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SOME PROPERTIES OF GENERALIZED SUBADDITIVE FUNCTIONS

Bogdan Choczewski[1], Zbigniew Powązka[2 §]

[1]Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30, Cracow, PL-30-059, POLAND
e-mail: smchocze@cyf-kr.edu.pl

[2]Institute of Mathematics
Pedagogical University of Cracow
Podchorążych 2, Cracow, PL-30-084, POLAND
e-mail: powazka@ap.krakow.pl

Abstract: By a generalized subadditive (GS) function we mean a solution ψ of the functional inequality $\psi(x+y) \leq F(\psi(x), \psi(y))$ $(F_{\leq})$. We are interested in those ψ which are bounded by a solution of the equation $(F_{=})$ (a generalized additive (GA) function).

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1. Introduction

We start with the following

Theorem 1. *If $f : \mathbf{R} \to \mathbf{R}$ satisfies the inequality*

$$f(x+y) \leq f(x) + f(y), \quad x, y \in \mathbf{R}, \tag{1}$$

and there exists an additive function $A : \mathbf{R} \to \mathbf{R}$, i.e.,

$$A(x+y) = A(x) + A(y), \quad x, y \in \mathbf{R}, \tag{2}$$

such that

$$f(x) \leq A(x). \quad x \in \mathbf{R}, \tag{3}$$

then $f = A$.

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§Correspondence author

Proof. On putting $x = 0$ in (1) we get $f(0) \geq 0$, whereas (3) yields $f(0) \leq A(0) = 0$ (cf. (2)), whence $f(0) = 0$. Take then $y = -x$ in (1) to get the inequality $0 = f(0) \leq f(x) + f(-x)$ and use this, together with (3), to estimate:

$$f(-x) \geq -f(x) \geq -A(x) = A(-x), \quad x \in \mathbf{R}$$

(thanks to (2)). This yields (in $\mathbf{R}$) the inequality opposite to (3), so that $f = A$. □

Remark 1. When $A = id_R$, the problem considered in Theorem 1 was proposed to solve on American Mathematical Olympiad in 1979, cf. Pawłowski [5], p. 79. This finding of the second author motivated us to come back to the topic we had studied in 1980, cf. Choczewski and Powązka [2].

2. Properties of GA Functions

In the whole paper we denote by I an open interval, $I := (\alpha, \beta) \subset \mathbf{R}$, and we assume that:

(H) The continuous function $F : I^2 \to I$ is a group operation on I, with e standing for the neutral element of the group (I, F).

By a *generalized additive (GA)* function is meant a solution $\varphi : \mathbf{R} \to I$ of the equation

$$\varphi(x + y) = F(\varphi(x), \varphi(y)), \quad x, y \in \mathbf{R}. \qquad (F_=)$$

Lemma 1. (see Aczél [1], p. 54) *Hypothesis* **(H)** *forces the selfmappings $F(x, \cdot)$ and $F(\cdot, y)$ of I to be strictly increasing functions, for every x, resp. y, from I.*

In the next lemma we collect a few properties of *CGA functions*, i.e., *GA* functions continuous in $\mathbf{R}$.

Lemma 2. (see Aczél [1], p. 57) *Let φ be a nonconstant CGA function. Then:*

i) Hypothesis (**H**) *is satisfied.*

ii) The function φ is monotonic (on $\mathbf{R}$).

iii) The function F is of the form

$$F(u, v) = \varphi(\varphi^{-1}(u) + \varphi^{-1}(v)), \quad u, v \in I. \qquad (4)$$

iv) All CGA functions φ_c are given by the formula

$$\varphi_c(x) = \varphi(cx), \quad x \in \mathbf{R}. \qquad (5)$$

Moreover, if hypothesis (**H**) *is satisfied, then there exists a nonconstant CGA function.*

3. A Property of CGS Functions

In this section we consider continuous solutions $\psi : \mathbf{R} \to I$ of the inequality (the *CGS* functions)

$$\psi(x+y) \leq F(\psi(x), \psi(y)), \quad x, y \in \mathbf{R}. \tag{$F_\leq$}$$

We start with recalling our result from [2]:

Lemma 3. *Assume* **(H)**. *If there exists a CGA function φ which is strictly increasing then every solution $\psi : \mathbf{R} \to I$ of inequality $(F_\leq)$ is of the form*

$$\psi(x) = \varphi(f(x)), \quad x \in \mathbf{R}, \tag{6}$$

where $f : \mathbf{R} \to I$ satisfies inequality (1) and is continuous on $\mathbf{R}$.

The following result is an analogue of Theorem 1.

Theorem 2. *Assume* **(H)**. *If ψ is a CGS function,*

$$\psi(0) = e, \tag{7}$$

and there is a CGA function φ such that

$$\psi(x) \leq \varphi(x), \quad x \in \mathbf{R}, \tag{8}$$

then $\psi = \varphi$.

