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A GENERALIZATION OF D.W. BRESTER'S FORMULAE

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Abstract: Let be $G = G(x, m)$ defined by $\left(\sum_{i=1}^{\mu} x_i^2\right)^m - \left(\sum_{i=\mu+1}^{\mu+\nu} x_i^2\right)^m$, where m is integer postive and $\mu + \nu = n$ dimension of the space. In this paper we give a sense to residue of $(c^2 + G)_+^\lambda$ and $(c^2 + G)_-^\lambda$ at $\lambda = -k$, $k = 1, 2, ...$, where $(c^2 + G)_+^\lambda$ is defined by (21) and $(c^2 + G)_-^\lambda$ by (22). Our formulae are generalization of the formulae (4) and (5) due to D.W. Bresters [2].

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1. Introduction

Let $c^2 + P$ be a quadratic form in n variables defined by

$$c^2 + P = c^2 + x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2 \,, \tag{1}$$

where c^2 is a real number and $\mu + \nu = n$ dimension of the space.

The distribution $(c^2 + P)_+^\lambda$ and $(c^2 + P)_-^\lambda$ are defined by

$$\left\langle (c^2 + P)_+^\lambda, \varphi \right\rangle = \int\limits_{c^2+P>0} (c^2 + P)^\lambda \varphi(x) dx \tag{2}$$

and

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$$\left\langle (c^2+P)_-^\lambda, \varphi \right\rangle = \int\limits_{c^2+P<0} (-(c^2+P))^\lambda \varphi(x)dx\,, \tag{3}$$

where λ is a complex numbers and φ is in K (set of all real functions with continuous derivatives of all order and bounded support (see [1], p. 2).

D.W. Brester in [2] showed that the following formulae

$$\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots}(c^2+P)_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(c^2+P) \tag{4}$$

and

$$\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots}(c^2+P)_-^\lambda = \frac{1}{(k-1)!}\delta^{(k-1)}(c^2+P). \tag{5}$$

The formula (5) can be rewrite in the following form

$$\delta^{(k-1)}(-c^2-P) = (-1)^{k-1}(k-1)!\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots}(c^2+P)_-^\lambda. \tag{6}$$

On the other hand, A. Kananthai and K. Nonloapan in (see [3], pp. 49-57) introduce the generalized function G^λ, where

$$G = G(m,x) = \left(\sum_{i=1}^{\mu} x_i^2\right)^m - \left(\sum_{i=\mu+1}^{\mu+\nu} x_i^2\right)^m \tag{7}$$

and m is positive integer.

They found that G^λ has two sets of singularities, namely

$$\lambda = -1. -2, \ldots \tag{8}$$

and

$$\lambda = -\frac{n}{2m}, -\frac{n}{2m} - 1, \ldots \tag{9}$$

Therefore from [3], pp. 49-55, we have the following result: *For odd n and for even if $k < \frac{n}{2m}$, the generalized function G_+^λ has simple pole at $\lambda = -k, k = 1, 2, \ldots$, where the residue are*

$$\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots} G_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!}\delta_1^{(k-1)}(G)\,, \tag{10}$$

where

$$\delta_1^{(k-1)}(G) = \delta^{(k-1)}(G) \tag{11}$$

and

$$\left\langle \delta^{(k-1)}(G), \varphi \right\rangle = \int\limits_o^\infty \left[\left(\frac{1}{2ms^{2m-1}} \cdot \frac{\partial}{\partial s} \right)^{k-1} \left\{ s^{q-2m} \frac{\Psi(r,s)}{2m} \right\} \right]_{s=r} r^{p-1} dr. \tag{12}$$

In (12)

$$\Psi(r,s)=\int\varphi d\Omega_\mu d\Omega_\nu, \tag{13}$$

$d\Omega_\mu$ and $d\Omega_\nu$ are elements of surface area on the unit sphere in R^μ and R^ν respectively.

