choose ε_1 and $\frac{\tilde\delta_1^6}{\tilde\delta_2^8}\eta^2 C$ arbitrary small. After that, with Gronwall Lemma, we get the assertion. □

It follows immediately and only from the bound of the first energy, especially by the ln-term, that w is bounded. Along this way, it is possible to add different types of reactions on the ln-energy, in most cases the maximum principles holds. This is contrary to the van der Waals energy, where the maximum principle can not be aware in the case that additional reactions occurs.

Lemma 9. *It is satisfied*

$$
-1 \;\le\; w(t,z) \;\le\; +1\,. \tag{31}
$$

An important problem in the theory of compressible Navier-Stokes equations is to guarantee, that in passing of the time t, non space point emerges in that the mass density is infinite. Otherwise, this points would constitute singularities for the system, and the equations are not well-posed, since the compressibility would be permitted in a too extensive degree. Now, we show that the mass density is bounded.

Theorem 10. *There exists C, such that*

$$
\rho(t,z) \le C \qquad \textit{for} \quad (t,z) \in [0,T_\infty) \times (0,1)\,.
$$

Proof. We know with equation (21)

$$
\frac{d}{dt}(\rho^\theta) = \theta\rho^{\theta-1}\rho_{|t} = -\theta\rho^{\theta+1}v_{|z}\,. \tag{32}
$$

Integrating the equation (22) from z to 1, and taking equation (32) and the boundary conditions into account, we get

$$\int_z^1 v_{|t}dz' - \rho^2\Big(\mathrm{dcp}\frac{1}{2}(1-w)+\rho a\Big)(.,z) - \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2(.,z)$$
$$+\rho^2\Big(\mathrm{dcp}\frac{1}{2}(1-w)+\rho a\Big)(.,1) + \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2(.,1)$$
$$= \int_z^1 \Big(-\frac{1}{\theta}\rho^{\theta+1}v_{|z}\Big)_{|z}dz = \big(\rho^{\theta+1}v_{|z}\big)(.,1) + \frac{1}{\theta}\frac{d}{dt}\rho^\theta(.,z)\,. \tag{33}$$

With the boundary conditions (24), (25) and integration over $\tau=0$ and $\tau=t$, it is

$$\frac{1}{\theta}\rho^\theta + \int_0^t \rho^2\Big(\,\mathrm{dcp}\,\frac{1}{2}(1-w)+\rho a\Big)\,d\tau + \int_0^t \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2 d\tau$$
$$= \frac{1}{\theta}\rho_0^\theta + \int_z^1 v\,dz - \int_z^1 v_0\,dz$$
$$\le \frac{1}{\theta}\rho_0^\theta + \Big(\int_0^1 v^2 dz\Big)^{\frac{1}{2}} + \Big(\int_0^1 v_0^2\;dz\Big)^{\frac{1}{2}}$$
$$\le C \qquad\qquad \text{because of (A.1) and (A.3)}\,.$$

Concerning $\rho \ge 0$ and $-1 \le w \le 1$ (Lemma 9), the left-hand side greater than 0. The right-hand side is bounded, due to Theorem 8. Therefore, we have $\rho^\theta \le C$, and because of $\theta > 0$, it holds

$$\rho(t,z) \le C \qquad\qquad \forall\;(t,z) \in [0,T_\infty)\times(0,1)\,. \qquad \square$$

Lemma 11. *Under assumptions (A.1)-(A.4), we have*

$$\int_0^t\int_0^1 \big|\rho_{|t}\big|^2\;dzd\tau \;\;\le\;\; C\,.$$

Proof. We multiply equation (21) with $\rho_{|t}$ and integrate over $(0,t)\times(0,1)$. By using Theorem 10, Young's inequality and the fact (Theorem 8)

$$\int_0^t\int_0^1 \rho^{\theta+1}\big|v_{|z}\big|^2 dzd\tau \;\;\le\;\; C\,,$$

it follows directly the assertion. $\square$

Lemma 12. *Under assumptions (A.1)-(A.4), it holds*

$$\int_0^1 \big|w_{|t}\big|^2(t,.)dz \;\;\le\;\; C(T) \qquad \textit{pointwise for} \quad 0\le t\le T\,, \tag{34}$$
$$\int_0^t\int_0^1 \rho^\gamma\big|w_{|tz}\big|^2 dzd\tau \;\;\le\;\; C(T)\,, \tag{35}$$

$$\int_0^t\int_0^1 |w_{|z}|^2 |w_{|tz}|^2 dz d\tau \;\leq\; C(T)\,, \tag{36}$$

$$\int_0^1 |g_N(w)|^2(t,.)dz \;\leq\; C(T) \quad \textit{pointwise for} \quad 0 \leq t \leq T. \tag{37}$$

Proof. Differentiate equation (23) by t

$$\eta w_{|tt} = -g_N'(w)w_{|t} + 2c_2 w_{|t} + 2\rho_{|t}\,\mathrm{dcp} + \tilde{\delta}_1^2\big(\rho^\gamma w_{|z}\big)_{|zt} + \frac{\tilde{\delta}_2^4}{4}\big(|w_{|z}|^2 w_{|z}\big)_{|zt}\,.$$

Then, multiplying it by $w_{|t}$ and integrate over $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\bigg(\int_0^1 \eta|w_{|t}|^2 dz\bigg) + \int_0^1 g_N'(w)|w_{|t}|^2 dz + \tilde{\delta}_1^2\int_0^1 \rho^\gamma |w_{|zt}|^2 dz$$
$$+ \frac{3\,\tilde{\delta}_2^4}{4}\int_0^1 |w_{|z}|^2|w_{|zt}|^2 dz$$
$$\leq (2c_2 + \mathrm{dcp}^2)\int_0^1 |w_{|t}|^2 dz + \int_0^1 |\rho_{|t}|^2 dz$$
$$+ \int_0^1 \frac{\tilde{\delta}_1^2}{2}\gamma\rho^{\gamma+1}|w_{|z}||v_{|z}||w_{|zt}|dz + C\,.$$

Now, integrate over $[0,t]$. Then

$$\frac{1}{2}\int_0^1 \eta|w_{|t}|^2(t,.)dz + \int_0^t\int_0^1 g_N'(w)|w_{|t}|^2 dz d\tau$$
$$+ \int_0^t\int_0^1 \tilde{\delta}_1^2\rho^\gamma|w_{|zt}|^2 dz d\tau + \frac{3\,\tilde{\delta}_2^4}{4}\int_0^t\int_0^1 |w_{|z}|^2|w_{|zt}|^2 dz$$
$$\leq \frac{1}{2}\int_0^1 \eta|w_{|t}|^2(0,.)dz + C\int_0^t\int_0^1 |w_{|t}|^2 dz d\tau + \int_0^t\int_0^1 |\rho_{|t}|^2 dz d\tau$$
$$+ \frac{1}{4\varepsilon}\int_0^t\int_0^1 \tilde{\delta}_1^2\Big(\frac{\theta-1}{2}\Big)^2\rho^{1+\theta}|v_{|z}|^2 dz d\tau$$
$$+ \frac{\varepsilon}{4}\int_0^t\int_0^1 \tilde{\delta}_1^2|w_{|z}|^2|w_{|zt}|^2 dz d\tau + C \;\leq\; C\,,$$

because of ε small, of Theorem 8 and Lemma 11. From equation (23) and assumption $(A.3)$ it holds

$$\int_0^1 |w_{|t}|^2(0,z)dz \;\leq\; C\,.$$

Then, we have

$$\int_0^1 |w_{|t}|^2(t,.)dz \;\leq\; C(T) \qquad \text{pointwise for} \quad 0 \leq t \leq T.$$

From that, we can achieve equation (37) by Lemma 7 and Theorem 10. □

Lemma 13. *It is bounded*

$$\tilde{\delta}_1^4\rho^{2\gamma}\big|w_{|z}\big|^2(t,z)+\frac{1}{2}\tilde{\delta}_1^2\tilde{\delta}_2^4\rho^{\gamma}\big|w_{|z}\big|^4(t,z)+\frac{\tilde{\delta}_2^8}{16}\big|w_{|z}\big|^6(t,z)\ \le\ C\,.$$

Proof. The proof follows from Lemma 6 in conjunction with Lemma 12 and Theorem 10. □

Lemma 14. *It holds*

$$\frac{9}{16}\tilde{\delta}_2^8\int_0^1\big|w_{|z}\big|^4\big|w_{|zz}\big|^2dz+\frac{6}{4}\tilde{\delta}_1^2\tilde{\delta}_2^4\int_0^1\rho^{\frac{\theta-1}{2}}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2dz \le\ C+C\int_0^1\rho^{2\theta-2}\big|\rho_{|z}\big|^2dz\,.$$

Proof. From Lemma 12 (inequality (34) and (37)) and Lemma 9, Theorem 10 we can conclude

$$\int_0^1\Big[\big(\tilde{\delta}_1^2\rho^{\gamma}w_{|z}\big)_{|z}+\big(\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^2w_{|z}\big)_{|z}\Big]^2dz\ \le\ C\,. \tag{38}$$

