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STRUCTURE THEOREMS AND STATISTICAL APPLICATION FOR MATRIX RINGS OVER MOORE-PENROSE TWO (MP2) RINGS

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Abstract: The mathematicians Edwin Moore [2] and Roger Penrose [3] authored the Moore-Penrose conditions which assert that given any nonzero matrix A over the complex field, there exists a nonzero matrix X such that: (1) AXA = A; (2) XAX = X; (3) $(XA)^* = XA$; (4) $(AX)^* = AX$. This paper generalizes the second Moore-Penrose condition to an arbitrary ring R which will be called MP2 as follows: Given any nonzero element a in R, there exists a nonzero x in R such that xax = x. Accordingly, the structure theorems for such MP2 rings are developed, as well as the structure theorems for matrix rings over them. Interestingly enough, MP2 rings appear frequently in physical chemistry for converting linear operators to symmetric ones, and in engineering applications for solving unstable linear systems, or in business demand-supply matrix models with ill-conditioned Leontif matrices.

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1. Introduction

Definition. R is MP2 if for each $a \in R$, $a \neq 0$, $\exists\, x \in R$, $x \neq 0$ such that $xax = x$.

Theorem 1. *Let R be an associative ring with unity. If every nonzero principal left ideal of R contains a nonzero idempotent, then R is MP2.*

Proof. Let a be a nonzero element of R. Then Ra is the principal left ideal generated by a. By hypothesis, Ra contains a nonzero idempotent $e = e^2$. For

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some r in R, e = ra. Set x = er.

Note. If er = 0 then 0 = era = e^2 = e = 0 which contradicts the fact that e is a nonzero idempotent. Then (er)a(er) = e(ra)(er) = eeer = er since e is idempotent and e = ra. Hence, x = er ≠ 0 and xax = x. Thus, R is MP2. □

Next some elementary examples of MP2 rings are listed.

2. Examples of MP2 RINGS

Example 1. If R is MP1, then R is MP2.

Proof. Let a ∈ R; since R is MP1 ∃ x ∈ R such that axa = a. Set y = xax then yay = y and R is MP2. □

Example 2. If F is a field then M_n(F) (n a positive integer) is MP2 since M_n(F) is MP1.

Example 3. Let R = $\oplus \sum_{i=1}^{n} Fi$, where F_i is a field, then R is MP2.

Example 4. Z/(n), where n is square-free, is MP2.

3. Idempotent Characterization of MP2 Rings

The upcoming result is also an equivalent condition for a ring to be MP2 but is stated only in terms of the behavior of idempotents.

Lemma 1. *Let R be an arbitrary ring. Then each nonzero right ideal contains a nonzero idempotent if and only if each nonzero left ideal of R contains a nonzero idempotent.*

Proof. (Necessity) Assume each nonzero right ideal of R contains a nonzero idempotent and let J be a nonzero left ideal of R. If ∀ b ∈ J, bR = 0 then J is a right ideal and by hypothesis contains a nonzero idempotnet. Otherwise, let b ∈ J such that bR ≠ 0. So for some r ∈ R, br = e, a nonzero idempotent. Then e' = reb is a nonzero idempotent of J. To see this, note that (reb)(reb) =(rbrb)(rbrb) = r(brbrbr) b = r(br) b since e = br is a nonzero idempotent. But r(br)b = reb = e' an idempotent. Since J is a left ideal and b ∈ J then obviously reb = e' ∈ J. Note that e' ≠ 0 since otherwise be'r = e = 0 contradicting the assumption that e is nonzero. Hence, each nonzero left ideal contains a nonzero idempotent.

The proof of sufficiency follows a similar pattern of construction. □

Theorem 2. *Let R be an arbitrary ring with unity. Then the following are equivalent:*

(a) R is MP2.

(b) Every nonzero principal left ideal of R contains a nonzero idempotent.

(c) Every nonzero left ideal of R contains a nonzero idempotent.

(d) Every nonzero right ideal of R contains a nonzero idempotent.