If $k-1<\frac{n}{2m}-1$, the meaning of $\delta_1^{(k-1)}(G)$ is given from the regularization of $\delta^{(k-1)}(G)$ (see [3], p. 53). Therefore we have the following formulae

$$\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots}G_+^\lambda=\frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(G) \tag{14}$$

under conditions:

$$\begin{array}{ll} \text{a)} & n \text{ odd and;} \\ \text{b)} & n \text{ even and } k<\frac{n}{2m}; \end{array} \tag{15}$$

and

$$\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots}G_+^\lambda=\frac{(-1)^{k-1}}{(k-1)!}\delta_1^{(k-1)}(G) \text{ if } n \text{ is even and } k\geq\frac{n}{2m}, \tag{16}$$

where $\delta_1^{(k-1)}(G)$ is in the sense of regularization of $\delta^{(k-1)}(G)$.

M. Agurre in [4] obtain the following formulae:

$$\begin{aligned}\operatorname{Re}s_{\lambda=-\frac{n}{2m}-k}G_+^\lambda&=\frac{\Gamma(\frac{n}{2m})}{(2m)^{2k}\Gamma(\frac{n}{2m}+k)}\cdot\frac{(-1)^k}{(\frac{2}{m}(m-1)-1)\ldots(\frac{2}{2}(m-1)-k)}\\&\times\frac{\Gamma(\frac{\mu}{2m})\Gamma(1-\frac{n}{2m})\pi^{\frac{n}{2}}}{m^2\Gamma(1-\frac{\mu}{2m})\Gamma(\frac{p}{2})}L_m^k\{\delta\} \text{ if } \mu=2ms+m \text{ and } n \text{ odd}\end{aligned} \tag{17}$$

(see [4], formula (31)),

$$\operatorname{Re}s_{\lambda=-\frac{n}{2m}-k}G_+^\lambda=0 \text{ if } n \text{ is odd, } \mu=2ms \text{ and } \nu \text{ odd} \tag{18}$$

(see [4], formula (37)),

$$\begin{aligned}&\operatorname{Res}_{\lambda=-\frac{n}{2m}-k,k=0,1,2,\ldots}G_+^\lambda\\&=\frac{(-1)^{\frac{n}{2m}+k-1}}{\Gamma(\frac{n}{2m}+k)}\delta_1^{(\frac{n}{2m}+k-1)}(G)+\frac{\pi^{\frac{n}{2}}\Gamma(\frac{\nu}{2m})(-1)^{\frac{\nu}{2m}}\Gamma(\frac{\mu}{2m})}{m^2\pi\Gamma(\frac{\mu}{2})\Gamma(\frac{\nu}{2})\Gamma(\frac{\mu+\nu}{2m})}\times\\&\frac{1}{k!(2m)^{2k}(\frac{n}{2m}+\frac{2(m-1)}{m})(\frac{n}{2m}+\frac{2(m-1)}{m}+1)\ldots(\frac{n}{2m}+\frac{2(m-1)}{m}+k-1)}L_m^k\{\delta\}\end{aligned} \tag{19}$$

if n is even, $\mu=2ms$, $\nu=2ml$; $s,l=0,1,2,\ldots$ (see [4], formula (106)) and

$$\begin{aligned}\operatorname*{Res}_{\lambda=-\frac{n}{2m}-k,k=0,1,2,\dots} G_+^\lambda &= \frac{(-1)^{\frac{n}{2m}+k-1}}{\Gamma(\frac{n}{2m}+k)}\delta_1^{*(\frac{n}{2m}+k-1)}(G)\\ &+\frac{\pi^{\frac{n}{2}-1}\Gamma(\frac{\nu}{2m})(-1)^{\frac{\nu-1}{2m}}\Gamma(\frac{\nu}{2m})}{m^2\pi\Gamma(\frac{\mu}{2})\Gamma(\frac{\nu}{2})\Gamma(\frac{\mu+\nu}{2m})}\times\left\{\left[\psi(\frac{p}{2m})-\psi(\frac{n}{2m})\right]\right\}\\ &\times\frac{1}{k!(2m)^{2k}(\frac{n}{2m}+\frac{2(m-1)}{m})(\frac{n}{2m}+\frac{2(m-1)}{m}+1)\dots(\frac{n}{2m}+\frac{2(m-1)}{m}+k-1)}\\ &\times L_m^k\left\{\delta\right\}. \quad (20)\end{aligned}$$

if n is even, $\mu=2ms+1$, $\nu=2ml+1$; $s,l=0,1,2,\dots$ (see [4], formula (108)).