Then

$$\begin{aligned}
&\Big[\big(\tilde{\delta}_1^2\rho^{\gamma}w_{|z}\big)_{|z}+\big(\frac{\tilde{\delta}_2^4}{4}\big|w_{|z}\big|^2w_{|z}\big)_{|z}\Big]^2\\
&=\tilde{\delta}_1^4\gamma^2\rho^{2\gamma-2}\big|\rho_{|z}\big|^2\big|w_{|z}\big|^2+\tilde{\delta}_1^4\rho^{2\gamma}\big|w_{|zz}\big|^2+\frac{9}{16}\,\tilde{\delta}_2^8\big|w_{|z}\big|^4\big|w_{|zz}\big|^2\\
&\quad+\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\rho^{\gamma}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2+2\,\tilde{\delta}_1^4\gamma\rho^{2\gamma-1}\rho_{|z}w_{|z}w_{|zz}\\
&\quad+\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\gamma\rho^{\gamma-1}\rho_{|z}\big|w_{|z}\big|^2w_{|z}w_{|zz}\\
&=\underbrace{\Big[\big(\tilde{\delta}_1^2\rho^{\gamma}w_{|z}\big)_{|z}\Big]^2}_{\ge 0}+\frac{9}{16}\,\tilde{\delta}_2^8\big|w_{|z}\big|^4\big|w_{|zz}\big|^2+\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\rho^{\gamma}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2\\
&\quad+\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\gamma\rho^{\gamma-1}\rho_{|z}\big|w_{|z}\big|^2w_{|z}w_{|zz}\,.
\end{aligned}$$

With $\gamma=\frac{\theta-1}{2}$, we have

$$\begin{aligned}
&\frac{9}{16}\,\tilde{\delta}_2^8\int_0^1\big|w_{|z}\big|^4\big|w_{|zz}\big|^2dz+\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\int_0^1\rho^{\frac{\theta-1}{2}}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2dz\\
&\qquad\le C-\int_0^1\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\big(\tfrac{\theta-1}{2}\big)\rho^{\frac{\theta-3}{2}}\rho_{|z}\big|w_{|z}\big|^2w_{|z}w_{|zz}dz\\
&\qquad\le C+\frac{1}{2\varepsilon}\int_0^1\frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4\big(\tfrac{\theta-1}{2}\big)^2\ \ \rho^{\frac{\theta-5}{2}}\big|w_{|z}\big|^4\big|\rho_{|z}\big|^2dz
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varepsilon}{2}\int_0^1 \frac{6}{4}\,\tilde{\delta}_1^2\tilde{\delta}_2^4 \quad \rho^{\frac{\theta-1}{2}}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2 dz\\
&= C + \frac{C}{2\varepsilon}\int_0^1 \quad \rho^{\frac{-7\theta+3}{2}}\cdot\rho^{2\theta-2}\big|w_{|z}\big|^4\cdot\rho^{2\theta-2}\big|\rho_{|z}\big|^2 dz\\
&+\frac{\varepsilon}{2}C\int_0^1 \rho^{\frac{\theta-1}{2}}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2 dz\,.
\end{aligned}
$$

Because of $0 \le \theta \le \frac{3}{7}$ and Theorem 10

$$
\rho^{\frac{-7\theta+3}{2}}(t,z) \;\le\; C \qquad \text{for} \quad (t,z)\in(0,T)\times(0,1)\,.
$$

From Lemma 13 we know, that

$$
\rho^{2\theta-2}\big|w_{|z}\big|^4(t,z) \;\le\; C \qquad \text{for} \quad (t,z)\in(0,T)\times(0,1)\,.
$$

For small ε we get the assertion. □

4. The Vacuum Problem

With equations (1)-(3) we model natural phenomena in those the initial state consists in a persistent medium core. This core is bordered from a free boundary, separating medium and vacuum. We are interested in the question, it is possible that vacuum regions develops within the solid core, in growing time. This problem is known as vacuum problem. In the next we can show, that we can exclude such islands without mass.

Simply, the conservation of mass in connection with $(A.1)$ implies $\rho \ge 0$.

Lemma 15. *It holds*

$$
\frac{d}{dt}\bigg(\int_0^z \frac{1}{\rho}dz'\bigg) \;=\; v(.,z)\,.
$$

The next lemma constitutes the central technical point in our calculation. If we write the assertion of the following lemma without our discretizing scheme in a continuous manner, it is to be claimed that $\frac{d}{dz}\rho^\theta$ can be controlled in the L^2-norm.

Lemma 16. *Assuming* $(A.1)$-$(A.4)$,

$$
\int_0^1 \big(\rho^\theta\big)^2_{|z}dz \;\le\; C(T)
$$

is satisfied.

Proof. From equation (22) and identity (32), we have

$$(\rho^\theta)_{|zt} = -\theta\Big(v_{|t} + \Big[\rho^2(\,\mathrm{dcp}\,\tfrac{1}{2}(1-w) + \rho a) + \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big]_{|z}\Big) \tag{39}$$

Multiplying with $(\rho^\theta)_{|z}$:

$$\frac{1}{2}\frac{d}{dt}(\rho^\theta)^2_{|z} = -\theta v_{|t}(\rho^\theta)_{|z} - \theta\Big(2\rho\rho_{|z}\,\mathrm{dcp}\,\tfrac{1}{2}(1-w) - \rho^2\,\mathrm{dcp}\,\tfrac{1}{2}w_{|z} + 3\rho^2\rho_{|z}a + \Big(\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|z}\Big)(\rho^\theta)_{|z}\,.$$

Integrating over $(0,t)\times(0,1)$, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (\rho^\theta)^2_{|z}(t,.)dz \\
&= \frac{1}{2}\int_0^1 \rho_0^{2\theta-2}\rho^2_{0|z}dz - \int_0^t\int_0^1 \theta v_{|t}(\rho^\theta)_{|z}dzd\tau \\
&- \int_0^t\int_0^1 \theta\Big(2\rho\rho_{|z}\,\mathrm{dcp}\,\tfrac{1}{2}(1-w) - \rho^2\,\mathrm{dcp}\,\tfrac{1}{2}w_{|z} + 3\rho^2\rho_{|z}a\Big)(\rho^\theta)_{|z}dzd\tau \\
&- \int_0^t\int_0^1 \theta\Big(\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|z}(\rho^\theta)_{|z}dzd\tau \\
&\le C - \int_0^1 \big(\theta v(\rho^\theta)_{|z}\big)(t,.)dz + \int_0^1 \theta v_0(\rho_0^\theta)_{|z}dz \\
&+ \int_0^t\int_0^1 \theta v(\rho^\theta)_{|zt}dzd\tau - \int_0^t\int_0^1 \theta^2\mathrm{dcp}\,\rho^\theta(1-w)\big|\rho_{|z}\big|^2dzd\tau \\
&+ \int_0^t\int_0^1 \tfrac{1}{2}\theta^2\mathrm{dcp}\,\rho^{1+\theta}w_{|z}\rho_{|z}dzd\tau - \int_0^t\int_0^1 3\,\theta^2a\,\rho^{1+\theta}\big|\rho_{|z}\big|^2dzd\tau \\
&- \int_0^t\int_0^1 \theta\Big(\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|z}(\rho^\theta)_{|z}dzd\tau\,.
\end{aligned}$$

We have

- $$+\int_0^t\int_0^1 \tfrac{1}{2}\theta^2\mathrm{dcp}\;\rho^{1+\theta}w_{|z}\rho_{|z}dzd\tau \le C\int_0^t\int_0^1 \rho^\gamma\big|w_{|z}\big|^2dzd\tau + C\int_0^t\int_0^1 \rho^{2\theta-2}\big|\rho_{|z}\big|^2dzd\tau$$

with Theorem 8 and the boundary conditions, it holds

$$\le C + C\int_0^t\int_0^1 \rho^{2\theta-2}\big|\rho_{|z}\big|^2dzd\tau.$$

With equation (39) and the assumption to the initial functions

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (\rho^\theta)^2_{|z}(t,.)dz + \int_0^t\int_0^1 \theta^2 \,\mathrm{dcp}\,\rho^\theta(1-w)\big|\rho_{|z}\big|^2 dzd\tau \\
&\quad + \int_0^t\int_0^1 3\ \theta^2 a\rho^{\theta+1}\big|\rho_{|z}\big|^2 dzd\tau \\
&\le C + \frac{\theta^2}{2\varepsilon}\int_0^1 v^2(t,.)dz + \frac{\varepsilon}{2}\int_0^1 (\rho^\theta)^2_{|z}(t,.)dz + C \\
&- \int_0^t\int_0^1 \theta^2 v\Big(v_{|t} + \Big[\rho^2(\,\mathrm{dcp}\,\tfrac{1}{2}(1-w)+\rho a) + \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big]_{|z}\Big)dzd\tau \\
&- \int_0^t\int_0^1 \theta\Big(\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|z}(\rho^\theta)_{|z}dzd\tau + C + C\int_0^t\int_0^1 \rho^{2\theta-2}\big|\rho_{|z}\big|^2 dzd\tau.
\end{aligned}$$

Now, we evaluate the fifth summand on the right-hand side

- $$+\int_0^t\int_0^1 \tfrac{1}{2}\theta^2\,\mathrm{dcp}\,\rho^2 w_{|z} v dzd\tau \le C\int_0^t\int_0^1 \rho^\gamma\big|w_{|z}\big|^2 dzd\tau + C\int_0^t\int_0^1 v^2 dzd\tau$$

- $$\begin{aligned}&-\int_0^t\int_0^1 \theta^2\Big(\,\mathrm{dcp}\,(1-w)\rho\,\rho_{|z} + 3a\rho^2\rho_{|z}\Big)v\ dzd\tau \\ &\le C\int_0^t\int_0^1 \rho^{2\theta-2}\big|\rho_{|z}\big|^2 dzd\tau + C\int_0^t\int_0^1 \rho^{4-2\theta}v^2 dzd\tau \\ &\quad + C\int_0^t\int_0^1 \rho^{6-2\theta}v^2 dzd\tau\,.\end{aligned}$$

We choose ε very small, recall that $4-2\theta > 0$ and $6-2\theta > 0$.

