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BRANCHING PROCESSES WITH A COMMON EXTINCTION PROBABILITY

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Abstract: We present a method of creating a class of branching processes with a common probability of extinction. For our specialized setting, one member of the class involves probabilities that involve Fibonacci numbers. By using another method to obtain the probabilities, we obtain a new expression for the n-th Fibonacci number.

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1. Introduction

For an introduction to branching processes, see Ross [3], for example. A branch-

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ing process with one particle in Generation 0 may generate $0, 1, 2, \ldots$ offspring in Generation 1 with probabilities $b_0, b_1, b_2, \ldots$ respectively. Let the generating function be $\beta(s) = \sum_{i=0}^{\infty} b_i s^i$, where $b_i \geq 0$ for all i, and $\sum_{i=0}^{\infty} b_i = 1$. Each particle in Generation 1 behaves the same way as the 1 item in Generation 0. This process continues forever or until extinction.

The Fundamental Theorem of Branching Processes states that the probability of extinction α is the smallest real non-negative root of the equation $s = \beta(s)$.

In Section 2 of this paper, we present two methods of finding a class of branching processes with a common probability of extinction. One member of our class of branching processes with common extinction probability involves the Fibonacci numbers. In Section 3, we generalize the results of Section 2, and use a two sided ballot theorem. This allows us to obtain a new expression for the n-th Fibonacci number, which appears as one of the special cases in Section 2.

2. Common Extinction Probability

Let us consider a the branching process with probability generating function $\beta(s) = b_0 + b_2 s^2$. Call this the Basic Branching Process Model. The quadratic generating function can be used to find the probability of ruin in the "gambler's ruin" problem. Assume a gambler begins with exactly 1 unit, and bets exactly 1 unit at each stage. He/she has probability p of winning and q of losing each bet, where $p + q = 1$. Then $\beta(s) = q + ps^2$. If the gambler plays against an infinitely rich opponent, the probability that the gambler eventually loses all his/her money (gambler's ruin) is the smallest positive root α of $s = q + ps^2$. If $p \leq q$, then $\alpha = 1$.

We can view the gambler's assets at any step as a random walk path. For example the gambler's assets at consecutive steps might be $1, 2, 1, 2, 3, 2, 3$, which would correspond to the path joining the points $(0, 1)$, $(1, 2)$, $(2, 1)$, $(3, 2)$, $(4, 3)$, $(5, 2)$, $(6, 3)$. This is illustrated in Figure 1.

The generating function $\beta(s) = q + ps^2$ implicitly defines the end of the first generation as the result after 1 step and consists of the values 0,2 with probabilities q and p respectively. However, for the same random walk, we can define the end of the first generation in many other ways. For example, we could define the end of the first generation to be the step on which the random walk takes a downward step for the first time. This happens on step 1 with

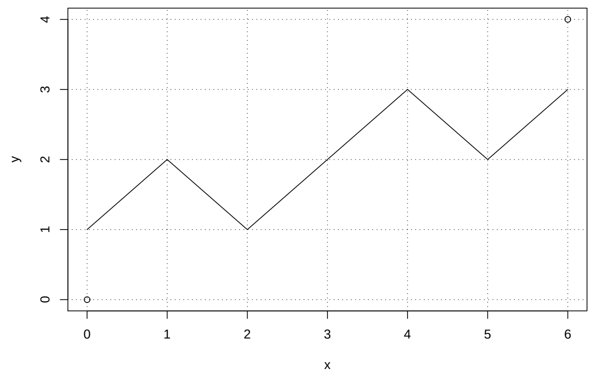

Figure 1: Random walk path

probability q, leaving an amount 0. It happens on step 2 with probability pq leaving an amount 1. It happens on step 3 with probability p^2q, leaving an amount 2, etc. Thus the generating function is $\beta(s) = q + qps + qp^2s^2 + \cdots = \frac{q}{1-ps}$ which is a different generating function that has the same probability of extinction.

We plot these two generating functions $\beta_1(s) = q+ps^2$ and $\beta_2(s) = \frac{q}{1-ps}$ for $q = .2, p = .8$ together with the line $\beta(s) = s$. The result appears in Figure 2. Note that all three curves meet at one point, which means that the probability of extinction for the 2 branching models is the same. The lower curve on the right is $\beta_2(s)$.

Another example of a definition for the end of the first generation might be to be the minimum of the sixth step or the step at which the random walk first reaches 0. For any definition we choose, we get a corresponding generating function. Each distinct generating function defines a different branching process, where the generating function gives the probabilities of number of individuals at the next generation. But since each process is describing the SAME random walk in a different way, the different branching processes (with their distinct generating functions) must have a common extinction probability. We look at three cases arising from the random walk model, each of interest on its own, and each with its own combinatorial structure.

