GEODESIC MAPPINGS OF WEAKLY BERWALD SPACES
AND BERWALD SPACES ONTO RIEMANNIAN SPACES

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Abstract: The present paper is devoted to the investigation of the geodesic mappings of weakly Berwald spaces and Berwald spaces which are special Finsler spaces $F_n$, onto the Riemannian spaces $\bar{V}_n$. We can see that not every Finsler space admits non-trivial geodesic mapping onto a Riemannian space.

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1. Geodesic Mappings of Finsler Spaces

It is known [4, 6, 9, 10], a Finsler space $F_n$, determined by the symmetric and regular metric tensor $g_{ij}(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)$, determined by a function $F(x, y) \equiv \frac{1}{2} L^2(x, y)$:

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The Berwald connection of $F_n$ is introduced by the formulas:

\[
G^h(x, y) = \frac{1}{2} g^{ij} \left( \frac{\partial^2 F(x, y)}{\partial y^i \partial x^k} y^k - \frac{\partial F(x, y)}{\partial y^i} \right),
\]

\[
G^h_i(x, y) = \frac{\partial G^h(x, y)}{\partial y^i}, \quad C^h_{ij}(x, y) = \frac{\partial^2 G^h(x, y)}{\partial y^i \partial y^j},
\]

where $g^{ij}$ are components of the matrix inverted to $g_{ij}(x, y)$. For metric tensor of $F_n$ we have the following formula:

\[
g_{ij}(x, y) = \frac{\partial^2 F(x, y)}{\partial y^i \partial y^j},
\]

where $P_{ijk} = C_{ijk} \alpha y^\alpha$ is the Landsberg tensor, and $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ is the Cartan tensor. Hereafter the semicolon “;” denotes the covariant derivative with respect to the Berwald connection of the space $F_n$.

We consider two Finsler spaces $F_n = (M_n, L)$ and $\bar{F}_n = (M_n, \bar{L})$ on the same underlying manifold $M_n$. If any geodesic of a Finsler space $F_n$ coincides with a geodesic of $\bar{F}_n$ as a set of points and vice versa, the change $L \rightarrow \bar{L}$ of the metrics is called projective, and the mapping $F_n \rightarrow \bar{F}_n$ is a geodesic mapping between $F_n$ and $\bar{F}_n$.

It is known that if a Finsler space $F_n$ admits a geodesic mapping onto a Finsler space $\bar{F}_n$ then in general with respect to the mapping system of coordinates $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n$ the objects of the Berwald connection of these spaces $G^h(x, y)$ and $\bar{G}^h(x, y)$ have the following relation [1, 2, 3, 5, 6, 9, 10]:

\[
\bar{G}^h_i(x, y) = G^h_i(x, y) + \delta^h_i \psi_i(x, y) y^h,
\]

\[
\bar{G}^h_{ij}(x, y) = G^h_{ij}(x, y) + \delta^h_i \psi_j + \delta^h_j \psi_i + \psi_{ij}(x, y) y^h,
\]

where $\psi_i = \frac{\partial \psi(x, y)}{\partial y^i}$, $\psi_{ij} = \frac{\partial \psi_i(x, y)}{\partial y^j}$, $\delta^h_i$ is the Kronecker symbol.

If $\psi_i \neq 0$, then a geodesic mapping is called nontrivial; otherwise it is said to be trivial.
From the above mentioned equations we obtain

\[ \tilde{G}_{ijk}^h(x, y) = G_{ijk}^h(x, y) + \delta_i^h \psi_{jk} + \delta_j^h \psi_{ik} + \delta_k^h \psi_{ij} + \psi_{ijk}(x, y) y^h, \]

where

\[ \psi_{ijk} = \frac{\partial \psi_{ij}(x, y)}{\partial y^k}. \]

Finsler space is a \textit{Berwald type} if \( G_{ijk}^h = 0 \), and Finsler space is a \textit{weakly Berwald type} if \( G_{ij\alpha}^h = G_{ij} = 0 \).

Hence \( \tilde{G}_{ij} = G_{ij} + (n+1)\psi_{ij} \), so if we consider the geodesic mapping between Berwald and weakly Berwald spaces then \( \psi_{ij} = 0 \).

It is well known, that every positive definite Berwald metric has common geodesics with some Riemannian metric [12]. The aim of this paper to show a new method, which in suitable for the investigation of geodesic mapping of Berwald spaces onto Riemannian spaces in any cases.

\[ \textbf{2. Geodesic Mapping of Weakly Berwald Spaces onto Riemannian Spaces} \]

We know, that the Douglas tensor is invariant under geodesic mapping, that is \( D = \tilde{D} \), where

\[ D_{ijk}^h = G_{ijk}^h - \frac{1}{(n+1)} (G_{ij} y^h \delta_k^h + G_{ik} \delta_j^h + G_{jk} \delta_i^h), \text{ and } G_{ijk} = \frac{\partial G_{ij}}{\partial y^k}. \]

The Douglas tensor vanishes in Riemannian spaces, so in the weakly Berwald space we get \( G_{ijk}^h = 0 \), where the weakly Berwald space has common geodesics with a Riemannian space. So we obtain the following

\textbf{Theorem 1.} \textit{If a weakly Berwald space has common geodesics with a Riemannian space, then the weakly Berwald space is a Berwald space.}

Berwald constructed a tensor, which is analogically of Weyl tensor of projective curvature. This tensor is also invariant under a geodesic mappings of Finsler spaces.