- $$\begin{aligned}&-\int_0^t\int_0^1 \theta^2\Big(\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|z} v\, dzd\tau \\ &= \int_0^t\int_0^1 \theta^2\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\frac{3\theta+5}{6}}\big|w_{|z}\big|^2 v_{|z}\, dzd\tau \\ &\le C\int_0^t\int_0^1 \rho^\gamma\big|w_{|z}\big|^4 dzd\tau + C\int_0^t\int_0^1 \rho^{1+\theta}\big|v_{|z}\big|^2 dzd\tau\,.\end{aligned}$$

The last term we have to evaluate is the most important

$$\begin{aligned}&-\int_0^t\int_0^1 \theta\Big(\frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|z}(\rho^\theta)_{|z}dzd\tau \\ &= -\int_0^t\int_0^1 \theta^2\frac{\tilde{\delta}_1^2}{8}\gamma\frac{3\theta+5}{6}\rho^{\frac{9\theta-7}{6}}\big|w_{|z}\big|^2\big|\rho_{|z}\big|^2 dzd\tau\end{aligned}$$

$$-\int_0^t\int_0^1 2\,\theta^2\frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\frac{9\theta-1}{6}}w_{|z}w_{|zz}\rho_{|z}\,dzd\tau\,.$$

Now, we consider the last term on the left-hand side

$$\begin{aligned}
&-\int_0^t\int_0^1 2\,\theta^2\frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\frac{9\theta-1}{6}}w_{|z}w_{|zz}\rho_{|z}\,dzd\tau\\
&\le C\frac{1}{2\varepsilon}\int_0^t\int_0^1\rho^{\frac{9\theta-7}{6}}\big|\rho_{|z}\big|^2dzd\tau+C\frac{\varepsilon}{2}\int_0^t\int_0^1\rho^{\frac{9\theta+5}{6}}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2dzd\tau\\
&\le C\frac{1}{2\varepsilon}\int_0^t\int_0^1\rho^{\frac{9\theta-7}{6}-(2\theta-2)}\cdot\rho^{2\theta-2}\big|\rho_{|z}\big|^2dzd\tau\\
&\quad+C\frac{\varepsilon}{2}\int_0^t\int_0^1\rho^{\frac{9\theta+5}{6}}\big|w_{|z}\big|^2\big|w_{|zz}\big|^2dzd\tau\,.
\end{aligned}$$

It holds by Theorem 10

$$\rho^{\frac{-3\theta+5}{6}}\le C\,,\qquad\qquad\text{because of}\quad\frac{-3\theta+5}{6}\ge 0\,.$$

And with Lemma 14 we have

$$-\int_0^t\int_0^1 2\,\theta^2\frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\frac{9\theta-1}{6}}w_{|z}w_{|zz}\rho_{|z}\,dzd\tau\le C\int_0^t\int_0^1\rho^{2\theta-2}\big|\rho_{|z}\big|^2dzd\tau+C.$$

Apply Theorem 8 and the Gronwall Lemma. Then, we get the assertion. □

Lemma 17. *For any positive integer n, we have*

$$\int_0^1 v^{2n}dz+n(2n-1)\int_0^t\int_0^1 v^{2n-2}\rho^{1+\theta}\big|v_{|z}\big|^2dzd\tau\ \le\ C(T)\,.$$

Proof. We multiply equation (22) with v^{2n-1} and integrate

$$\begin{aligned}
&\int_0^1 v^{2n}(t,.)dz+2n(2n-1)\int_0^t\int_0^1\rho^{1+\theta}v^{2n-2}\big|v_{|z}\big|^2dzd\tau\\
&\quad=\int_0^1 v_0^{2n}dz+2n(2n-1)\int_0^t\int_0^1\Big(\rho^2\,\mathrm{dcp}\,\tfrac{1}{2}(1-w)\\
&\qquad\qquad+\rho^3a+\frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)v^{2n-2}v_{|z}dzd\tau\\
&\quad\le C+n(2n-1)\int_0^t\int_0^1 v^{2n-2}\Big[\rho^{3-\theta}\,\mathrm{dcp}^{\,2}\tfrac{1}{4}(1-w)^2\\
&\qquad\qquad+\rho^{5-\theta}a^2+\frac{\tilde{\delta}_1^4}{64}\gamma^2\rho^{2\gamma+2\beta-1-\theta}\big|w_{|z}\big|^2\Big]dzd\tau\\
&\qquad+n(2n-1)\int_0^t\int_0^1 v^{2n-2}\rho^{1+\theta}\big|v_{|z}\big|^2dzd\tau
\end{aligned}$$

$$\leq C + (2n-1)\int_0^t\int_0^1 \Big[\rho^{(3-\theta)n}\,\mathrm{dcp}^{\,2n}\tfrac{1}{4^n}(1-w)^{2n} + \rho^{(5-\theta)n}a^{2n} + \frac{\tilde{\delta}_1^{4n}}{64^n}\gamma^{2n}\rho^{(2\beta-2)n}\big|w_{|z}\big|^{2n}\Big]dzd\tau$$
$$+ (2n-1)(n-1)\int_0^t\int_0^1 v^{2n}dzd\tau\,.$$

In the last step we have used, that $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$, where $p = n$ and $q = n/(n-1)$. With Theorem 10, Lemma 13 and the Gronwall Lemma, we get the assertion. ◻

Lemma 18. *For any positive integer n, it is fulfilled*
$$\int_0^1 \rho^{\alpha_n}v^2dz + C\int_0^t\int_0^1 \rho^{1+\theta+\alpha_n}\big|v_{|z}\big|^2dzd\tau \;\leq\; C(T)\,,$$
where $\alpha_n = (1-\frac{1}{2^n})(\theta-1)\,.$

Proof. View
$$(\rho^{\tilde{\alpha}_n}v^{2^n})_{|t} = \tilde{\alpha}_n\rho^{\tilde{\alpha}_n-1}\rho_{|t}v^{2^n} + \rho^{\tilde{\alpha}_n}2^nv^{2^n-1}v_{|t}$$
$$= -\tilde{\alpha}_n\rho^{\tilde{\alpha}_n+1}v^{2^n}v_{|z} + 2^n\rho^{\tilde{\alpha}_n}v^{2^n-1}\big(\rho^{1+\theta}v_{|z}\big)_{|z}$$
$$- 2^n\rho^{\tilde{\alpha}_n}v^{2^n-1}\Big[\rho^2\,\mathrm{dcp}\,\tfrac{1}{2}(1-w) + \rho^3a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big]_{|z}$$

and integrate over $(0,t)\times(0,1)$
$$\int_0^1 \rho^{\tilde{\alpha}_n}v^{2^n}dz + 2^n(2^n-1)\int_0^t\int_0^1 \rho^{1+\theta+\tilde{\alpha}_n}v^{2^n-2}\big|v_{|z}\big|^2dzd\tau$$
$$= \int_0^1 \rho_0^{\tilde{\alpha}_n}v_0^{2^n}dz - \tilde{\alpha}_n\int_0^t\int_0^1 \rho^{\tilde{\alpha}_n+1}v^{2^n}v_{|z}dzd\tau$$
$$- 2^n\tilde{\alpha}_n\int_0^t\int_0^1 \rho^{\theta+\tilde{\alpha}_n}v^{2^n-1}\rho_{|z}v_{|z}dzd\tau$$
$$+ 2^n(2^n-1)\int_0^t\int_0^1 \rho^{2+\tilde{\alpha}_n}v^{2^n-2}\Big(\mathrm{dcp}\,\tfrac{1}{2}(1-w) + \rho a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta-2}\big|w_{|z}\big|^2\Big)v_{|z}dzd\tau$$
$$+ 2^n\tilde{\alpha}_n\int_0^t\int_0^1 \rho^{\tilde{\alpha}_n+1}\rho_{|z}\Big(\mathrm{dcp}\,\tfrac{1}{2}(1-w) + \rho a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta-2}\big|w_{|z}\big|^2\Big)v^{2^n-1}$$
$$=: \sum_{\kappa=1}^{5} I_{n,\kappa}.$$

Now, we estimate each summand

- $I_{n,1} = \int_0^1 \rho_0^{\tilde{\alpha}_n} v_0^{2^n} dz \le C\,,$
- $$\begin{aligned} I_{n,2} &= -\tilde{\alpha}_n \int_0^t\int_0^1 \rho^{\tilde{\alpha}_n+1} v^{2^n} v_{|z} dz d\tau \\ &\le C \int_0^t\int_0^1 v^{2^{n+1}} dz d\tau + C \int_0^t\int_0^1 \rho^{2\tilde{\alpha}_n+2} \big|v_{|z}\big|^2 dz d\tau \\ &\le C\,, \qquad \text{iff} \quad 2\tilde{\alpha}_n + 2 = 1+\theta\,, \end{aligned}$$
- $$\begin{aligned} I_{n,3} &= -2^n\tilde{\alpha}_n \int_0^t\int_0^1 \rho^{\theta+\tilde{\alpha}_n} v^{2^n-1} \rho_{|z} v_{|z} dz d\tau \\ &\le C \int_0^t\int_0^1 \rho^{2\theta-2}\big|\rho_{|z}\big|^2 dz d\tau + C \int_0^t\int_0^1 \rho^{2\tilde{\alpha}_n+2} v^{2^{n+1}-2} \big|v_{|z}\big|^2 dz d\tau \end{aligned}$$

  It holds $I_{n,3} \le C$, iff $2\tilde{\alpha}_n + 2 = 1+\theta$. Then
  $$\tilde{\alpha}_n = \frac{\theta-1}{2}\,.$$
- $$\begin{aligned} I_{n,4} &= 2^n(2^n-1) \int_0^t\int_0^1 \rho^{2+\tilde{\alpha}_n} v^{2^n-2} \big(\,\mathrm{dcp}\,\tfrac{1}{2}(1-w) \\ &\qquad + \rho a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta-2}\big|w_{|z}\big|^2\big) v_{|z} dz d\tau \\ &\le C \int_0^t\int_0^1 v^{2^{n+1}-4} dz d\tau + C \int_0^t\int_0^1 \rho^{4+2\tilde{\alpha}_n}\big|v_{|z}\big|^2 dz d\tau \\ &\quad + C \int_0^t\int_0^1 \rho^{2\gamma+2\beta+2\tilde{\alpha}_n}\big|w_{|z}\big|^4\big|v_{|z}\big|^2 dz d\tau\,. \end{aligned}$$

It holds $I_{n,4} \le C$, because of $\frac{\theta-1}{2} \le 2\gamma + 2\beta + 2\tilde{\alpha}_n - (1+\theta)$ (Theorem 10) and Lemma 13.