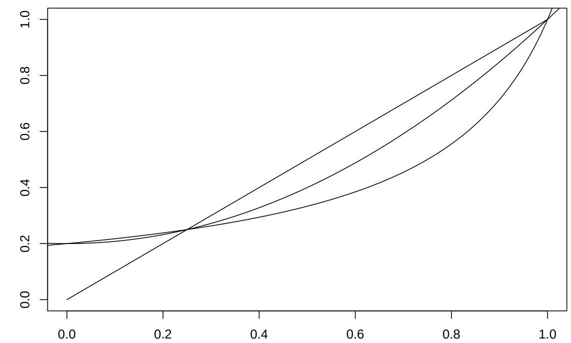

Figure 2: Two generating functions with same extinction probability

2.1. Case 1

For the random walk of Section 1, define the end of Generation 1 to be the step at which the random walk first reaches either level 0 or 3 given that the process starts at level 1. A typical path might end after the transitions $(0,1),(1,2),(2,1),(3,2),(4,1),(5,0)$. The probability of that particular path is $pqpqq = p^2q^3$. We can count the number of such restricted paths from $(0,1)$ to $(2i-1,0)$ or from $(0,1)$ to $(2i,3)$, for $i = 1,2,\ldots$. The number of paths from $(0,1)$ to $(1,0)$, $(3,0)$, $(5,0)$, $\ldots$ is 1, 1, 1, $\ldots$ respectively. The number of paths from $(0,1)$ to $(2,3)$, $(4,3)$, $(6,3)$, $\ldots$ is 1, 1, 1, $\ldots$ respectively. Thus the probability of eventually reaching level 0 is $a = 1q + 1pq^2 + 1p^2q^3 + \cdots = \frac{q}{1-pq}$, $0 < p < 1$. The probability of eventually reaching level 3 is $b = 1p^2 + 1p^3q + 1p^4q^2 + \cdots = \frac{p^2}{1-pq}$. of course $a + b = 1$.

Proposition 2.1. *The two branching processes defined by their two generating functions*

$$\beta^{(1)}(s) = q + ps^2\,, \quad \beta^{(2)}(s) = \frac{q}{1-pq} + \frac{p^2}{1-pq}s^3, \tag{1}$$

have a common extinction probability.

Proof. The result is true since we are describing the same random walk process and merely changing our definition of the end of a generation. We observe that the smallest root of the first equation in (1) is q/p if $q < p$. It is easy to check that this is the smallest positive root of the second equation in

(1). □

There is another simple way of getting a class of branching processes with the same limiting probability as our Basic Branching Process Model. This involves changing the probability of 1 offspring and adjusting the other probabilities accordingly.

Proposition 2.2. *Let $\beta_1(s) = b_0 + b_1 s + \sum_{i=2}^{n} b_i s^i$ be the probability generating function of a branching process with probability of extinction α. Then the branching process with probability generating function $\beta_2(s) = \frac{1-k}{1-b_1} b_0 + ks + \sum_{i=2}^{n} \frac{1-k}{1-b_1} b_i s^i$ has the same probability of extinction.*

Proof. If there is a single particle in one generation and it generates exactly one particle in the next generation, there is no change in the number of particles there will be no effect on the probability of extinction. Thus we preserve the probability of extinction by changing the b_1 coefficient of s in $\beta_1(s)$ to any other value k in $[0, 1)$ and adjusting the other probabilities proportionately (to each other) so that the sum of the new probabilities is still 1. □

Example 2.1. $\beta_1(s) = (1/3) + (1/2)s + (1/6)s^2$ is the pgf of a branching process with the same limiting probability as the branching process with pgf $\beta_2(s) = (1/6) + (3/4)s + (1/12)s^2$.

2.2. Case 2

Define the end of Generation 1 to be the step at which the random walk first reaches either level 0 or 4 given that the process starts at level 1. A typical path might be $1, 2, 1, 2, 1, 2, 3, 4$. The probability of that particular path is $pqpqppp = p^5 q^2$. We can count the number of such restricted paths from $(0, 1)$ to $(2i - 1, 0)$, for $i = 1, 2, \ldots$, or from $(0, 1)$ to $(2i - 1, 4)$, for $i = 2, 3, \ldots$. The number of paths from $(0, 1)$ to $(1, 0)$, $(3, 0)$, $(5, 0)$, $(7, 0)$, $(9, 0)$, $\ldots$ is 1, 1, 2, 4, 8 $\ldots$ respectively. The number of paths from $(0, 1)$ to $(3, 4)$, $(5, 4)$, $(7, 4)$, $(9, 4)$, $\ldots$ is 1, 2, 4, 8, $\ldots$ respectively. Thus the probability of eventually reaching level 0 is $a = 1q + 1pq^2 + 2p^2q^3 + 4p^3q^4 + \cdots = \frac{q - pq^2}{1 - 2pq}$, $0 < p < 1$. The probability of eventually reaching level 4 is $b = 1p^3 + 2p^4 q + 4p^5 q^2 + \cdots = \frac{p^3}{1 - 2pq}$.