(e) Every nonzero principal right ideal contains a nonzero idempotent.

Proof. (a) $\Rightarrow$ (b) Let I be a principal left ideal of R. Then I = Rv for some arbitrary v in R. Since R is MP2, rvr = r for some nonzero r in R. But rv which lies in I is idempotent since rvrv = rv $\neq 0$.

(b) $\Rightarrow$ (c) Let I be a nonzero left ideal of R. Let a be a nonzero element of I. Then Ra, the principal left ideal generated by a is contained in I. But Ra contains a nonzero idempotent e which is an element of I.

(c) $\Rightarrow$ (b) Obvious since a nonzero principal ideal of R is a nonzero left ideal.

(b) $\Rightarrow$ (a) Demonstrated by the Lemma 1.

(c) $\Leftrightarrow$ (d) Demonstrated by Lemma 1.

(d) $\Rightarrow$ (e) Obvious.

(e) $\Rightarrow$ (d) Let J be a nonzero right ideal of R. Let a be a nonzero element of J. Then aR, the principal right ideal generated by a is contained in J. But aR contains a nonzero idempotent e which is an element of J.

Note. Every principal left ideal of R contains a nonzero idempotent is equivalent to the condition that every nonzero element of R has a multiple which is idempotent. For let a be a nonzero element of R. Then by hypothesis ra = e = $e^2 \neq 0$. Moreover, ra is in the left ideal generated by a, i.e., ra belongs to Ra. On the other hand, suppose Ra contains a nonzero idempotent e = e^2. Then for some s in R, sa = e. Hence, some multiple of a is idempotent.

MP2 rings can be manufactured or cut from other MP2 rings as follows.

4. MP2 Rings and Direct Sums

Theorem 3. *(a) A direct sum of MP2 rings is MP2.*

(b) A direct summand of an MP2 ring is MP2.

Proof. (a) Set R = $\oplus \sum_{i=1}^{n} R_i$ where each R_i is MP2. Let a ∈ R. Then a = $a_1 + a_2 + \ldots + a_n$. Since $a_i \in R_i$, ∃ $x_i \in R_i$ such that $x_i a_i x_i = x_i$. Set x = $x_1 + x_2 + \ldots + x_n$. Then xax = x and R is MP2.

(b) Let S be a direct summand of R an MP2 ring. Then R = R'⊕ S for some R'. Then a left ideal I of S is a left ideal of R and I contains a nonzero idempotent. □

5. Generalization Theorem of an MP2 Ring

Theorem 4. *Let R be a ring in which every nonzero principal left ideal of R contains a nonzero idempotent, then R is MP2.*

Proof. If R contains a unity (or identity), then R is MP2 by Theorem 1. Now assume R does not contain a unity. Suppose a ∈ R, a ≠ 0. According to Neal H. McCoy in his book [1], the principal left ideal generated by a is given by the set {na + sa | n ∈ Z, the ring of integers; s, t ∈ R} which he denotes by $< a >_L$. McCoy notes that Ra = { ra | r ∈ R} is itself a principal left ideal contained in $< a >_L$. Now assume Ra = 0, the zero ideal. By hypothesis, $< a >_L$ contains a nonzero idempotent e, where e = na + sa for some n ∈ Z and s ∈ R. Since e is idempotent, e^2 must equal e. But e^2 = e = (na + sa)(na + sa) = n^2 a^2 + nasa + sana + sasa = 0, since Ra = 0 implies that a^2 = 0 and sa = 0. This contradicts the fact that e is a nonzero idempotent. Thus, Ra ≠ 0. Furthermore, Ra is a nonzero principal left ideal and must contain a nonzero idempotent, say f such that f^2 = f. Then f ∈ Ra $\subseteq < a >_L$. Now f = ta for some t ∈ R; the result follows now by mimicking the elegant proof given in Theorem 1. □

6. Matrix Ring over MP2 Ring

Theorem 5. *Let R be a ring with identity. $M_n(R)$ is MP2 provided that R is MP2.*