- $$\begin{aligned} I_{n,5} &= 2^n\tilde{\alpha}_n \int_0^t\int_0^1 \rho^{\tilde{\alpha}_n+1}\rho_{|z} \big(\,\mathrm{dcp}\,\tfrac{1}{2}(1-w) \\ &\qquad + \rho a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta-2}\big|w_{|z}\big|^2\big) v^{2^n-1} dz d\tau \\ &\le C \int_0^t\int_0^1 v^{2^{n+1}-2} dz d\tau + C \int_0^t\int_0^1 \rho^{2\tilde{\alpha}_n+2}\big|\rho_{|z}\big|^2 dz d\tau \\ &\quad + C \int_0^t\int_0^1 \rho^{2\gamma+2\beta+2\tilde{\alpha}_n-2}\big|w_{|z}\big|^4\big|\rho_{|z}\big|^2 dz d\tau\,. \end{aligned}$$

For the last summand we have $\rho^{2\beta-2}\big|w_{|z}\big|^4 \cdot \rho^{2\theta-2}\big|\rho_{|z}\big|^2$ and with Lemma 13

$\rho^{2\beta-2}\big|w_{|z}\big|^4 \leq C$. Therefore $I_{n,5} \leq C$. From that it follows

$$\int_0^1 \rho^{\tilde{\alpha}_n} v^{2^n} dz + 2^n(2^n-1)\int_0^t\!\!\int_0^1 \rho^{1+\theta+\tilde{\alpha}_n} v^{2^n-2}\big|v_{|z}\big|^2 dz d\tau \leq C(T)\,.$$

Now, we do the same procedure for $n-1$:

$$\begin{aligned}
&\int_0^1 \rho^{\tilde{\alpha}_{n-1}} v^{2^{n-1}} dz + 2^{n-1}(2^{n-1}-1)\int_0^t\!\!\int_0^1 \rho^{1+\theta+\tilde{\alpha}_{n-1}} v^{2^{n-1}-2}\big|v_{|z}\big|^2 dz d\tau \\
&\quad= \int_0^1 \rho_0^{\tilde{\alpha}_{n-1}} v_0^{2^{n-1}} dz - \tilde{\alpha}_{n-1}\int_0^t\!\!\int_0^1 \rho^{\tilde{\alpha}_{n-1}+1} v^{2^{n-1}} v_{|z} dz d\tau \\
&\qquad - 2^{n-1}\tilde{\alpha}_{n-1}\int_0^t\!\!\int_0^1 \rho^{\theta+\tilde{\alpha}_{n-1}} v^{2^{n-1}-1}\rho_{|z} v_{|z} dz d\tau \\
&\qquad + 2^{n-1}(2^{n-1}-1)\int_0^t\!\!\int_0^1 \rho^{2+\tilde{\alpha}_{n-1}} v^{2^{n-1}-2}\Big(\operatorname{dcp}\tfrac{1}{2}(1-w) \\
&\qquad\qquad + \rho a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta-2}\big|w_{|z}\big|^2\Big) v_{|z} dz d\tau \\
&\qquad + 2^{n-1}\tilde{\alpha}_{n-1}\int_0^t\!\!\int_0^1 \rho^{\tilde{\alpha}_{n-1}+1}\rho_{|z}\Big(\operatorname{dcp}\tfrac{1}{2}(1-w) \\
&\qquad\qquad + \rho a + \frac{\tilde{\delta}_1^2}{8}\gamma\,\rho^{\gamma+\beta-2}\big|w_{|z}\big|^2\Big) v^{2^{n-1}-1} dz d\tau \\
&\quad =: \sum_{\kappa=1}^{5} I_{n-1,\kappa}.
\end{aligned}$$

For $I_{n-1,3}$, it is satisfied:

- $$\begin{aligned}
I_{n-1,3} &= -2^{n-1}\tilde{\alpha}_{n-1}\int_0^t\!\!\int_0^1 \rho^{\theta+\tilde{\alpha}_{n-1}} v^{2^{n-1}-1}\rho_{|z} v_{|z} dz d\tau \\
&\leq C\int_0^t\!\!\int_0^1 \rho^{1+\theta+\tilde{\alpha}_n} v^{2^n-2}\big|v_{|z}\big|^2 dz d\tau \\
&\quad + C\int_0^t\!\!\int_0^1 \rho^{\theta-1+2\tilde{\alpha}_{n-1}-\tilde{\alpha}_n}\big|\rho_{|z}\big|^2 dz d\tau.
\end{aligned}$$

It has to be

$$\theta - 1 + 2\tilde{\alpha}_{n-1} - \tilde{\alpha}_n = 2\theta - 2,$$

i.e.,

$$\tilde{\alpha}_{n-1} = \frac{\tilde{\alpha}_n}{2} + \frac{\theta-1}{2} \qquad \text{for } n \geq 2.$$

With that we have

$$I_{n-1,\kappa} \le C(T), \quad \kappa = 1, \ldots, 5\,.$$

Finally, we have

$$\int_0^1 \rho^{\tilde{\alpha}_{n-1}} v^{2^{n-1}} dz + 2^{n-1}(2^{n-1}-1)\int_0^t\int_0^1 \rho^{1+\theta+\tilde{\alpha}_{n-1}} v^{2^{n-1}-2}\left|v_{|z}\right|^2 dz d\tau \le C(T)\,.$$

By going back with $\tilde{\alpha}_n$ from fixed n^* to 1, we have

$$\int_0^1 \rho^{\tilde{\alpha}_1} v^2 dz + C\int_0^t\int_0^1 \rho^{1+\theta+\tilde{\alpha}_1}\left|v_{|z}\right|^2 dz d\tau \le C(T)\,.$$

We denote $\alpha_{n^*} := \tilde{\alpha}_1$ and we have the assertion. □

Lemma 19. *For* $\beta_1 := (2-\frac{1}{2^n})(\theta-1) < 0$ *it is valid*

$$\int_0^1 \rho^{\beta_1} dz \le C(T)\,.$$

Proof. Can be shown with equation (21), the Gronwall Lemma and Lemma 18. □

Now, we can conclude

Theorem 20. *There exists* $\beta_2 < 0$*, so that*

$$\rho^{\beta_2} \;\le\; C(T)\,.$$

Lemma 21. *Under the assumption* (*A*.1)-(*A*.4)*, it holds*

$$\int_0^1 \left|v_{|t}\right|^2(t,.)dz + \int_0^t\int_0^1 \rho^{\theta+1}\left|v_{|zt}\right|^2 dz d\tau \;\le\; C(T)\,, \tag{40}$$

$$\left|\left(\rho^{\theta+1}v_{|z}\right)(t,.)\right| \;\le\; C(T)\,. \tag{41}$$

Proof. Differentiate equation (22) by t, multiply it by $v_{|t}$ and integrate from 0 to 1:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left|v_{|t}\right|^2 dz\right)\\
&+\int_0^1 \left(\rho^2 \mathrm{dcp}\,\tfrac{1}{2}(1-w) + \rho^3 a + \frac{\tilde{\delta}_1^2}{8}\gamma\rho^{\gamma+\beta}\left|w_{|z}\right|^2\right)_{|zt} v_{|t}\,dz\\
&=\int_0^1 \left(\rho^{\theta+1}v_{|z}\right)_{|zt} v_{|t} dz = -\int_0^1 \left(\rho^{\theta+1}v_{|z}\right)_{|t} v_{|zt} dz + \left[\left(\rho^{\theta+1}v_{|z}\right)_{|t} v_{|t}\right](t,1)\\
&= -\int_0^1 (\theta+1)\rho^{\theta}\rho_{|t}v_{|z}v_{|zt}dz - \int_0^1 \rho^{\theta+1}\left|v_{|zt}\right|^2 dz + \left[\left(\rho^{\theta+1}v_{|z}\right)_{|t} v_{|t}\right](t,1).
\end{aligned}$$

Further, we have

$$\int_0^1 \Big(\rho^2 \mathrm{dcp}\,\tfrac12(1-w)+\rho^3 a+\frac{\tilde\delta_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\Big)_{|zt}\,v_{|t}\,dz$$
$$= -\int_0^1 \big(\rho^2 \mathrm{dcp}\,\tfrac12(1-w)+\rho^3 a+\frac{\tilde\delta_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\big)_{|t} v_{|tz} dz$$
$$+\Big[\big(\rho^2 \mathrm{dcp}\,\tfrac12(1-w)+\rho^3 a+\frac{\tilde\delta_1^2}{8}\gamma\rho^{\gamma+\beta}\big|w_{|z}\big|^2\big)_{|t} v_{|t}\Big](t,1)\,.$$

