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ON FRAMES IN CONJUGATE BANACH SPACES

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Abstract: Finite sum of frames for conjugate Banach spaces have been studied. A sufficient condition under which finite sum of retro Banach frames is a retro Banach frame has been given. Also, perturbation of retro Banach frames by a non-zero vector has been considered and a necessary and sufficient condition for the perturbed sequence to be a retro Banach frame has been given. Finally, a necessary and sufficient condition for the stability of a retro Banach frame has been given.

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1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [7] while addressing some problems in the theory of non-harmonic Fourier-series. Later, Daubechies, Graossman and Meyer [6] found a new fundamental application to wavelet and Gabor transforms in which frames played an important role.

Casazza [3] and Benedetto and Fickus [2] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. For results on signal reconstruction without phase information, one may refer to [1].

Coifman and Weiss [5] introduced the notion of *atomic decomposition* for

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function spaces. Feichtinger and Gröchenig [8] extended the notion of atomic decomposition to Banach spaces. Gröchenig [9] introduced a more general concept for Banach spaces called Banach frame. *Banach frames* were further studied in [4, 11, 12].

In the present paper, we study frames for conjugate Banach spaces called retro Banach frames and obtained a sufficient condition under which finite sum of retro Banach frames is a retro Banach frame. Perturbation of a retro Banach frame by a non-zero vector has been considered and observed that such a perturbation need not be a retro Banach frame. Also, a necessary and sufficient condition for this perturbed sequence to be a retro Banach frame has been obtained. Finally, a necessary and sufficient condition for the stability of a retro Banach frame has been obtained.

2. Preliminaries

Throughout this paper E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, π the canonical isomorphism of E into E^{**}, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, $[\widetilde{f_n}]$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology of E^*, E_d and $(E^*)_d$, respectively, the associated Banach spaces of the scalars-valued sequences indexed by $\mathbb{N}$.

A sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0,\ n \in \mathbb{N}\} = \{0\}$.

The following result which is refered in this paper is listed in the form of a lemma.

Lemma 2.1. (see [13]) *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Finally, in this section, we give the definition of a retro Banach frame introduced in [10].

Definition 2.2. Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar valued sequences associated with E^* indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* (RBF) for E^* with respect to $(E^*)_d$ if:

(i) $\{f(x_n)\} \in (E^*)_d$, for each $f \in E^*$,

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*} \quad f \in E^* \tag{2.1}$$

(iii) T is a bounded linear operator such that $T(\{f(x_n)\}) = f$, $f \in E^*$.

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \to E^*$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *retro frame inequality.*

3. Finite Sum of Retro Banach Frames

Let $E = c_0$ and let $\{x_n\}$ be a sequence of unit vectors in E^*. Then, by Lemma 2.1, there exists an associated Banach space

$$(E^*)_d = \{\{f(x_n)\} : f \in E^*\} \quad (= \ell^1)$$

with norm given by $\|\{f(x_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$, $f \in E^*$ and a bounded linear operator $T : (E^*)_d \to E^*$ with $T(\{f(x_n)\}) = f$, $f \in E^*$. Then T is a bounded linear operator such that $(\{x_n\}, T)$ is a RBF for E^* with respect to $(E^*)_d$. Define a sequence $\{y_n\} \in E$ by $y_1 = x_1$, $y_2 = x_1 - x_2$ and $y_n = x_n$, $n = 3, 4, \ldots$. Then, by Lemma 2.1 again, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ and a reconstruction operator $T_0 : (E^*)_{d_0} \to E^*$ given by $T_0(\{f(y_n)\}) = f$, $f \in E^*$ such that $(\{y_n\}, T_0)$ is a RBF for E^* with respect to $(E^*)_{d_0}$. Also $\{f(y_n)\} \in (E^*)_d$. Put $z_n = x_n + y_n$, $n \in \mathbb{N}$. Then, there exists no associated Banach space $(E^*)_{d_1}$ and hence no reconstruction operator $T_1 : (E^*)_{d_1} \to E^*$ such that $(\{z_n\}, T_1)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_1}$. Indeed, if $(\{z_n\}, T_1)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_1}$. Then, there exist positive constants A_z and B_z, such that

$$A_z\|f\|_{E^*} \leq \|\{f(z_n)\}\|_{(E^*)_{d_1}} \leq B_z\|f\|_{E^*} \quad \text{for all} \quad f \in E^*. \tag{3.1}$$

Now $f = (0, 1, 0, 0, \ldots)$ is a non-zero functional in E^* such that $f(z_n) = 0$, for all $n \in \mathbb{N}$. So, by (3.1), $f = 0$. This is a contradiction.