The fundamental equation (1) of geodesic mapping between Finsler spaces \( F_n \) and \( \tilde{F}_n \) equivalent to following formula:

\[ \overline{\eta}_{ijk} = 2\psi_k \overline{\eta}_{ij} + \psi_i \overline{\eta}_{jk} + \psi_j \overline{\eta}_{ik} + \overline{\eta}_{i\alpha} y^\alpha \psi_{jk} + \overline{\eta}_{j\alpha} y^\alpha \psi_{ik} - 2 \overline{P}_{ijk}. \]

If case that space \( F_n \) and \( \tilde{F}_n \) are weakly Berwald spaces this formula have
following simly form:
\[ g_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik} - 2\bar{P}_{ijk}. \] (2)
In this case the functions \( \psi_i \) are independent of \( y \), i.e. \( \psi_i = \psi_i(x) \).

3. Geodesic Mappings of Berwald Spaces onto Riemannian Spaces

**Theorem 2.** The Berwald space \( F_n \) admits a non-trivial geodesic mapping onto a Riemannian space \( \bar{V}_n \) with the metric tensor \( \bar{g}_{ij}(x) \) if and only if the following system of differential equations with covariant derivatives of Cauchy type has a solution with respect to the symmetric tensor \( \bar{g}_{ij}(x) \) (det \( ||\bar{g}_{ij}(x)|| \neq 0 \), the non-zero vector \( \psi_i(x) \) and the invariant \( \mu(x) \):

\[
\begin{align*}
(a) \quad \bar{g}_{ij,k} &= 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}; \\
(b) \quad n\psi_{i;j} &= n\psi_i \psi_j + \mu \bar{g}_{ij} - \bar{g}_{i\alpha} \bar{g}^{\beta\gamma} H^\alpha_{\beta\gamma} - \frac{2}{n+1} H^\alpha_{\alpha i}; \\
(c) \quad (n-1)\mu_{;i} &= 2(n-1)\psi_i \bar{g}^{\beta\gamma} H^\alpha_{\beta\gamma} \\
&+ \psi_\alpha \bar{g}^\alpha(5H_{\beta i} + \frac{6}{n+1} H^\gamma_{\gamma\beta i} - H_{i\beta}) \\
&+ \bar{g}^\alpha(7H_{\alpha i;\gamma} - H_{\alpha i;\beta} - \frac{2}{n+1} H^\gamma_{\gamma\alpha i;\beta}),
\end{align*}
\] (3)

where the semicolon denotes covariant derivative with respect to the space connection \( F_n \), \( \bar{g}^{ij}(x) \) are components of the matrix inverted to \( ||\bar{g}_{ij}(x)|| \), \( H_{ijk}^h \) and \( H_{ij} \) are respectively curvature and Ricci tensors of the space \( F_n \).

**Proof.** Suppose that \( F_n \) admits non trivial geodesic mapping onto \( \bar{V}_n \) with metric tensor \( \bar{g}_{ij}(x) \). Then the connections \( F_n \) and \( \bar{V}_n \) have the relation (2) with \( \bar{P}_{ijk}^h = 0 \). If we consider the fundamental tensor \( \bar{g}_{ij}(x) \) of \( \bar{V}_n \), then we get the conditions which are sufficient for \( F_n \) to admit non trivial geodesic mapping onto \( \bar{V}_n \).

Let us consider integrability conditions of the equations (3a)
\[ \bar{g}_{i\alpha} H^\alpha_{ijk} + \bar{g}_{\alpha i} H^\alpha_{hijk} = 2\bar{g}_{hi} \psi_{j[k]} + \bar{g}_{j(h} \psi_{i)k} - \bar{g}_{k(h} \psi_{i)j}, \] (4)
where \( \psi_{ij} = \psi_{ij} - \psi_i \psi_j, \ [i \ j] \) and \( (i \ j) \) denotes alternation and symmetrization with respect to \( i \) and \( j \), respective.

Transvecting (4) by \( \bar{g}^{ij} \), we get \( \psi_{[jk]} = \frac{1}{n+1} H^\alpha_{\alpha ijk} \). Excluding \( \psi_{[jk]} \) from (4),
we obtain
\[ \bar{g}_\alpha(hH^\alpha_{i}k) - \frac{2}{n + 1}g_{ij}(h\psi)i_k = \bar{g}_{(h\psi)}j_k - \bar{g}_k(h\psi)i_j. \] (5)

After the transvecting (5) by \( g^{ik} \) we easily obtain the conditions (3b) with \( \mu \equiv \psi^{\alpha\beta}\bar{g}^{\alpha\beta} \).

If we consider the equation \( \bar{g}^{ij}k_j = \delta^i_j \), it is not difficult to show that the equations (3a) are equivalent to the relations
\[ \bar{g}^{ij}k_k = -2\psi_k\bar{g}^{ij} - \delta_k^i\psi^j - \delta_j^i\psi^k, \] (6)
where \( \psi^i \equiv \psi^{\alpha}\bar{g}^{\alpha i} \).

We covariantly differentiate the conditions (3b) and then alternate the result with respect to the indices \( j \) and \( k \) taking into account (3a), (3b), (6) and transvecting by \( \bar{g}^{ik} \), and finally we get equations (3c).

The theorem has been proved.

From Theorem 2 we may conclude that the set of all Riemannian spaces \( \bar{V}_n \), the given Berwald space \( F_n \) admits non trivial geodesic mapping onto \( \bar{V}_n \), is dependent on \( r \leq r_0 = (n + 1)(n + 2)/2 \) parameters.

Finding of all the solutions of (3) requires a consideration of their integrability conditions and differential extensions, which form a set of algebraic equations with respect to the unknown functions \( \bar{g}_{ij}, \psi, \) and \( \mu \) with coefficients from \( F_n \). But this set is not linear and its solution is certainly difficult.


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