Proof. Assume R is MP2. Let A ∈ M_n(R) be nonzero. Suppose a_{kp} (k, p ≤ n) is the first nonzero entry of A. Let E be the matrix with a 1 in the (k, k) position and zeros elsewhere. Then

$$EA = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & a_{kp}a_{kp+1} \quad \dots & a_{kn} \\ 0 & \dots & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & 0 \end{bmatrix}.$$

Certainly, EA is in the principal ideal of $M_n(R)$ generated by A. Now postmultiply the matrix EA by the matrix F which has a 1 in the (p, p) position and zeros elsewhere.

Then

$$EAF = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & a_{kp}\ 0 \quad \dots & a_{kn} \\ 0 & \dots & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & 0 \end{bmatrix}.$$

Now premultiply EAF by the matrix E_{pk} which is derived from the n × n identity matrix by switching the kth and pth rows. Then E_{pk} EAF is the matrix whose (p, p) th entry is a_{kp} and zeros elsewhere. Since R is MP2, there is an element r ∈ R such that ra_{kp} = e an idempotent. Hence, let X be the n × n matrix with r in the (p, p) th position, 1's along the main diagonal except in the (p, p) th position, and zeros elsewhere. Then XE_{pk} EAF = I a matrix with e in the (p, p)-th position.

Note that $I^2 = I$ since e is idempotent. Now premultiply XE_{pk} EAF = I by F and postmultiply XE_{pk} EAF = I by XE_{pk} E so that FXE_{pk} $EAFXE_{pk}$ E = $FIXE_{pk}$ E. Evenmore, FXE_{pk} $EAFXE_{pk}$ E = $FIXE_{pk}$ $EAFIXE_{pk}$ E since I is idempotent. Set J = $FIXE_{pk}$ E so that JAJ = J and $M_n(R)$ is MP2. Of course, J ≠ 0, since otherwise XE_{pk} EAJAF = $I^3 = 0$ if and only if e = 0, contradicting the fact that e is a nonzero idempotent. □

Conversely:

Theorem 6. *Let R be an associative ring with identity. If $M_n(R)$ is MP2, then R is MP2.*

Proof. Let a $\in$ R be arbitrary. In M_n(R) let the matrix A be defined as:

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since M_n(R) is MP2, $\exists$ X $\in$ M_n(R) (X = (x_{ij}) and X $\neq$ 0) such that XAX = X. Postmultiplying both sides of this equation by the matrix A yields XAXA = XA or

$$\begin{bmatrix} x_{11}ax_{11}a0...0 \\ x_{21}ax_{11}a0...0 \\ ... \\ ... \\ x_{n1}ax_{11}a0 \end{bmatrix} = \begin{bmatrix} x_{11}a0...0 \\ x_{21}a0...0 \\ ... \\ ... \\ x_{n1}a0 \end{bmatrix}.$$

Now using the rule of matrix equality, it follows that x_{11} ax_{11} a = x_{11} a. If e = x_{11} a, then e^2 = e. Notice that e $\neq$ 0 since otherwise X = 0 contrary to the MP2 premise. By setting x' = ex_{11}, it becomes the case that x'ax' = x' (or descriptively, ex_{11} aex_{11} = eeex_{11} = ex_{11}). Note that x' $\neq$ 0 since otherwise x' = 0 implies x'a = e = 0. Hence, R is MP2. □

Now here is an interesting alternative proof of Theorem 6.

Proposition 1. *If M_n(R) is MP2, then R is MP2.*

Proof. Let Λ be a left ideal in R. Consider the left ideal Λ' in M_n(R), where

$$\Lambda' = \left\{ \begin{bmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & 0 & \dots & 0 \end{bmatrix} |a_i \in \Lambda \right\}.$$

Since M_n(R) is MP2, then there exists an e'$\in$ Λ', say

$$e' = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix} \neq 0,$$

such that

$$\begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix}.$$

Then: $b_1^2 = b_1$, $b_2 b_1 = b_2$, $\ldots$, $b_n b_1 = b_n$.