In view of the above discussion, it is natural to ask for conditions under which a finite sum of retro Banach frames is a retro Banach frame. We prove the following result.

Theorem 3.1. *Let* $(\{x_{i,n}\}, T_i)$ $(\{x_{i,n}\} \subset E, T_i : (E^*)_d \to E^*$, $i = 1, 2, \ldots, k)$ *are retro Banach frames for* E^* *with respect to* $(E^*)_d$. *Then there exists an associated Banach space* $(E^*)_{d_0}$ *and a recons-truction operator* T_0 :

$(E^*)_{d_0} \to E^*$ such that $\left(\left\{\sum_{i=1}^{k} x_{i,n}\right\}, T_0\right)$ is retro Banach frame for E^* with respect to $(E^*)_{d_0}$ provided

$$\|\{f(x_{j,n})\}\|_{(E^*)_d} \le \left\|\left\{f\left(\sum_{i=1}^{k} x_{i,n}\right)\right\}\right\|_{(E^*)_d}, \quad \text{for some } j \in \{1,2,\ldots,k\}.$$

Proof. Note that for each $j \in \{1,2,\ldots,k\}$,

$$\|T_j\|^{-1}\|f\|_{E^*} \le \|\{f(x_{j,n})\}\|_{(E^*)_d} \le \left\|\left\{f\left(\sum_{i=1}^{k} x_{i,n}\right)\right\}\right\|_{(E^*)_d} \tag{3.2}$$

Put $\phi_n = \pi\left(\sum_{i=1}^{k} x_{i,n}\right)$, $n \in \mathbb{N}$, where $\pi : E \to E^{**}$ is the canonical isomorphism. Let $\phi_n(f) = 0$, for all $n \in \mathbb{N}$. Then $f\left(\sum_{i=1}^{k} x_{i,n}\right) = 0$, $n \in \mathbb{N}$. So, by (3.2), $f = 0$. Therefore, by Lemma 2.1,

$$(E^*)_{d_0} = \left\{\left\{f\left(\sum_{i=1}^{k} x_{i,n}\right)\right\} : f \in E^*\right\}$$

is a Banach space of scalar-valued sequences with norm given by

$$\left\|\left\{f\left(\sum_{i=1}^{k} x_{i,n}\right)\right\}\right\|_{(E^*)_{d_0}} = \|f\|_{E^*}, \quad f \in E^*.$$

Define $T_0 : (E^*)_{d_0} \to E^*$ by

$$T_0\left(\left\{f\left(\sum_{i=1}^{k} x_{i,n}\right)\right\}\right) = f, \quad f \in E^*.$$

Then T_0 is a bounded linear operator such that $\left(\left\{\sum_{i=1}^{k} x_{i,n}\right\}, T_0\right)$ is a RBF for E^* with respect to $(E^*)_{d_0}$. □

4. Perturbation of Retro Banach Frames

Let $(\{x_n\}, S)$ $(\{x_n\} \subset E, S : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Let $0 \ne x_0 \in E$. Then, there exists, in general, no reconstruction operator U such that $(\{x_n + x_0\}, U)$ is a retro Banach frame for E^*. Indeed, if $E = \ell^1$ and $\{y_n\}$ in E be such that $y_n = (0, 0, \ldots, \underset{\substack{\downarrow \\ n\text{-th place}}}{1}, 0, 0, \ldots)$,

$n \in \mathbb{N}$. Then there exists an associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\}, f \in E^*\}$ with norm given by $\|\{f(y_n)\}\|_{(E^*)_{d_0}} = \|f\|_{E^*}$, $f \in E^*$ and a reconstruction operator $T : (E^*)_{d_0} \to E^*$ given by $T(\{f(y_n)\}) = f$, $f \in E^*$ such that $(\{y_n\}, T)$ is a RBF for E^* with respect to $(E^*)_{d_0}$. Let $y_0 = (-1, 0, 0, \ldots) \in E$. Then there exists no reconstruction operator T_0 such that $(\{y_n + y_0\}, T_0)$ is a RBF for E^*.

We now give a necessary and sufficient condition under which perturbation of a RBF by a non-zero vector is again a RBF.