Proof. Put $x = y = 0$ in $(F_=)$ to get $\varphi(0) = F(\varphi(0), \varphi(0))$, whence $\varphi(0) = e$, i.e.,

$$\psi(0) = \varphi(0). \tag{9}$$

We shall prove that in (8) only equality is the case. For, assume there is a $t \in \mathbf{R}$ for which $\varphi(t) > \psi(t)$. Put $x = t$ and $y = -t$ in $(F_\leq)$ and $(F_=)$. Thanks to (9) and (7) we obtain

$$F(\psi(t), \psi(-t)) \geq \psi(0) = \varphi(0) = F(\varphi(t), \varphi(-t)) > F(\psi(t), \varphi(-t)),$$

as $F(\cdot, \varphi(-t))$ is strictly increasing (cf. Lemma 1). But $F(\psi(t), \cdot)$ is also strictly increasing, consequently $\psi(-t) > \varphi(-t)$ for the $-t \in \mathbf{R}$, which contradicts inequality (8). Thus $\psi = \varphi$, as claimed.

Remark 2. Theorem 2 has a counterpart for generalized superadditive functions ψ – those which satisfy inequality $(F_\geq)$). To get the same assertion for such a ψ it is enough to assume that ψ is bounded from below by a *CGA* function.

However, Theorem 2 itself would not be true if in (8) the inequality sign ”$\leq$” were replaced by ”$\geq$”. For, the subadditive continuous function $f(x) = |x|, x \in \mathbf{R}$, is bounded below by the continuous additive function $A(x) = \frac{1}{2}x, x \in \mathbf{R}$, but f is not additive.

4. Differentiability of CGA Functions

In Powązka [5] the following fact is proved.

Proposition 1. *Assume* (**H**). *A CGA function is differentiable (in* **R***) if and only if it is differentiable at* $x = 0$.

We supply a condition for the function F equivalent to the differentiability of a CGA function at a point.

Theorem 3. *Assume* (**H**). *A CGA function* φ *is differentiable at a point* $t \in \mathbf{R}$ *if and only if the function* $\Phi_t : \mathbf{R} \to I$ *given by the formula*

$$\Phi_t(x) = F(\varphi(t), \varphi(x)), \quad x \in \mathbf{R}, \tag{10}$$

is differentiable at $x = 0$.

Proof. Since $\varphi(0) = e$ for every GA function φ, we obtain from $(F_=)$ and (10):

$$\varphi'(t) = \lim_{h\to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \lim_{h\to 0} \frac{F(\varphi(t), \varphi(h)) - F(\varphi(t), \varphi(0))}{h}$$

$$= \lim_{h\to 0} \frac{\Phi_t(h) - \Phi_t(0)}{h} = \Phi_t'(0).$$

Thus the existence of $\varphi'(t)$ is equivalent to the existence of $\Phi_t'(0)$. □

5. Bounds for CGS Functions

Finally, we look for conditions under which a continuous solution ψ of inequality $(F_\leq)$ (a *CGS* function) is bounded by a *CGA* function φ.

In Ilse and al [3] we find the following properties of *CGA* functions.

Lemma 4. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a continuous subadditive function (i.e., a solution to inequality (1)) such that* $f(0) = 0$ *and*

$$\sup\{t^{-1}f(t), t > 0\} = a \quad (\textit{resp.}\ \inf\{t^{-1}f(t), t < 0\} = b), \tag{11}$$

with some $a, b \in \mathbf{R}$. *Then there exists the limit*

$$\lim_{x\to 0+} t^{-1}f(t) = a \quad (\textit{resp.}\ \lim_{x\to 0-} t^{-1}f(t) = b) \tag{12}$$

and $a \le b$.

We shall prove the following

Theorem 3. *Assume* (**H**). *Let a function* $\psi, \psi(0) = e$ *satisfy inequality* $(F_{\le})$, *i.e., be a CGS. If there is a strictly increasing CGA function* φ *such that the function* $f = \varphi^{-1} \circ \psi$ *satisfies the first (resp. the second) condition (11), then*

$$\psi(x) \le \varphi(ax), x > 0 \quad (resp.\ \psi(x) \ge \varphi(bx), x < 0). \tag{13}$$

Proof. Because of Lemma 2, the function $f = \varphi^{-1} \circ \psi$ is subadditive, so that Lemma 4 applies. Thus we get from (12)

$$\frac{\varphi^{-1}(\psi(x))}{x} \le \frac{f(x)}{x} \le \sup\{t^{-1}f(t), t > 0\} = a\,.$$

Multiplying both sides of these inequalities by $x > 0$, because of the strict monotonicity of φ we obtain the first inequality in (13). The proof of the second inequality in (13), based on the second inequality in (12), is analogous. □

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