Both together and using the boundary condition (25), we have

$$\frac12\frac{d}{dt}\Big(\int_0^1 \big|v_{|t}\big|^2 dz\Big)+\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz$$
$$= +\int_0^1 \Big\{2\rho\rho_{|t}\mathrm{dcp}\,\tfrac12(1-w)-\rho^2\mathrm{dcp}\,\tfrac12 w_{|t}+3\rho^2\rho_{|t}a$$
$$-\frac{\tilde\delta_1^2}{8}\gamma(\gamma+\beta)\rho^{\gamma+\beta+1}v_{|z}\big|w_{|z}\big|^2+\frac{\tilde\delta_1^2}{8}\gamma\rho^{\gamma+\beta}2w_{|z}w_{|zt}\Big\}v_{|zt}\;dz$$
$$+\int_0^1 (\theta+1)\rho^{\theta+2}\big|v_{|z}\big|^2 v_{|zt}\;dz\quad;$$

by Young's inequality, we have

$$\le\;\frac{\varepsilon}{2}\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz+\frac{1}{2\varepsilon}\int_0^1 \rho^{5-\theta}\big(\,\mathrm{dcp}\,\tfrac12(1-w)+3\rho\,a\big)^2_{\le C},$$

$(\,\text{Lemma 9 and Theorem 8}\,)\big|v_{|z}\big|^2 dz$

$$+\frac{\varepsilon}{2}\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz+\frac{1}{2\varepsilon}\int_0^1 \tfrac14\mathrm{dcp}^2\rho^{3-\theta}\big|w_{|t}\big|^2 dz$$
$$+\frac{\varepsilon}{2}\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz+\frac{C}{2\varepsilon}\int_0^1 \rho^{2\gamma+2\beta+1-\theta}\big|w_{|z}\big|^4\big|v_{|z}\big|^2 dz$$
$$+\frac{\varepsilon}{2}\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz+\frac{C}{2\varepsilon}\int_0^1 \rho^{2\gamma+2\beta-1-\theta}\big|w_{|z}\big|^2\big|w_{zt}\big|^2 dz$$
$$+\frac{\varepsilon}{2}\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz+\frac{1}{2\varepsilon}\int_0^1 (\theta+1)^2\rho^{3+\theta}\big|v_{|z}\big|^4 dz\,.$$

From equations (32), (33) and (24), (25) we can conclude

$$\rho^{2+2\theta}\big|v_{|z}\big|^2\;\le\;C\Big(\int_0^1 \big|v_{|t}\big|^2 dz+C\Big),$$
$$\rho^{5-\theta}\;\le\;C\rho^{1+\theta}\qquad\qquad(\text{Theorem 10})\;.$$

From above, by choosing ε small, we apply Lemma 12 and take into account, that

$$\frac{1}{C(T)} \;\le\; \rho \;\le\; C \quad \text{and} \quad -1 \;\le\; w \;\le\; 1\,.$$

Then we derive

$$\begin{aligned}\frac{d}{dt}\Big(\int_0^1 \big|v_{|t}\big|^2 dz\Big) + C\int_0^1 \rho^{\theta+1}\big|v_{|zt}\big|^2 dz \\ \le\; C\Big(\int_0^1 \rho^{\theta+1}\big|v_{|z}\big|^2 dz\Big)\Big(C\int_0^1 \big|v_{|t}\big|^2 dz + C\Big).\end{aligned}$$

Due to Theorem 8, the first factor is in $L^1(0,T)$. Then with the Gronwall Lemma we get the assertion. □

5. Construction of the Weak Solution

Now we try to construct a weak solution. Our handling goes in an usual way. Lets N an arbitrary integer and $\triangle := 1/N$. We discretize the equation in z. This leads to a system of ordinary differential equations:

$$\frac{d}{dt}\rho_{n-1} + \rho_{n-1}^2\frac{v_n - v_{n-1}}{\triangle} = 0\,, \tag{42}$$

$$\frac{d}{dt}v_n + \frac{p_n - p_{n-1}}{\triangle} = \Big[\mu_n\rho_n\frac{v_{n+1}-v_n}{\triangle} - \mu_{n-1}\rho_{n-1}\frac{v_n - v_{n-1}}{\triangle}\Big]\frac{1}{\triangle}\,, \tag{43}$$

$$\begin{aligned}\eta\frac{d}{dt}w_n &= -g_N(w_n) + 2c_2 w_n + 2\rho_{n-1}\mathrm{dcp} \\ &\quad + \tilde{\delta}_1^2\Big[\rho_n^{\gamma}\frac{w_{n+1}-w_n}{\triangle} - \rho_{n-1}^{\gamma}\frac{w_n - w_{n-1}}{\triangle}\Big]\frac{1}{\triangle} \\ &\quad + \frac{\tilde{\delta}_2^4}{4}\Big[\Big|\frac{w_{n+1}-w_n}{\triangle}\Big|^2\frac{w_{n+1}-w_n}{\triangle} - \Big|\frac{w_n - w_{n-1}}{\triangle}\Big|^2\frac{w_n - w_{n-1}}{\triangle}\Big]\frac{1}{\triangle}.\end{aligned} \tag{44}$$

Here

$$p_{n-1} = \rho_{n-1}^2\mathrm{dcp}\tfrac{1}{2}(1-w_n) + \rho_{n-1}^3 a + \frac{\tilde{\delta}_1^2}{8}\gamma\rho_{n-1}^{\gamma+\beta}\Big|\frac{w_n - w_{n-1}}{\triangle}\Big|^2\,.$$

The boundary conditions are

$$v_0(t) = 0 \qquad \text{and} \qquad \Big(p_N - \mu_N\rho_N\frac{v_{N+1}-v_N}{\triangle}\Big)(t) = 0\,, \tag{45}$$

$$\frac{w_1 - w_0}{\triangle}(t) = 0 \qquad \text{and} \qquad \frac{w_{N+1}-w_N}{\triangle}(t) = 0\,, \tag{46}$$

and the initial conditions are

$$\rho_{n-1}(0) = \rho_0((n-1)\triangle) \geq 0\,, \tag{47}$$

$$v_n(0) = v_0(n\triangle)\,, \tag{48}$$

$$w_n(0) = 1 - 2\Phi_0(n\triangle)\,. \tag{49}$$

By the theory of the ordinary differential equations, the Cauchy problem (42)-(49) admits a temporarily local solution in the domain $\{(\rho_{n-1}, v_n, w_n)_{n=1,\ldots,N}\} \in \mathbb{R}^{3N}$ in the class of regularity

$$\left[C^0([0,T_\infty)) \times C^0([0,T_\infty)) \times C^0([0,T_\infty))\right] \cap \left[C^1((0,T_\infty)) \times C^1((0,T_\infty)) \times C^1((0,T_\infty))\right],$$

where $[0, T_\infty)$ be the right maximal interval of existence of this solution. By the equation (42) and the initial condition (A.1), we see $\rho_{n-1}(t) > 0$ for $0 < t < T_\infty$.

Theorem 22. *The first energy inequality (Theorem 8) holds in the discrete sense for each chosen $N \in \mathbb{N}$. Further the same is valid for Lemma 9, Theorem 10 and Theorem 20.*

Now, we want to show, that the solution w_n of that N-problem exists for $0 \leq t < T_\infty = +\infty$.

Theorem 23. *We have:*

(i) $\sum_{n=1}^{N} |\rho_n(t) - \rho_{n-1}(t)| \leq C(T)\,,$

(ii) $\sum_{n=1}^{N} |v_n(t) - v_{n-1}(t)| \leq C(T)\,,$

(iii) $\sum_{n=1}^{N} |w_n(t) - w_{n-1}(t)| \leq C\,,$

(iv) $\sum_{n=1}^{N} \left|\frac{w_{n+1}-w_n}{\triangle}(t) - \frac{w_n - w_{n-1}}{\triangle}(t)\right| \leq C(T)\,,$

(v) $\sum_{n=1}^{N} \left|\rho_n^{\gamma+\beta}\left|\frac{w_{n+1}-w_n}{\triangle}\right|^2(t) - \rho_{n-1}^{\gamma+\beta}\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2(t)\right| \leq C(T)\,,$

(vi) $\sum_{n=1}^{N} \left|\rho_n^{\theta+1}\frac{v_{n+1}-v_n}{\triangle}(t) - \rho_{n-1}^{\theta+1}\frac{v_n-v_{n-1}}{\triangle}(t)\right| \leq C(T)\,,$

(vii) $\sum_{n=1}^{N} \left|\rho_n^{\gamma}\frac{w_{n+1}-w_n}{\triangle}(t) - \rho_{n-1}^{\gamma}\frac{w_n-w_{n-1}}{\triangle}(t)\right| \leq C(T)\,,$

(viii) $\sum_{n=1}^{N} \left|\left|\frac{w_{n+1}-w_n}{\triangle}\right|^2\frac{w_{n+1}-w_n}{\triangle}(t) - \left|\frac{w_n-w_{n-1}}{\triangle}\right|^2\frac{w_n-w_{n-1}}{\triangle}(t)\right| \leq C(T)\,.$

Proof. (i) is valid, because of Theorem 10 and Lemma 16. (ii) is valid, due to equation (41) and Theorem 20. Further, we prove (iii): With Theorem 8, we get

$$\sum_{n=1}^{N}\Big|\frac{w_n - w_{n-1}}{\triangle}\Big|\triangle \;\le\; C\Big[\sum_{n=1}^{N}\Big|\frac{w_n - w_{n-1}}{\triangle}\Big|^4\triangle\Big]^{1/4}\Big[\sum_{n=1}^{N}\triangle\Big]^{4/3} \;\le\; C.$$