Proposition 2.3. *Let $0 < p < 1$ and $q = 1 - p$. The branching processes* *defined by their generating functions*

$$\beta^{(1)}(s) = q + ps^2, \quad \beta^{(2)}(s) = \frac{q}{1-pq} + \frac{p^2}{1-pq}s^3, \\ \beta^{(3)}(s) = \frac{q-pq^2}{1-2pq} + \frac{p^3}{1-2pq}s^4, \tag{2}$$

have a common extinction probability.

2.3. Case 3

Define the end of Generation 1 to be the step at which the random walk first reaches either level 0 or 5 given that the process starts at level 1. This is the case in which the Fibonacci numbers appear. A typical path might be $1,2,3,2,1,0$. The probability of that particular path is $ppqqq = p^2q^3$. We can count the number of such restricted paths from $(0,1)$ to $(2i+1,0)$, for $i = 1,2,\ldots,$ or from $(0,1)$ to $(2i,5)$, for $i = 2,3,\ldots$. The number of paths from $(0,1)$ to $(1,0)$, $(3,0)$, $(5,0)$, $(7,0)$, $(9,0)$, $\ldots$ is 1, 1, 2, 5, 13, $\ldots$ respectively. The number of paths from $(0,1)$ to $(4,5)$, $(6,5)$, $(8,5)$, $(10,5)$, $\ldots$ is 1, 3, 8, 21, $\ldots$ respectively. We observe that the Fibonacci numbers show up in the counts of the number of paths. We now show why this is true. See Vorob'yev [4], for more information on Fibonacci numbers.

Proposition 2.4. *The number of paths from* $(1,0)$ *to* $(2i+1,0)$*, for* $i = 1,2,\ldots,$ *which do not touch level 5 or level 0 except on the last step is* F_{2i-1}*. The number of paths from* $(1,0)$ *to level* $(2i,5)$*, for* $i = 2,3,\ldots,$ *which do not touch level 5 or level 0 except on the last step is* F_{2i-2}*.*

Proof. Let a_{2i+1} be the number of paths from $(0,1)$ to $(2i+1,0)$ which are strictly below level 5 and strictly above level 0, except on the last step. Let a_{2i} be the number of paths from $(0,1)$ to $(2i,5)$ which are strictly below level 5 and strictly above level 0, except on the last step. Consider a path which ends at $(2i+1,0)$. At the previous step the path must have been at $(2i,1)$. At the step previous to that, the path must have been at $(2i-1,2)$. At the step previous to that, the process must have been at either $(2i-2,1)$ or $(2i-2,3)$ (see Figure 3).

But all allowable paths from $(0,1)$ to $(2i-1,0)$ must pass through $(2i-2,1)$, and the number of such paths is a_{2i-1}. All allowable paths from $(0,1)$ to $(2i,5)$ must pass through $(2i-1,4)$ and hence through $(2i-2,3)$. The number of such paths is a_{2i}. Hence $a_{2i+1} = a_{2i} + a_{2i-1}$. We can easily check that $a_3 = 1$ and $a_4 = 1$. This is the defining relationship for the Fibonacci numbers (shifted).

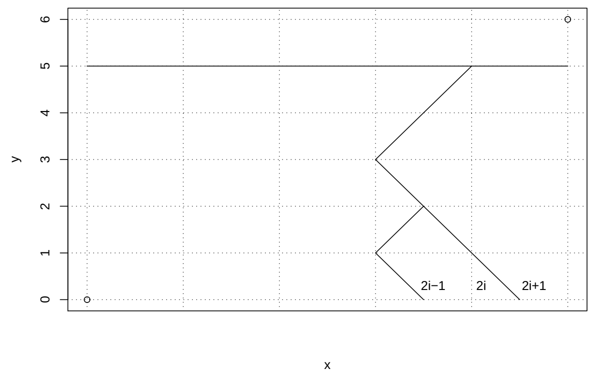

Figure 3: Path counting recursion

Thus $a_3 = F_1$, $a_4 = F_2$, and in general $a_i = F_{i-2}$ for $i = 3, 4, \ldots$. □