If $b_1 = 0$, then $b_2 = b_3 = \cdots = b_n = 0$, contradicting the fact that e' $\neq 0$. Hence, $b_1 \neq 0$ is an idempotent in Λ. Thus, R is MP2. □

Once again, the question is raised if the existence of an identity is necessary in the statement of Theorem 5. The response is negative as the following proof demonstrates:

7. Generalization of a Matrix Ring over an MP2 Ring

Theorem 7. *Let R be an arbitrary ring. $M_n(R)$ is MP2 provided that R is MP2.*

Proof. Assume R is MP2. Let A $\in M_n$(R) be nonzero. Suppose a_{kp} (k, p $\leq$ n) is the first nonzero entry of A. Since R is MP2, there exists a nonzero r $\in$ R such that ra_{kp} = e a nonzero idempotent. Let E be the matrix with this r in the (k, k) position and zeros elsewhere. Then

$$EA = \begin{bmatrix} 0 & \ldots & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & ea_{kp+1} & \ldots & a_{kn} \\ 0 & \ldots & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & & & 0 \end{bmatrix},$$

where e is in the (k, p) position. Certainly, EA is in the principal ideal of M_n(R) generated by A. Now postmultiply the matrix EA by the matrix F which has the e in the (p, p) position and zeros elsewhere.

Then

$$EAF = \begin{bmatrix} 0 & \ldots & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & e0 & \ldots & 0 \\ 0 & \ldots & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & & & 0 \end{bmatrix}.$$

Now premultiply EAF by the matrix E_{pk} which has the given e in the (k, p) position and the same e in the (p, k) position. This will switch the k-th and p-th row. Then E_{pk} EAF is the matrix whose (p, p)-th entry is e and zeros elsewhere. Then E_{pk} EAF = I an idempotent matrix. Note that I^2 = I since e is idempotent. Now premultiply E_{pk} EAF = I by F and postmultiply E_{pk} EAF = I by E_{pk} E so that FE_{pk} $EAFE_{pk}$ E = FIE_{pk} E. Evenmore, FE_{pk} $EAFE_{pk}$ E = FIE_{pk} $EAFIE_{pk}$ E since I is idempotent. Set X = FIE_{pk} E so that XAX = X and M_n(R) is MP2. Of course, X $\neq$ 0, since otherwise E_{pk} EAXAF = I^3 = 0 if and only if e = 0, contradicting the fact that e is a nonzero idempotent. □

8. Statistical Application of MP2 Matrix Ring

As an illustration of the MP2 matrix ring concept, consider the linear regression model (full rank) with intercept.

The equation of the full rank linear model is y = Xb + ε, where

$$X = \begin{bmatrix} x_{10} & x_{11} & \dots & x_{1k} \\ x_{20} & x_{21} & \dots & x_{2k} \\ \vdots & \vdots & & \vdots \\ x_{N0} & x_{N1} & \dots & x_{Nk} \end{bmatrix}$$

is the matrix of observations, and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$ – the dependent observations vector; $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$ – the vector of model parameters, and $\varepsilon = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$ – the error vector (residuals), where E(y) = Xb and e = y - E(y). The solution is b' = $(X^T X)^{-1} X^T$ y, where $X^+ = (X^T X)^{-1} X^T$ is MP2.

This follows immediately since $X^+ X^+ = X^+$.

Here the ' symbol denotes "estimated". Note that $I_L = XX^+$ is the orthogonal projection onto the row-space of X^T. Now b' is known as the least squares estimator of b in the sense that it minimizes $(y - y')^T(y - y')$, the residual error sum of squares which is the sum of the squares of the observed y_i's from their estimated expected values. The residual error sum of squares,

$$SSE = y^T(I - X(X^TX)^{-1}X^T)y = y^T(I - I_L)y = y^Ty - b'^TX^Ty = SST - SSR$$

(the total sum of squares minus the sum of squares due to regression, respectively).

References

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