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MAPS SATISFYING GENERALIZED CONTRACTIVE CONDITIONS OF INTEGRAL TYPE FOR WHICH $F(T) = F(T^n)$

B.E. Rhoades[1 §], M. Abbas[2]

[1]Department of Mathematics
Indiana University
Bloomington, IN 47405-7106, USA
e-mail: rhoades@indiana.edu

[2]Centre for Advanced Studies in Mathematics
Department of Mathematics
Lahore University of Management Sciences
Lahore, 54792, PAKISTAN
e-mail: mujahid@lums.edu.pk

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1. Introduction and Preliminaries

It is an obvious fact that if T is a map which has a fixed point p, then p is also a fixed point of T^n for every natural number n. However the converse is false. For example, consider, $X = [0, 1]$, and T defined by $Tx = 1-x$. Then T has a unique fixed point at $\frac{1}{2}$, but every even iterate of T is the identity map, which has every point of $[0, 1]$ as a fixed point. On the other hand, if $X = [0, \pi]$, $Tx = \cos x$, then every iterate of T has the same fixed point as T (see [4], [6], [13]). If a map satisfies $F(T) = F(T^n)$ for each $n \in N$, where $F(T)$ denotes the set of all fixed points of T, then it is said to have property P. The first author and G.S. Jeong [6] showed that maps satisfying many contractive conditions have property P. They have [7] also shown that $F(S) \cap F(T) = F(S^n) \cap F(T^n)$ for a number of contractive conditions involving pairs of maps. The first author and Akgun [1]

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§Correspondence author

have shown that a map satisfying a Meir–Keeler type contractive condition of integral type has property P. Branciari [3] obtained a fixed point theorem for a single valued mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. The first author [12] proved two fixed point theorems involving more general contractive condition of integral type (see, also [2], [14], [15] and references therein). The aim of this paper is to present several results for maps defined on a metric space satisfying a contractive condition of integral type satisfying property P.

2. Periodic Point Theorems

Define $\digamma = \{\varphi : R^+ \longrightarrow R^+ : \varphi$ is a Lebesgue integral mapping which is summable, nonnegative and satisfies $\int_0^{\epsilon} \varphi(t)dt > 0$, for each $\epsilon > 0\}$. Let T be a function mapping a metric space X into itself. For $A \subseteq X$, set $\delta(A) = \sup\{d(x,y) : x, y \in A\}$ and $O(x,n) = \{x, Tx, \dots, T^n x\}$. The set $O(x,\infty) = \{x, Tx, T^2x, \dots\}$ is called the *orbit of* x, see [5]. We say that T is orbitally continuous at $p \in X$ if $\lim_{k\to\infty} T^{n_k}x = p$ implies that $\lim_{n\to\infty} T^{n_k+1}x = Tp$. A function G mapping X into the reals is $T-$ orbitally lower semicontinuous at p if $\{x_n\}$ is a sequence in $O(x,\infty)$ and $x_n \to p$ implies that $G(p) \le \liminf G(x_n)$.

Theorem 1. *Let T be a self map of a complete metric space (X, d) satisfying*

$$\int_0^{d(Tx,T^2x)} \varphi(t)dt \le \lambda \int_0^{d(x,Tx)} \varphi(t)dt \tag{2.1}$$

for all $x \in X$, where $\varphi \in \digamma$ and $\lambda \in [0,1)$. Then there exists a p in X such that $p \in F(T)$

Proof. Consider,

$$\int_0^{d(T^2x,T^3x)} \varphi(t)dt \le \lambda \int_0^{d(Tx,T^2x)} \varphi(t)dt \le \lambda^2 \int_0^{d(x,Tx)} \varphi(t)dt.$$

Using induction,

$$\int_0^{d(T^nx,T^{n+1}x)} \varphi(t)dt \le \lambda^n \int_0^{d(x,Tx)} \varphi(t)dt,$$

which implies that $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$. Following an argument similar to that given in [3], we conclude that $\{T^n x\}$ is a Cauchy sequence in X and hence there exists a point p in X such that $T^n x \to p$ as $n \to \infty$. Now suppose that T is orbitally lower semicontinuous at p. Then, $d(p, Tp) \leq \liminf d(T^n x, T^{n+1}x)$, which implies that

$$\int_0^{d(p,Tp)} \varphi(t)dt \leq \int_0^{\liminf d(T^n x, T^{n+1}x)} \varphi(t)dt \to 0,$$

as $n \to \infty$. Hence $p \in F(T)$. The converse is immediate. □

Remark 1. Theorem 1 generalizes the main result of [5]. Moreover if T is a self map of a complete metric space $(X,\ d)$ satisfying $\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \lambda \int_0^{d(x,y)} \varphi(t)dt$ for all $x, y \in X$, where $\varphi \in F$ and $\lambda \in [0, 1)$, then T has property P. Also, if there exists a point u in X such that the orbit $O(x_0, \infty)$ has a cluster point $p \in X$, T is orbitally continuous at p and Tp, and

$$\int_0^{d(Tx,Ty)} \varphi(t)dt < \int_0^{d(x,y)} \varphi(t)dt$$

for all $x, y = Tx \in \overline{O(x_0, \infty)}$, $x \neq y$, where $\varphi \in F$, then $p \in F(T)$.