Proposition 2.5. *Let $0 < p < 1$ and $q = 1 - p$. Let $a = \frac{q-2pq^2}{1-3pq+p^2q^2}$ and $b = 1 - a$. Then the branching processes defined by their generating functions*

$$\beta^{(1)}(s) = q + ps^2, \quad \beta^{(2)}(s) = \frac{q}{1-pq} + \frac{p^2}{1-pq}s^3,$$
$$\beta^{(3)}(s) = \frac{q-pq^2}{1-2pq} + \frac{p^3}{1-2pq}s^4, \quad \beta^{(4)}(s) = a + bs^5, \tag{3}$$

have a common extinction probability.

Proof. The probability of eventually reaching level 0 is $a = 1q + 1pq^2 + 2p^2q^3 + 5p^3q^4 + 13p^4q^5 + \cdots = q + pq^2 \sum_{i=0}^{\infty} F_{2i+1}p^iq^i$. The probability of eventually reaching level 4 is $b = 1p^4 + 3p^5q + 8p^6q^2 + 21p^7q^3 = p^4 \sum_{i=0}^{\infty} F_{2i+2}p^iq^i$. We can sum the series for a and b if we use the following well known expression for the Fibonacci numbers (see [4]). $F_i = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right]$. Thus

$$a = q + pq^2 \sum_{i=0}^{\infty} F_{2i+1}p^iq^i = q + pq^2 \sum_{i=0}^{\infty} (pq)^i \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2i+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2i+1}\right]$$
$$= \frac{q-2pq^2}{1-3pq+p^2q^2},$$

$$b = p^4 \sum_{i=0}^{\infty} F_{2i+2} p^i q^i = p^4 \sum_{i=0}^{\infty} (pq)^i \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{2i+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{2i+2} \right]$$
$$= \frac{1 - q - 3pq + 2pq^2 + p^2q^2}{1 - 3pq + p^2q^2}.$$

Again we apply the fact that the two generating functions for the two branching processes are really simply descriptions of the same random walk process with different definitions of the end of the first generation. □

Note: The value for a in the previous proposition can also be obtained by using results for probabilities of entering absorbing states and the fundamental matrix $(I-Q)^{-1}$ for Markov chains.

We have presented four branching processes with common extinction probabilities and it is clear that we can continue our technique and find infinitely many members in the class.

3. The General Case

In Section 2, we considered, for $k = 3, 4, 5$, random walks that began at $(0, 1)$ and reached level k on their last step, with the restriction that stay above level 0 and below level k until the last step. We will now consider arbitrary positive values of k.

We use the two sided ballot theorem (see Feller [1], p. 96, #3 or Narayana [2], p. 12). It is restated using our notation. In the following result, we use the convention that $\binom{n}{r}$ is 0 if $r < 0$ or $r > n$.

Proposition 3.1. (The Two Sided Ballot Theorem) *Let a and b be positive integers. Let n and x be positive integers, such that $-a < x < b$. Then the number of paths from $(0, 0)$ to (n, x) such that the path never touches or crosses level $-a$ or level b, is given by*

$$k = \binom{n}{(n+x)/2} - \binom{n}{(n+x+2a)/2} - \binom{n}{(n+x-2b)/2}$$
$$+ \binom{n}{(n+x+2a+2b)/2} + \binom{n}{(n+x-2a-2b)/2} - \binom{n}{(n+x+4a+2b)/2}$$
$$- \binom{n}{(n+x-2a-4b)/2} + \binom{n}{(n+x+4a+4b)/2} + \binom{n}{(n+x-4a-4b)/2}$$
$$- \binom{n}{(n+x+6a+4b)/2} - \binom{n}{(n+x-4a-6b)/2} + \dots . \quad (4)$$

The proof appears in Appendix. By using this two sided ballot theorem, it is easy to count the number of paths from $(1,0)$ to (n,k) which are above level 0 and below level k until the last step. We could then find generating functions of the type given in (3), with coefficients expressed in terms of a summation involving the counts. Alternatively, we could use the fundamental matrix approach to obtain the coefficients of the generating function. We next apply the two sided ballot theorem result to Case 3 of Section 2.