Theorem 4.1. *Let $(\{x_n\}, S)$ $(\{x_n\} \subset E, S : (E^*)_d \to E^*)$ be a RBF for E^* with respect to $(E^*)_d$. Then there exists no reconstruction operator $S_0 : (E^*)_d \to E^*$ such that, for a given non-zero element $x_0 \in E$, $(\{x_n + x_0\}, S_0)$ is a RBF for E with respect to $(E^*)_d$ if and only if there exists a non-zero functional g in E^* such that $g(x_n) = \lambda$, $n \in \mathbb{N}$, where λ is some non-zero scalar.*

Proof. Let $x \in E$ be a non-zero element such that $g(x) \neq 0$. Let $x_0 = -\lambda x / g(x)$. Suppose that there exists a reconstruction operator $S_0 : (E^*)_d \to E^*$ such that $(\{x_n + x_0\}, S_0)$ is a RBF for E^* with respect to $(E^*)_d$. Since $g(x_n + x_0) = 0$, for all $n \in \mathbb{N}$, by frame inequality, for the RBF $(\{x_n + x_0\}, S_0)$, it follows that $g = 0$. This is a contradiction.

Conversely, by hypotheses, there exists a non-zero $f \in E^*$ such that $f(x_n + x_0) = 0$, for all $n \in \mathbb{N}$. If $f(x_0) = 0$, then $f(x_n) = 0$, for all $n \in \mathbb{N}$. So, by frame inequality for the RBF $(\{x_n\}, S)$, $f = 0$. This is a contradiction. Also, if $f(x_0) \neq 0$. Then, $g = -\lambda f / f(x_0)$ is a non-zero functional in E^* such that

$$g(x_n) = \frac{-\lambda}{f(x_0)} f(x_n) = \lambda, \text{ for all } \ n \in \mathbb{N}. \qquad \square$$

5. Stability of Retro Banach Frames

In this section, we shall give a necessary and sufficient condition for the stability of a retro Banach frame.

Theorem 5.1. *Let $(\{x_n\}, S)$ $(\{x_n\} \subset E, S : (E^*)_d \to E^*)$ be a RBF for E^* with respect to $(E^*)_d$ and let $\{y_n\}$ be a sequence in E such that $\{f(y_n)\} \in (E^*)_d$, $f \in E^*$ and let $W : (E^*)_d \to (E^*)_d$ be a bounded linear operator such that $W(\{f(y_n)\}) = \{f(x_n)\}$, $f \in E^*$. Then there exists a reconstruction operator T such that $(\{y_n\}, T)$ is a RBF for E^* with respect to $(E^*)_d$ if and only if there exists a constant $M > 0$ such that*

$$\|\{f(x_n - y_n)\}\|_{(E^*)_d} \leq \gamma M \,,$$

where $\gamma = \min\{\|\{f(x_n)\}\|_{(E^*)_d}, \|\{f(y_n)\}\|_{(E^*)_d}\}$.

Proof. Suppose first that $(\{y_n\}, T)$ is a RBF for E^* with respect to $(E^*)_d$ and with bounds A_0 and B_0. Let A and B be the bounds of the RBF $(\{x_n\}, S)$. Then

$$\|\{f(x_n - y_n)\}\|_{(E^*)_d} \leq \left(1 + \frac{A_0}{A}\right) \|\{f(x_n)\}\|_{(E^*)_d}, \quad f \in E^* .$$

Similarly, we obtain

$$\|\{f(x_n - y_n)\}\|_{(E^*)_d} \leq \left(1 + \frac{A}{A_0}\right) \|\{f(y_n)\}\|_{(E^*)_d}, \quad f \in E^* .$$

Choose $M = \left(1 + \frac{A_0}{A}\right)$ or $\left(1 + \frac{A}{A_0}\right)$ according as

$$\min\{\|\{f(x_n)\}\|_{(E^*)_d}, \|\{f(y_n)\}\|_{(E^*)_d}\} = \|\{f(x_n)\}\|_{(E^*)_d}$$

or $\|\{f(y_n)\}\|_{(E^*)_d}$, respectively.

Conversely, if A and B are the bounds of the RBF $(\{x_n\}, S)$. Then, for each $f \in E^*$,

$$\begin{aligned} A\|f\|_{E^*} &\leq \|\{f(x_n - y_n)\}\|_{(E^*)_d} + \|\{f(y_n)\}\|_{(E^*)_d} \\ &\leq (1+M)\|\{f(y_n)\}\|_{(E^*)_d} \\ &\leq (1+M)(\|\{f(x_n)\} - \{f(y_n - x_n)\}\|_{(E^*)_d} \\ &\leq (1+M)^2 B\|f\|_{E^*} . \end{aligned}$$

Put $T = SW$. Then $T : (E^*)_d \to E^*$ is a bounded linear operator such that $T(\{f(y_n)\}) = f$, $f \in E^*$. Hence, $(\{y_n\}, T)$ is a RBF for E^* with respect to $(E^*)_d$ and with frame bounds $\frac{A}{(1+M)}$ and $(1+M)B$. □

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