Theorem 2. *Let T be an orbitally lower semicontinuous self map of a complete metric space $(X,\ d)$ satisfying (2.1) either (i) for all $x \in X$, or (ii) with strict inequality, $\lambda = 1$, and for all $x \in X$, $x \neq Tx \in \overline{O(x_0, \infty)}$, where x_0 is a point in X such that the orbit $O(x_0, \infty)$ has a cluster point $p \in X$, T is orbitally continuous at p and Tp, then T has property P.*

Proof. If (i) is satisfied, then T has a fixed point by Theorem 1. If (ii) is satisfied, then T has a fixed point by the remark following Theorem 1. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $u \in F(T^n)$. Suppose that T satisfies (i). Then

$$\begin{aligned} \int_0^{d(u,Tu)} \varphi(t)dt &= \int_0^{d(T(T^{n-1}u),T^2(T^{n-1}u))} \varphi(t)dt \leq \lambda \int_0^{d(T^{n-1}u,T^n u)} \varphi(t)dt \\ &\leq \lambda^2 \int_0^{d(T^{n-2}u,T^{n-1}u)} \varphi(t)dt \leq \cdots \leq \lambda^n \int_0^{d(u,Tu)} \varphi(t)dt, \end{aligned}$$

which, on taking the limit as $n \to \infty$, implies that $d(u, Tu) = 0$, and hence $u = Tu$. Suppose that T satisfies (ii). If $u = Tu$, then there is nothing to prove. Suppose, if possible, that $u \neq Tu$. Then a repetition of the argument for case (i) leads to

$$\int_0^{d(u,Tu)} \varphi(t)dt < \int_0^{d(u,Tu)} \varphi(t)dt,$$

which implies that $u = Tu$, a contradiction. Therefore, in all cases, $u = Tu$ and $F(T^n) = F(T)$. □

Theorem 3. *Let T be a self map of a metric space (X, d) satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \int_0^{d(x,y)} \varphi(t)dt - \int_0^{\Psi(d(x,y))} \varphi(t)dt \tag{2.2}$$

for all $x, y \in X$, $x \neq y$, where $\varphi \in F$ and $\Psi : R^+ \longrightarrow R^+$ is continuous and nondecreasing such that Ψ is positive on $R^+, \Psi(0) = 0$. Then $F(T) \neq \phi$ and T has a property P.

Proof. First, we show that T has a fixed point. For this, suppose that x_0 is an arbitrary point of X, and $\{x_n\}$ is a Picard iterative sequence; that is, $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$. Suppose that, for any n, $x_{n+1} \neq x_n$, since, otherwise, T has a fixed point. From (2.2), we have

$$\int_0^{d(x_{n+1},x_n)} \varphi(t)dt = \int_0^{d(Tx_n,Tx_{n-1})} \varphi(t)dt$$
$$\leq \int_0^{d(x_n,x_{n-1})} \varphi(t)dt - \int_0^{\Psi(d(x_n,x_{n-1}))} \varphi(t)dt < \int_0^{d(x_n,x_{n-1})} \varphi(t)dt,$$

which implies that $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$. Therefore $\{d(x_{n+1}, x_n)\}$ is a monotone decreasing positive sequence with limit $\alpha \geq 0$. Suppose that $\alpha > 0$. Then,

$$0 < \int_0^{\alpha} \varphi(t)dt \leq \int_0^{d(x_{n+1},x_n)} \varphi(t)dt$$
$$\leq \int_0^{d(x_n,x_{n-1})} \varphi(t)dt - \int_0^{\Psi(d(x_n,x_{n-1}))} \varphi(t)dt \leq \int_0^{d(x_n,x_{n-1})} \varphi(t)dt - \int_0^{\Psi(\alpha)} \varphi(t)dt\,.$$