There we had counts involving the Fibonacci numbers and considered paths from $(0,1)$ to $(2i+1,0)$, $i=1,2,\ldots,$ which were above level 0 and below level 5. We observe that each such path passes through $(2i-1,2)$. Shifting down one unit gives paths from $(0,0)$ to $(2i-1,1)$ which are above -1 and below 4. So take $a=1$, $b=4$, $n=2i-1$, and $x=1$ in the two sided ballot theorem. This gives an expression for F_{2i-1}. Similarly we get an expression for F_{2i-2} by taking $a=1$, $b=4$, $n=2i-2$, and $x=2$ in the two sided ballot theorem.

Proposition 3.2. *For $i=1,2,\ldots,$ we have*

$$F_{2i-1}=\binom{2i-1}{i}-\binom{2i-1}{i+1}-\binom{2i-1}{i-4}+\binom{2i-1}{i+5}+\binom{2i-1}{i-5}-\binom{2i-1}{i+6}-\binom{2i-1}{i-9}+\ldots. \tag{5}$$

For $i=2,3,\ldots,$ we have

$$F_{2i-2}=\binom{2i-2}{i}-\binom{2i-2}{i+1}-\binom{2i-2}{i-4}+\binom{2i-2}{i+5}+\binom{2i-2}{i-5}-\binom{2i-2}{i+6}-\binom{2i-2}{i-9}+\ldots. \tag{6}$$

The expression in Property 3.2 is an infinite sum but becomes finite for any given value of i because the binomial coefficients all become zero after some number of steps.

Example 3.1.

$$\begin{aligned}F_{50}=F_{2i-2}|_{i=26}&=\binom{50}{26}-\binom{50}{27}-\binom{50}{22}+\binom{50}{31}+\binom{50}{21}-\binom{50}{32}-\binom{50}{17}\\&+\binom{50}{36}+\binom{50}{16}-\binom{50}{37}-\binom{50}{12}+\binom{50}{41}+\binom{50}{11}-\binom{50}{42}-\binom{50}{7}\\&+\binom{50}{46}+\binom{50}{6}-\binom{50}{47}-\binom{50}{2}+\binom{50}{1}=12586269025\,.\end{aligned}$$

4. Conclusion

We have given methods of generating a class of branching processes which all have the same limiting probability. We recognize that there are many more branching processes with the same limiting probability but finding them is a subject of future research. We have successfully applied the two sided ballot theorem to our setting. We have found counts of the number of restricted paths from $(0,1)$ to $(2i+1,0)$ or (2i, 5) subject to the restriction that the path is above 0 an below 5 until the last step. Surprisingly, these counts turn out to be Fibonacci numbers. We then used the two sided ballot theorem to obtain a new expression for Fibonacci numbers.

Acknowledgments

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References

[1] William Feller, *An Introduction to Probability Theory and its Applications*, Volume 1, Third Edition, Revised, John Wiley and Sons (1970).

[2] T.V. Narayana, *Lattice Path Combinatorics with Statistical Applications*, University of Toronto Press (1979).

[3] Sheldon Ross, *Introduction to Probability Models*, Ninth Edition, Academic Press (2006).

[4] N.N. Vorob'ev, *Fibonacci Numbers* Blaisdell Publishers (1961).

Appendix

Proof of the Two Sided Ballot Theorem. Let N, N_a, N_b, N_{ab}, N_{ba}, N_{aba}, $\ldots$ represent the number of paths from $(0,0)$ to (n,x) with restrictions of the following types respectively:

(a) N has no restrictions;

(b) N_a requires paths to touch $-a$ at some time;

(c) N_b requires paths to touch b at some time;

(d) N_{ab} requires paths to touch $-a$ followed by b at some time;

(e) N_{aba} requires paths to touch $-a$, b, $-a$ at some time in that order. This would include paths such as $\dots, b, \dots, -a, \dots, -a, \dots, b, \dots, b, \dots, -a, \dots$.

With this notation, the total number of paths from $(0,0)$ to (n,x) which are always below b and above $-a$ is

$$\begin{aligned} k =& N - N_a - N_b + N_{ab} + N_{ba} - N_{aba} - N_{bab} + N_{abab} + N_{baba} - \dots \\ =& \binom{n}{(n+x)/2} \\ &- \binom{n}{(n+x+2a)/2} - \binom{n}{(n+x-2b)/2} \\ &+ \binom{n}{(n+x+2a+2b)/2} + \binom{n}{(n+x-2a-2b)/2} \\ &- \binom{n}{(n+x+4a+2b)/2} - \binom{n}{(n+x-2a-4b)/2} \\ &+ \binom{n}{(n+x+4a+4b)/2} + \binom{n}{(n+x-4a-4b)/2} \\ &- \binom{n}{(n+x+8a+4b)/2} - \binom{n}{(n+x-4a-8b)/2}. \end{aligned}$$

□