Now considering the distribution $(c^2+G)_+^\lambda$ and $(c^2+G)_-^\lambda$ defined by

$$\left\langle (c^2+G)_+^\lambda,\varphi\right\rangle = \int\limits_{c^2+G>0}(c^2+G)^\lambda\varphi(x)dx \quad (21)$$

and

$$\left\langle (c^2+G)_-^\lambda,\varphi\right\rangle = \int\limits_{c^2+G<0}(-(c^2+G))^\lambda\varphi(x)dx\,, \quad (22)$$

where

$$(c^2+G)=(c^2+G)(x)=c^2+G(x,m) \quad (23)$$

and $G=G(x,m)$ is defined by (7).

In this paper we give a sense to residue of $(c^2+G)_+^\lambda$ and $(c^2+G)_-^\lambda$ at $\lambda=-k, k=1,2,\dots$.

Our formulae are generalization of the formulae (4) and (5).

In order to do it we need the following formula

$$\delta^{(k-1)}(u(x)-t)=\sum_{l\geq 0}\frac{1}{l!}\delta^{(k-1+l)}(u(x))(-t)^l \quad (24)$$

(see [5], p. 90, formula (46)), where $u(x)=u(x_1,x_2,\dots x_n)\in C^\infty(R^n)$, t is a real number, $u(x)-t=0$ has not critical point and $\delta^{(k-1)}(v(x))$ is defined by

$$\begin{aligned}\left\langle \delta^{(k-1)}(v(x)),\varphi\right\rangle &= (-1)^{k-1}\int\limits_{\nu(x)=0} f_{u_1}^{(k-1)}(0,u_2,\dots u_n)du_2\dots du_n\\ &= (-1)^{k-1}\int\limits_{\nu(x)=0}\left[\frac{\partial^{k-1}}{\partial u_1^{k-1}}f_{u_1}(0,u_2,\dots u_n)\right]_{u_1=0}du_2\dots du_n, \quad (25)\end{aligned}$$

(see [5], p. 87, formula (19)),

$$f(u_1, u_2, \dots u_n) = \varphi_1(u_1, u_2, \dots u_n) D\binom{x}{u}, \tag{26}$$

$$\varphi_1(u_1, u_2, \dots u_n) = \varphi(x_1, x_2, \dots x_n), \tag{27}$$

$$D\binom{x}{u} = \frac{1}{\frac{\partial\nu}{\partial x_1}} \text{ with } \frac{\partial\nu}{\partial x_1} > 0 \tag{28}$$

and

$$\begin{array}{c} u_1 = \nu(x_1, x_2, \dots x_n)\,, \\ u_2 = x_2\,, \dots\,, u_n = x_n\,. \end{array} \tag{29}$$

Also, we need the following formula,

$$(c^2 + G)^\lambda = \sum_{l \geq 0} \binom{\lambda}{l} (c^2)^l G^{\lambda - l} \tag{30}$$

if $c^2 < G$, where λ is a complex numbers,

$$\binom{\lambda}{l} = \frac{\Gamma(\lambda + 1)}{l!\Gamma(\lambda - l + 1)} = \frac{(-1)^l \Gamma(-\lambda + l)}{l!\Gamma(-\lambda)} \tag{31}$$

and

$$\operatorname{Re} s_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!} \tag{32}$$

$k = 0, 1, 2, \dots$ (see [6], p. 2).

2. Residue of $(c^2 + G)_+^\lambda$ and $(c^2 + G)_-^\lambda$ at $\lambda = -k, k = 1, 2, \dots$

Putting $u(x_1, x_1, ..X_n) - t = G + c^2 = G(x, m) + c^2$ in (24) we have

$$\delta^{(k-1)}(G + c^2) = \sum_{l \geq 0} \frac{1}{l!} \delta^{(k+l-1)}(G)(c^2)^l. \tag{33}$$

Now taking into account the formulae (14)-(20), from (33) we have

$$\delta^{(k-1)}(G + c^2) = \sum_{l \geq 0} \frac{(c^2)^l}{l!} \frac{(k + l - 1)!}{(-1)^{k+l-1}} \operatorname{Re} s_{\lambda = -(k+l)} G_+^\lambda. \tag{34}$$