Now, we show (iv):

$$\begin{aligned}
&\sum_{n=1}^{N}\Big|\frac{w_{n+1}-w_n}{\triangle}(t) - \frac{w_n - w_{n-1}}{\triangle}(t)\Big| \\
&\le \sum_{n=1}^{N}\Big|\frac{dv_n}{dt}\Big|\triangle + \sum_{n=1}^{N}\big|g_N(w_n)\big|\triangle + \sum_{n=1}^{N}2c_2\big|w_n\big|\triangle + \sum_{n=1}^{N}2\,\mathrm{dcp}\,\big|\rho_{n-1}\big|\triangle \\
&\le \Big[\sum_{n=1}^{N}\Big|\frac{dw_n}{dt}\Big|^2\triangle\Big]^{1/2}\Big[\sum_{n=1}^{N}\triangle\Big]^{1/2} \\
&\qquad + \Big[\sum_{n=1}^{N}\big|g_N(w_n)\big|^2\triangle\Big]^{1/2}\Big[\sum_{n=1}^{N}\triangle\Big]^{1/2} + C \\
&\le C(T)\,, \qquad\qquad \text{because of Lemma 12}\,.
\end{aligned}$$

(v) can be mainly shown, because of Lemma 14 and Lemma 16. We show (vi): From equation (43), we have

$$\begin{aligned}
&\sum_{n=1}^{N}\Big|\rho_n^{\theta+1}\frac{v_{n+1}-v_n}{\triangle}(t) - \rho_{n-1}^{\theta+1}\frac{v_n - v_{n-1}}{\triangle}(t)\Big| \\
&\le \sum_{n=1}^{N}\Big|\frac{dv_n}{dt}\Big|\triangle \\
&\quad + \sum_{n=1}^{N}\Bigg|\frac{\rho_n^2(\mathrm{dcp}\,\frac{1}{2}(1-w_{n+1}) + \rho_n a) + \frac{\tilde{\delta}_1^2}{8}\gamma\rho_n^{\gamma+\beta}\Big|\frac{w_{n+1}-w_n}{\triangle}\Big|^2}{\triangle} \\
&\qquad\quad \frac{-\rho_{n-1}^2(\mathrm{dcp}\,\frac{1}{2}(1-w_n) + \rho_{n-1}a) - \frac{\tilde{\delta}_1^2}{8}\gamma\rho_{n-1}^{\gamma+\beta}\Big|\frac{w_n - w_{n-1}}{\triangle}\Big|^2}{}\Bigg|\triangle \\
&\le C(T)\,, \qquad \text{mainly because of } (v) \text{ and Lemma 21}\,.
\end{aligned}$$

Items (vii) and $(viii)$ go in the same way. □

Theorem 24. *We have:*

(i) $\sum_{n=1}^{N}|\rho_{n-1}(t) - \rho_{n-1}(s)|^2\triangle \;\le\; C(T)\,|t-s|^2\,,$

(ii) $\sum_{n=1}^{N}|v_n(t) - v_n(s)|^2\triangle \;\le\; C(T)\,|t-s|^2\,,$

(iii) $\sum_{n=1}^{N}\left|w_n(t)-w_n(s)\right|^2\triangle \;\le\; C(T)\,|t-s|^2\,,$

(iv) $\sum_{n=1}^{N}\left|\rho_{n-1}^{\theta+1}\frac{v_n-v_{n-1}}{\triangle}(t)-\rho_{n-1}^{\theta+1}\frac{v_n-v_{n-1}}{\triangle}(s)\right|^2\triangle \;\le\; C(T)\,|t-s|\,,$

(v) $\sum_{n=1}^{N}\left|\frac{w_n-w_{n-1}}{\triangle}(t)-\frac{w_n-w_{n-1}}{\triangle}(s)\right|^2\triangle \;\le\; C(T)|t-s|\,,$

(vi) $\sum_{n=1}^{N}\left|\rho_{n-1}^{\gamma+\beta}\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2(t)-\rho_{n-1}^{\gamma+\beta}\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2(s)\right|^2\triangle$
$\le\; C(T)\,|t-s|\,,$

(vii) $\sum_{n=1}^{N}\left|\rho_{n-1}^{\gamma}\frac{w_n-w_{n-1}}{\triangle}(t)-\rho_{n-1}^{\gamma}\frac{w_n-w_{n-1}}{\triangle}(s)\right|^2\triangle \;\le\; C(T)\,|t-s|\,,$

(viii) $\sum_{n=1}^{N}\left|\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2\frac{w_n-w_{n-1}}{\triangle}(t)-\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2\frac{w_n-w_{n-1}}{\triangle}(s)\right|^2\triangle$
$\le\; C(T)\,|t-s|\,.$

Proof. We exemplary show (iii): From Lemma 12 follows that

$$\sum_{n=1}^{N}\left|\frac{d}{dt}w_n\right|^2\triangle \;\le\; C(T),$$

which implies (iii). We show (v):

$$\begin{aligned}\sum_{n=1}^{N}&\left|\frac{w_n-w_{n-1}}{\triangle}(t)-\frac{w_n-w_{n-1}}{\triangle}(s)\right|^2\triangle\\ &\le\; \int_0^T\sum_{n=1}^{N}\left|\frac{d}{dt}\Big(\frac{w_n-w_{n-1}}{\triangle}\Big)\right|^2\triangle d\tau\,|t-s|\\ &\le\; C(T)\,|t-s| \qquad \text{(Lemma 12, equation (35))}\,.\end{aligned}$$

Further $(viii)$:

$$\begin{aligned}&\sum_{n=1}^{N}\left|\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2\frac{w_n-w_{n-1}}{\triangle}(t)-\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2\frac{w_n-w_{n-1}}{\triangle}(s)\right|^2\triangle\\ &=\sum_{n=1}^{N}\left|\int_s^t\frac{d}{d\tau}\Big(\left|\frac{w_n-w_{n-1}}{\triangle}\right|^2\frac{w_n-w_{n-1}}{\triangle}\Big)(\tau)d\tau\right|^2\triangle\\ &\le\; \int_0^T\sum_{n=1}^{N}3\left|\frac{w_n-w_{n-1}}{\triangle}\right|^4\left|\frac{d}{d\tau}\Big(\frac{w_n-w_{n-1}}{\triangle}\Big)\right|^2\triangle d\tau\,|t-s|\\ &\le\; C(T)\int_0^T\sum_{n=1}^{N}\left|\frac{d}{d\tau}\Big(\frac{w_n-w_{n-1}}{\triangle}\Big)\right|^2\triangle d\tau\,|t-s|\end{aligned}$$

(Lemma 13, Theorem 20)

$$\leq C(T)|t-s|\,. \qquad \square$$

Now, we interpolate ρ, v and w by ρ_{n-1}, v_n and w_n in the following way. Define $\rho_\triangle(t,z)$, $v_\triangle(t,z)$ and $w_\triangle(t,z)$ by

$$\rho_\triangle(t,z) = \rho_{n-1}(t)\,,$$
$$v_\triangle(t,z) = \frac{1}{\triangle}\big[(z-(n-1)\triangle)v_n(t) + (n\triangle - z)v_{n-1}(t)\big]\,,$$
$$w_\triangle(t,z) = \frac{1}{\triangle}\big[(z-(n-1)\triangle)w_n(t) + (n\triangle - z)w_{n-1}(t)\big]\,,$$

for $(n-1)\triangle \leq z < n\triangle$. Further

$$v_{\triangle|z}(t,z) = \frac{1}{\triangle}\big(v_n(t) - v_{n-1}(t)\big)\,,$$
$$w_{\triangle|z}(t,z) = \frac{1}{\triangle}\big(w_n(t) - w_{n-1}(t)\big)$$

for $(n-1)\triangle \leq z < n\triangle$ and any $t \in [0,T]$.

We define

$$v^R_\triangle(t,z) = v_n(t)\,, \qquad v^L_\triangle(t,z) = v_{n-1}(t) \qquad \text{for } (n-1)\triangle \leq z < n\triangle\,,$$
$$w^R_\triangle(t,z) = w_n(t)\,, \qquad w^L_\triangle(t,z) = w_{n-1}(t) \qquad \text{for } (n-1)\triangle \leq z < n\triangle\,,$$

and, further, we define

$$g_\triangle(x) = g_{[1/\triangle]}(x).$$

6. Convergence from Approximate Solutions to a Global Weak Solution

Lemma 25. *The approximate solution $(\rho_\triangle, v_\triangle, w_\triangle)$ constructed above satisfy:*

(*i*) $1/C(T) \leq \rho_\triangle(t,z) \leq C(T)\,,$ *for all* (t,z);

(*ii*) $|v_\triangle| \leq C(T)\,,$ *for all* (t,z);

(*iii*) $-1 \leq w_\triangle \leq 1\,,$ *for all* (t,z);

(*iv*) $\big|v_{\triangle|z}(t,z)\big| \leq C(T)\,,$ *for all* (t,z);

(*v*) $\big|(\mu_\triangle \rho_\triangle v_{\triangle|z})(t,z)\big| \leq C(T)\,,$ *for all* (t,z);

(*vi*) $\big|w_{\triangle|z}(t,z)\big| \leq C(T)\,,$ *for all* (t,z);

(vi) $\left|(\rho^\gamma w_{\triangle|z})(t,z)\right| \leq C(T)\,,$ *for all* $(t,z)\,.$ □

Lemma 26. *There is a sequence* $\triangle = \triangle_\ell \to 0$ *such that:*

(i) $\rho_\triangle \to \rho$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,;$

(ii) $v_\triangle \to v$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,;$

(iii) $w_\triangle \to w$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,;$

(iv) $w^R_\triangle \to w$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,;$

(v) $\mu_\triangle \rho_\triangle v_{\triangle|z} \to \mu\rho v_{|z}$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,;$

(vi) $\rho^\gamma_\triangle w_{\triangle|z} \to \rho^\gamma w_{|z}$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,;$

(vii) $\rho^{\gamma+\beta}_\triangle \left|w_{\triangle|z}\right|^2 \to \rho^{\gamma+\beta}\left|w_{|z}\right|^2$ *almost everywhere on* $\{(t,z) : 0 < z < 1,\, t > 0\}\,.$