Taking the limit of both sides of the above inequality as $n \to \infty$ yields

$$\int_0^{\alpha} \varphi(t)dt \le \int_0^{\alpha} \varphi(t)dt - \int_0^{\Psi(\alpha)} \varphi(t)dt,$$

a contradiction. Therefore $\alpha = 0$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose, if possible, that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ and subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k < m_k$ such that $d(x_{n_k}, x_{m_k}) \ge 2\varepsilon$ for each k. Since $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$, for large enough k, we have $d(x_{n_k+1}, x_{n_k}) < \frac{\varepsilon}{2}$ and $d(x_{m_k+1}, x_{m_k}) < \frac{\varepsilon}{2}$. Now,

$$\begin{aligned} d(x_{n_k+1}, x_{m_k}) &\ge d(x_{n_k}, x_{m_k}) - d(x_{n_k+1}, x_{n_k}) > \varepsilon, \\ d(x_{n_k}, x_{m_k-1}) &\ge d(x_{n_k}, x_{m_k}) - d(x_{m_k-1}, x_{m_k}) > \varepsilon, \end{aligned}$$

and

$$d(x_{n_k+1}, x_{m_k-1}) \ge d(x_{n_k}, x_{m_k}) - d(x_{m_k-1}, x_{m_k}) - d(x_{n_k+1}, x_{n_k}) > \varepsilon.$$

We may assume that the n_k are even and that the m_k are odd and, that $d(x_{n_k}, x_{m_k}) > \varepsilon$. for all k. Set $r_k = \min\{m_k : d(x_{n_k}, x_{m_k}) > \varepsilon$ and m_k is odd number$\}$. Now,

$$\varepsilon < d(x_{n_k}, x_{r_k}) \le d(x_{n_k}, x_{r_k-2}) + d(x_{r_k-2}, x_{r_k-1},) + d(x_{r_k-1}, x_{r_k}) \to \varepsilon$$

as $k \to \infty$. Thus $d(x_{n_k}, x_{r_k}) \to \varepsilon$, as $k \to \infty$. Also,

$$\begin{aligned} &d(x_{n_k}, x_{r_k}) - d(x_{n_k}, x_{n_k+1}) - d(x_{r_k}, x_{r_k+1}) \\ &\qquad \le d(x_{n_k+1}, x_{r_k+1}) \le d(x_{n_k}, x_{r_k}) + d(x_{n_k}, x_{n_k+1}) + d(x_{r_k}, x_{r_k+1}), \end{aligned}$$

and, $d(x_{n_k+1}, x_{r_k+1}) \to \varepsilon$, as $k \to \infty$. Now,

$$\begin{aligned} \int_0^{d(x_{n_k+1}, x_{r_k+1})} \varphi(t)dt &= \int_0^{d(Tx_{n_k}, Tx_{r_k})} \varphi(t)dt \\ &\le \int_0^{d(x_{n_k}, x_{r_k})} \varphi(t)dt - \int_0^{\Psi(d(x_{n_k}, x_{r_k}))} \varphi(t)dt \le \int_0^{d(x_{n_k}, x_{r_k})} \varphi(t)dt - \int_0^{\Psi(\varepsilon)} \varphi(t)dt\,. \end{aligned}$$

Taking the limit as $k \to \infty$, yields

$$\int_0^{\varepsilon} \varphi(t)dt \le \int_0^{\varepsilon} \varphi(t)dt - \int_0^{\Psi(\varepsilon)} \varphi(t)dt,$$

a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a point in p in X such that $x_n \to p$ as $n \to \infty$. Then $\lim_{n\to\infty} \Psi(d(x_n, p)) = 0$ and inequality (2.2) imply that p is fixed point of T.

Moreover, p is the unique fixed point of T. The proof that T has property P is the same as the corresponding proof in Theorem 2. □

Corollary 4. *Let T be a self map of symmetric space $(X,\, d)$ satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \lambda \int_0^{M(x,y)} \varphi(t)dt, \tag{2.3}$$

where, $\varphi \in F, 0 \leq \lambda < 1$, and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\},$$

then T has a property P.

Proof. That T has a unique fixed point follows from [12]. Note that,

$$M(x,Tx) = \max\{d(x,Tx), d(Tx,T^2x)\}.$$

If $M(x,Tx) = d(Tx,T^2x)$ for some x, then substituting into (2.3) yields a contradiction. Therefore $M(x,Tx) = d(x,Tx)$, and (2.3) becomes (2.1). The result then follows form Theorem 1. □

Remark 2. Every contractive condition of integral type automatically includes a corresponding contractive condition, not involving integrals, by setting $\varphi(t) = 1$ over R^+. So this paper generalizes Theorem 1 of [8], Theorem 1 of [11], Theorem 1.1, Corollaries 1.1-1.10, Propositions $1.4 - 1.7$ of [6] along with the references in [6] associated with these results.

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