On the other hand from (30) and using (31) and (32) we have

$$\operatorname{Re} s_{\lambda=-k} (c^2 + G)_+^\lambda = \sum_{l \geq 0} (c^2)^l \left. \binom{\lambda}{l} \right|_{\lambda=-k} \operatorname{Re} s_{\lambda=-k} G_+^{\lambda - l}$$

$$= \sum_{l\geq 0} \frac{(-1)^l (c^2)^l (k+l-1)!}{l!(k-1)!} \operatorname{Re} s_{\alpha=-(k+l)} G_+^{\alpha} \,, \quad (35)$$

where

$$\left\langle G_+^{\lambda}, \varphi \right\rangle = \int_{G>0} G^{\lambda} \varphi(x) dx \quad (36)$$

(see [4]).

From(34) and (35) we arrive at the following formula

$$\operatorname{Re} s_{\lambda=-k} (c^2+G)_+^{\lambda} = \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(G+c^2) \,, \quad (37)$$

where $G = G(x, m)$ is defined by (7).

Our formula (37) can be also obtained using the Dfinitions 2 and (12), and by making the transformation to bipolar coordinate defined by

$$\begin{aligned} &x_1 = r\omega_1 \,, \dots \,, x_\mu = r\omega_\mu \,, \\ &x_{\mu+1} = s\omega_{\mu+1} \,, \dots \,, x_{\mu+\nu} = s\omega_{\mu+\nu} \,, \end{aligned} \quad (38)$$

where

$$r = \sqrt[2]{x_1^2 + x_2^2 + \cdots + x_\mu^2} \quad (39)$$

and

$$s = \sqrt[2]{x_{\mu+1}^2 + x_{\mu+2}^2 + \cdots + x_{\mu+\nu}^2}. \quad (40)$$

Similarly from (24) we have

$$\delta^{(k-1)}(-G-c^2) = \sum_{l\geq 0} \frac{1}{l!} \delta^{(k+l-1)}(-G)(-c^2)^l. \quad (41)$$

and considering all the above G_+^{λ} remains true also for G_-^{λ} except that p and q must be interchange, and in all the formulae $\delta_1^{(k-1)}(G)$ must be replaced by $\delta_2^{(k-1)}(-G) = (-1)^{k-1}\delta_1^{(k-1)}(G)$ and L_m by $(-L_m)$ (see [4]), from (14)-(20) we have

$$\delta^{(k-1)}(-G-c^2) = \sum_{l\geq 0} \frac{(-1)^l (c^2)^l}{l!} \frac{(k+l-1)!}{(-1)^{k+l-1}} \operatorname{Re} s_{\lambda=-(k+l)} G_-^{\lambda} \,, \quad (42)$$

where

$$\left\langle G_-^{\lambda}, \varphi \right\rangle = \int_{-G<0} (-G)^{\lambda} \varphi(x) dx \,. \quad (43)$$

On the other hand, using (30), (31) and (32) we have

$$\operatorname{Re}s_{\lambda=-k}(c^2+G)_-^\lambda = \sum_{l\geq 0}(c^2)^l \binom{\lambda}{l}\Big|_{\lambda=-k} \operatorname{Re}s_{\lambda=-k} G_-^{\lambda-l}$$
$$= \sum_{l\geq 0}\frac{(c^2)^l(k+l-1)!}{l!(k-1)!}\operatorname{Re}s_{\alpha=-(k+l)} G_-^{\alpha}. \quad (44)$$

From (42) and (44) we arrive at the following formula

$$\delta^{(k-1)}(-G-c^2) = \frac{(k-1)!}{(-1)^{k-1}}\operatorname{Re}s_{\lambda=-k}(c^2+G)_-^\lambda \quad (45)$$

or equivalently we have the following formula

$$\operatorname{Re}s_{\lambda=-k}(c^2+G)_-^\lambda = \frac{1}{(k-1)!}\delta^{(k-1)}(G+c^2)\,, \quad (46)$$

where $G=G(x,m)$ is defined by (7).

Our formulae (37) and (46) are generalization of the formulae (4) and (5) respectively. In fact, letting $m=1$ in (37) and (43) and considering (7) we obtaing the formulae (4) and (5) respectively. The formulae (4) and (5) are due D.W. Bresters (see [2]).

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