Proof. We know from Theorem 23 that the function $\rho_\triangle$ has uniformly bounded total variation with respect to $\triangle$ for any fixed t. Let $\{t_m\}$, $(m = 1,2,\ldots)$ in $[0,T]$ a countable dense set. By Helly's Theorem and a diagonal process on the family $\{(\rho_\triangle, v_\triangle, w_\triangle)\}$, we can select a subsequence, which converges bounded and pointwise, almost everywhere in $z \in [0,1]$ for each t_m. Hereby, $(\rho, v, w) \in L^\infty((0,1))$ for each t_m. By Lebesgue's Theorem, the subsequence also converges in the L^2-norm on $t = t_m$. With usage of Lemma 24 one can see that $(\rho_\triangle, v_\triangle, w_\triangle)$ converge in $L^2([0,1])$ uniformly in $t \in [0,T]$. Then we can choose a subsequence tends almost everywhere in $(t,z) \in [0,T] \times [0,1]$. □

Now, we show, that the limit functions are weak solutions. At first, let us show, that the weak formulation of the second equation holds. Let $\varphi \in C_0^\infty((0,1])$ be an arbitrary test function. Multiplying equation (43) with $\varphi_n(t)\triangle = \varphi(t, n\triangle)\triangle$ and summing up from $n = 1$ to $n = N$, we get

$$\begin{aligned} 0 \; &= \sum_{n=1}^N \varphi_n \frac{d}{dt} v_n \triangle \\ &\quad + \sum_{n=1}^N \varphi_n \frac{1}{\triangle}\Bigg[(p_n - p_{n-1}) - \Big(\mu_n \rho_n \frac{v_{n+1} - v_n}{\triangle} \\ &\qquad\qquad\qquad - \mu_{n-1}\rho_{n-1}\frac{v_n - v_{n-1}}{\triangle}\Big)\Bigg]\triangle\,. \end{aligned}$$

Since φ is smooth and vanishes at $z=0$, we have

$$\sum_{n=1}^{N} \varphi_n \frac{d}{dt} v_n \triangle \;=\; \int_0^1 \varphi\, v^R_{\triangle|t} dz + o(1)\,,$$

$$\sum_{n=1}^{N} \varphi_n \frac{1}{\triangle}\Bigg[(p_n - p_{n-1}) - \Bigg(\mu_n \rho_n \frac{v_{n+1}-v_n}{\triangle}$$

$$- \mu_{n-1}\rho_{n-1}\frac{v_n - v_{n-1}}{\triangle}\Bigg)\Bigg]\triangle = \; -\int_0^1 \varphi_{|z}(p_\triangle - \mu_\triangle \rho_\triangle v_{\triangle|z}) dz + o(1)\,,$$

as $\triangle \to 0$. Therefore, to the limit and because of Lebesgue dominated convergence, we see that equation (19) holds for the limit function. Further, it is easy to see, that equation (18) holds.

Now, let us show, that equation (20) is valid. From Lemma 12 follows

$$\int_0^1 \left|g_\triangle(w^R_\triangle(t,z))\right|^2 dz \;=\; \sum_{n=1}^{N} \left|g_N(w_n)\right|^2 (t)\triangle \;\le\; C(T)\,.$$

Then, there exists g such that

$$g_\triangle(w^R_\triangle(t,z)) \longrightarrow g \qquad \text{in } L^\infty(0,T;L^2(0,1)) \quad \text{weak}^* \text{ for } \triangle_j \to 0\,.$$

We choose $\{t_k\}$ dense in $(0,T)$. For arbitrary small $\epsilon \in (0,1)$ and for every $t \in (0,T)$, we denote

$$\mathcal{N}^\epsilon_\triangle(t) := \{z \in (0,1) : |w^R_\triangle(t,z)| > 1-\epsilon\} \qquad \text{and}$$
$$\mathcal{N}^\epsilon(t) := \{z \in (0,1) : |w(t,z)| > 1-\epsilon\}\,.$$

With inequality (37), we have

$$C \ge \left[\int_{\mathcal{N}^\epsilon_\triangle(t)} g^2_\triangle(w^R_\triangle(t,.))\right]^{\frac{1}{2}}$$
$$\ge |\mathcal{N}^\epsilon_\triangle(t)|^{\frac{1}{2}} \left[\inf_{z\in\mathcal{N}^\epsilon_\triangle(t)} \left(4c_1 \sum_{k=0}^{N} \frac{(w^R_\triangle(t,z))^{2k+1}}{2k+1}\right)^2\right]^{\frac{1}{2}}$$
$$\ge 2c_1\, |\mathcal{N}^\epsilon_\triangle(t)|^{\frac{1}{2}} 2 \sum_{k=0}^{N} \frac{(1-\epsilon)^{2k+1}}{2k+1} = 2c_1\, |\mathcal{N}^\epsilon_\triangle(t)|^{\frac{1}{2}}\; \ln\left(\frac{2-\epsilon}{\epsilon}\right)$$

and from that

$$|\mathcal{N}^\epsilon_\triangle(t)|^{\frac{1}{2}} \le \frac{C}{2c_1\, \ln\left(\frac{2-\epsilon}{\epsilon}\right)} \qquad \text{and} \qquad \frac{C}{2c_1\, \ln\left(\frac{2-\epsilon}{\epsilon}\right)} \overset{\epsilon\to 0}{\longrightarrow} 0\,.$$

Further with Fatou's Lemma, we conclude

$$\begin{aligned} 0 \;\leq\; |\mathcal{N}^\epsilon(t)|^{\frac{1}{2}} \;&=\; \int_0^1 \chi_{\mathcal{N}^\epsilon(t)}(z)dz \\ &\leq\; \int_0^1 \liminf_{\triangle\to 0} \chi_{\mathcal{N}^\epsilon_\triangle(t)}(z)dz \;\leq\; \liminf_{\triangle\to 0} \left|\mathcal{N}^\epsilon_\triangle(t)\right| \,. \end{aligned}$$

As $\epsilon \to 0$, it follows

$$|\{z \in (0,1) : |w(t,z)| \geq 1\}| = 0\,.$$

Now, we have to prove that $g = 2c_1 \ln\left(\frac{1+w}{1-w}\right)$, a.e. in $(0,T)\times(0,1)$. We regard

$$\begin{aligned} \Big|g_\triangle(w^R_\triangle)(t,z) - 2c_1 \ln\big(\frac{1+w^R_\triangle}{1-w^R_\triangle}\big)(t,z)\Big| &\leq 4c_1 \sum_{k=N+1}^{\infty} \frac{|w^R_\triangle(t,z)|^{2k+1}}{2k+1} \\ &\leq 4c_1 \frac{1}{2N+3} \sum_{k=N+1}^{\infty} (1-\epsilon)^{2k+1} \\ &\leq 4c_1 \frac{1}{2N+3}\frac{1}{\epsilon} \quad \xrightarrow{N\to\infty} \quad 0\,. \end{aligned}$$

Therefore

$$\begin{aligned} &\Big|g_\triangle(w^R_\triangle) - 2c_1 \ln\big(\frac{1+w}{1-w}\big)\Big| \\ &\quad \leq \Big|g_\triangle(w^R_\triangle) - 2c_1 \ln\big(\frac{1+w^R_\triangle}{1-w^R_\triangle}\big)\Big| + \Big|2c_1 \ln\big(\frac{1+w^R_\triangle}{1-w^R_\triangle}\big) - 2c_1 \ln\big(\frac{1+w}{1-w}\big)\Big| \\ &\quad \xrightarrow{\triangle\to 0} \quad 0\,. \end{aligned}$$

With the theorem of Lebesgue

$$\int_0^1 g_\triangle(w^R_\triangle)dz \xrightarrow{\triangle\to 0} \int_0^1 2c_1 \ln\left(\tfrac{1+w}{1-w}\right)dz\,.$$

Because of the uniqueness of above convergence, it holds $g = 2c_1 \ln\left(\frac{1+w}{1-w}\right)$. □

We could further regard

$$\sum_{n=1}^{N} \varphi_n \frac{d}{dt} w_n \triangle = \int_0^1 \varphi w^R_{\triangle|t} dz + o(1)\,,$$

$$\begin{aligned} &\sum_{n=1}^{N} \varphi_n \Big\{ \tilde{\delta}_1^2 \frac{1}{\triangle}\Big[\rho_n^\gamma \frac{w_{n+1}-w_n}{\triangle} - \rho_{n-1}^\gamma \frac{w_n - w_{n-1}}{\triangle}\Big] \\ &+ \frac{\tilde{\delta}_2^4}{4}\frac{1}{\triangle}\Big[\Big|\frac{w_{n+1}-w_n}{\triangle}\Big|^2 \frac{w_{n+1}-w_n}{\triangle} - \Big|\frac{w_{n+1}-w_n}{\triangle}\Big|^2 \frac{w_n-w_{n-1}}{\triangle}\Big]\Big\}\triangle \end{aligned}$$

$$= -\int_0^1 \varphi_{|z} \left(\tilde{\delta}_1^2 \rho_\triangle^\gamma w_{\triangle|z} + \frac{\tilde{\delta}_2^4}{4} \left|w_{\triangle|z}\right|^2 w_{\triangle|z}\right) dz + o(1)\,,$$

$$\sum_{n=1}^{N} \varphi_n \big(g_N(w_n) - 2c_2 w_n - 2\rho_{n-1}\mathrm{dcp}\big)\triangle$$

$$= \int_0^1 \varphi\big(g_\triangle(w_\triangle^R) - 2c_2 w_\triangle^R - 2\rho_\triangle \mathrm{dcp}\big) dz + o(1)\,.$$

Regard that $\varPhi_\triangle := 1 - 2w_\triangle$, then equation (20) is valid. And further, the Main Theorem 2 is proved. □

7. Uniqueness of the Weak Solution

Theorem 27. *Assume (A.1)-(A.4). Let* $(\rho_1, v_1, \varPhi_1)$ *and* $(\rho_2, v_2, \varPhi_2)$ *be solutions of (7)-(13) satisfying (14)-(20). Then we have* $\rho_1 = \rho_2$, $v_1 = v_2$ *and* $\varPhi_1 = \varPhi_2$.

Proof. Taking the difference of (9), it yields

$$\int_0^1 \varphi(\varPhi_2 - \varPhi_1)_{|t} dz + \int_0^1 \varphi\, c_1 \Big[\ln\Big(\frac{\varPhi_2}{1-\varPhi_2}\Big) - \ln\Big(\frac{\varPhi_1}{1-\varPhi_1}\Big)\Big] dz$$

$$= \int_0^1 \varphi\, c_2 (\varPhi_2 - \varPhi_1) dz - \int_0^1 \varphi \;\mathrm{dcp}\,(\rho_2 - \rho_1) dz$$

$$- \int_0^1 \varphi_{|z} \Big\{ \tilde{\delta}_1^2 (\rho_2^\gamma \varPhi_2 - \rho_1^\gamma \varPhi_1)_{|z} + \tilde{\delta}_2^4 \Big(\left|\varPhi_{2|z}\right|^2 \varPhi_{2|z} - \left|\varPhi_{1|z}\right|^2 \varPhi_{1|z} \Big) \Big\} dz,$$

for an arbitrary $\varphi \in C_0^\infty((0,1])$. Then, there exists a sequence $\varphi_n \in C_0^\infty((0,1])$ such that $\varphi_n \to (\varPhi_2 - \varPhi_1)(t,.)$ in $C([0,1])$ and $\varphi_{n|z} \to (\varPhi_2 - \varPhi_1)_{|z}(t,.)$ in $L^1((0,1))$. Therefore, going to the limit, we can replace φ by $\varPhi_2 - \varPhi_1$. Then, we get

$$\frac{1}{2} \int_0^t \int_0^1 \frac{d}{dt} \left|\varPhi_2 - \varPhi_1\right|^2 (t,.) dz d\tau$$

$$+ c_1 \int_0^t \int_0^1 \Big[\ln\Big(\frac{\varPhi_2}{1-\varPhi_2}\Big) - \ln\Big(\frac{\varPhi_1}{1-\varPhi_1}\Big)\Big] (\varPhi_2 - \varPhi_1)\, dz d\tau$$

$$+ C\tilde{\delta}_1^2 \int_0^t \int_0^1 \rho_2^\gamma \left|(\varPhi_2 - \varPhi_1)_{|z}\right|^2 dz d\tau$$

$$+ \tilde{\delta}_2^4 \int_0^t \int_0^1 \Big(\left|\varPhi_{2|z}\right|^2 \varPhi_{2|z} - \left|\varPhi_{1|z}\right|^2 \varPhi_{1|z} \Big) \big(\varPhi_{2|z} - \varPhi_{1|z}\big) dz d\tau$$

$$\leq C\int_0^t\int_0^1(\Phi_2-\Phi_1)^2dzd\tau+C\int_0^t\int_0^1\left(\rho_2^\gamma-\rho_1^\gamma\right)^2\left|\Phi_{1|z}\right|^2dzd\tau$$
$$+\operatorname{dcp}^2\int_0^t\int_0^1(\rho_2-\rho_1)^2dzd\tau\,.$$

It holds

$$c_1\int_0^t\int_0^1\left[\ln\left(\frac{\Phi_2}{1-\Phi_2}\right)-\ln\left(\frac{\Phi_1}{1-\Phi_1}\right)\right](\Phi_2-\Phi_1)\,dzd\tau$$
$$=c_1\int_0^t\int_0^1 g'(\Phi_1+\beta(\Phi_2-\Phi_1))\left|\Phi_2-\Phi_1\right|^2dzd\tau\ \geq\ 0\,,$$

where $\beta\in[0,1]$ and $g'(x)=\frac{1}{x}\frac{1}{1-x}\ \geq\ 0$ for $0\leq x\leq 1$, and because of the monotony of the p-laplace operator further

$$\tilde{\delta}_2^4\int_0^t\int_0^1\left(\left|\Phi_{2|z}\right|^2\Phi_{2|z}-\left|\Phi_{1|z}\right|^2\Phi_{1|z}\right)\left(\Phi_{2|z}-\Phi_{1|z}\right)dzd\tau\ >\ 0\,.$$

Finally, by applying the Gronwall Lemma, we get

$$\int_0^1\left|\Phi_2-\Phi_1\right|^2(t,.)dz+C\int_0^t\int_0^1(\Phi_2-\Phi_1)_{|z}^2dzd\tau \qquad (50)$$
$$\leq\ C\int_0^t\int_0^1\left|\rho_2-\rho_1\right|^2dzd\tau\,.$$

We do with equation (8) the same procedure as before. Then

$$\frac{1}{2}\int_0^1\frac{d}{dt}\left|v_2-v_1\right|^2dz+\int_0^1\rho_1^{1+\theta}(v_2-v_1)_{|z}^2dz$$
$$\leq\ \frac{1}{2\varepsilon}\int_0^1\left[(\rho_2^2-\rho_1^2)\operatorname{dcp}\Phi_2+(\Phi_2-\Phi_1)\operatorname{dcp}\rho_1^2+(\rho_2^3-\rho_1^3)a\right]^2dz$$
$$+\varepsilon\int_0^1(v_2-v_1)_{|z}^2dz+\frac{\tilde{\delta}_1^4\gamma^2}{2}\int_0^1\left(\rho_2^{\gamma+\beta}-\rho_1^{\gamma+\beta}\right)^2\left|\Phi_{2|z}\right|^4dz$$
$$+\frac{\tilde{\delta}_1^4\gamma^2}{2}\int_0^1\rho_1^{2\gamma+2\beta}\left(\left|\Phi_{2|z}\right|^2-\left|\Phi_{1|z}\right|^2\right)^2dz$$
$$+\frac{1}{2\varepsilon}\int_0^1(\rho_2^{1+\theta}-\rho_1^{1+\theta})^2\left|v_{2|z}\right|^2dz+\frac{\varepsilon}{2}\int_0^1(v_2-v_1)_{|z}^2dz\,.$$

Because of $\frac{1}{C(T)}\leq\rho_1$, we choose ε as small as that $\left(\frac{1}{C(T)}\right)^{1+\theta}-\ \ \varepsilon>0$

$$\frac{1}{2}\int_0^1\left|v_2-v_1\right|^2dz+\int_0^t\int_0^1\rho_1^{1+\theta}(v_2-v_1)_{|z}^2dzd\tau$$
$$\leq\ C\int_0^t\int_0^1\operatorname{dcp}^2\Phi_2^2(\rho_2-\rho_1)^2dzd\tau+\int_0^t\int_0^1\operatorname{dcp}^2\rho_1^4(\Phi_2-\Phi_1)^2dzd\tau$$

$$+ C\int_0^t\int_0^1 a^2(\rho_2-\rho_1)^2 dzd\tau + C\int_0^t\int_0^1 (\rho_2-\rho_1)^2\big|\Phi_{2|z}\big|^4 dzd\tau$$
$$+ C\int_0^t\int_0^1 \rho_1^{2\gamma+2\beta}(\Phi_2-\Phi_1)^2_{|z} dzd\tau + C\int_0^t\int_0^1 (\rho_2-\rho_1)^2\big|v_{2|z}\big|^2 dzd\tau$$
$$\leq C(T)\int_0^t\int_0^1 (\rho_2-\rho_1)^2 dzd\tau \tag{51}$$
$$\big(\text{due to, (50) and } \big|v_{2|z}\big| \leq C\,, \big|\Phi_{2|z}\big| \leq C(T)\,, |\rho_1| \leq C(T)\big)\,.$$

Now, we rewrite equation (7) by $\frac{\partial}{\partial t}\frac{1}{\rho} = \frac{\partial v}{\partial z}$, take the difference and multiply it by $\frac{1}{\rho_2} - \frac{1}{\rho_1}$. Then, we have
$$\frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{1}{\rho_2}-\frac{1}{\rho_1}\Big)^2 = (v_2-v_1)_{|z}\Big(\frac{1}{\rho_2}-\frac{1}{\rho_1}\Big)\,.$$
We integrate over $[0,t]\times[0,1]$ respectively τ and z and we get
$$\begin{aligned}\frac{1}{2}\int_0^1\Big(\frac{1}{\rho_2}-\frac{1}{\rho_1}\Big)^2 dz &= \int_0^t\int_0^1 (v_2-v_1)_{|z}\Big(\frac{1}{\rho_2}-\frac{1}{\rho_1}\Big) dzd\tau \\ &\leq \int_0^t\int_0^1 \rho_1^{1+\theta}(v_2-v_1)^2_{|z} dzd\tau \\ &\quad + \frac{1}{2}\int_0^t\int_0^1 \frac{1}{\rho_1^{1+\theta}}\Big(\frac{1}{\rho_2}-\frac{1}{\rho_1}\Big)^2 dzd\tau\,.\end{aligned}$$
Since $\frac{1}{\rho_2} - \frac{1}{\rho_1} = \frac{\rho_1-\rho_2}{\rho_1\rho_2}$ and (51), we get
$$\int_0^1 (\rho_2-\rho_1)^2 dz \leq C(T)\int_0^t\int_0^1 (\rho_2-\rho_1)^2 dzd\tau\,.$$
So $\rho_2 = \rho_1$, a.e. And with equations (50) and (51), it follows $v_2 = v_1$ and $\Phi_2 = \Phi_1$, a.e. □

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