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NONLINEAR LIMIT-POINT/LIMIT-CIRCLE PROPERTIES OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract: The authors consider the nonlinear differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)\,f(y) = 0,$$

where $p > 0$, $a(t) > 0$, $r(t) > 0$, and $xf(x) > 0$ for $x \neq 0$ and prove some new nonlinear limit-point and nonlinear limit-circle results. Their results are illustrated with some examples.

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1. Introduction

In this paper, we examine the asymptotic properties of the solutions of the second order nonlinear differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)\,f(y) = 0, \tag{1}$$

where $p > 0$, $a \in C^1(\mathbb{R}_+)$, $r \in C^1(\mathbb{R}_+)$, $a > 0$, $r > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is

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continuous, nondecreasing, and satisfies $xf(x) > 0$ for $x \neq 0$. We are especially interested in the nonlinear limit-point/limit-circle properties as defined below. An important special case of equation (1) is the equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)\,|y|^{\lambda}\operatorname{sgn} y = 0 \tag{2}$$

where $\lambda > 0$. If $\lambda = p$, then equation (2) is the well-known *half-linear* equation. Where it is convenient, we will refer to equation (1) as being of the *super-half-linear* type if $\lambda > p$ and of the *sub-half-linear* type if $\lambda < p$. By Theorem 3 in [1], we know that every solution of (1) is defined on $\mathbb{R}_+$.

In [4], [5], the authors investigated similar questions to the ones explored in this paper, but in those papers, it was assumed that

$$\int_0^{\infty} (a^{-\frac{1}{p}}(t) + r(t))\,dt = \infty.$$

Here, we will often assume that both

$$\int_0^{\infty} a^{-\frac{1}{p}}(t)\,dt < \infty \tag{3}$$

and

$$\int_0^{\infty} r(t)\,dt < \infty. \tag{4}$$

In that sense, the results here compliment those in [4] and [5].

We wish to introduce the notation

$$y^{[1]}(t) = a(t)|y'(t)|^{p-1}y'(t)$$

and define the function $R\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$R(t) = a^{1/p}(t)\,r(t)\,.$$

We also define the constants α, β and δ by

$$\alpha = \frac{p+1}{(\lambda+2)p+1}\,, \quad \beta = \frac{(\lambda+1)p}{(\lambda+2)p+1}\,, \quad \text{and} \quad \delta = \frac{p+1}{p}\,.$$

As indicated above, we are especially interested in the nonlinear limit-point/limit-circle properties of solutions which are defined as follows.

Definition 1.1. A solution y of equation (1) defined on $\mathbb{R}_+$ is said to be of the *nonlinear limit–circle* type if

$$\int_0^{\infty} y(\sigma)f(y(\sigma))\,d\sigma < \infty, \tag{NLC}$$

and it is said to be of the *nonlinear limit–point* type otherwise, i.e., if

$$\int_0^{\infty} y(\sigma)f(y(\sigma))\,d\sigma = \infty. \tag{NLP}$$

Equation (1) will be said to be of the *nonlinear limit–circle* type if every solution y of (1) defined on $\mathbb{R}_+$ satisfies (NLC) and to be of the *nonlinear limit–point* type if there is at least one solution y defined on $\mathbb{R}_+$ for which (NLP) holds.

The notion of limit-point and limit-circle type solutions goes back to the fundamental work of Weyl [16] on second order linear equations which has generated a great deal of interest over the last 100 years. When equation (1) is linear, Definition 1.1 reduces to the (linear) limit-point and limit-circle definitions of Weyl. The extension of these ideas to the nonlinear case as described in the above definitions began with the papers of Graef and Spikes [11, 12, 13]. For a survey of known results on the linear and nonlinear problems as well as their relationships to other properties of solutions such as boundedness, oscillation, and convergence to zero, we refer the reader to the recent monograph by Bartušek, Došlá, and Graef [3] as well as the recent papers of Bartušek and Graef [4–8].

We are also be interested in the *strong nonlinear limit-point* and *strong nonlinear limit-circle* types of solutions of (1) as first introduced in [6], [5], and [4].

Definition 1.2. A solution y of (1) defined on $\mathbb{R}_+$ is said to be of the strong nonlinear limit-point type if

$$\int_0^\infty y(\sigma)f(y(\sigma))\,d\sigma = \infty \quad \text{and} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = \infty\,.$$

Equation (1) is said to be of the strong nonlinear limit-point type if every nontrivial solution is of the strong nonlinear limit-point type.

Similarly, we have the following definition.

Definition 1.3. A solution y of (1) defined on $\mathbb{R}_+$ is said to be of the strong nonlinear limit-circle type if

$$\int_0^\infty y(\sigma)f(y(\sigma))\,d\sigma < \infty \quad \text{and} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma < \infty\,.$$

Equation (1) is said to be of the strong nonlinear limit-circle type if every solution is of the strong nonlinear limit-circle type.

We define the oscillation of solutions in the usual way.

Definition 1.4. A nontrivial solution y of (1) is called oscillatory if there exists a sequence of its zeros tending to ∞. Otherwise, it is called nonoscillatory.

The next section of this paper contains some preliminary lemmas and our main results are presented in Section 3. Examples of our results are given in Section 4.

2. Preliminary Lemmas

If $h:\mathbb{R}_+\to\mathbb{R}$ is a continuous function, we let $h_+(t)=\max\{h(t),0\}$ and $h_-(t)=\max\{-h(t),0\}$ so that $h(t)=h_+(t)-h_-(t)$. For any solution $y:\mathbb{R}_+\to\mathbb{R}$ of (1), we set

$$Z(t)=\frac{a(t)}{r(t)}\,|y'(t)|^{p+1}+\delta\int_0^{y(t)}f(s)\,ds=\frac{|y^{[1]}(t)|^\delta}{R(t)}+\delta\int_0^{y(t)}f(s)\,ds\,. \tag{5}$$

Theorem 3 in [1] ensures that $Z>0$ on $\mathbb{R}_+$ for every nontrivial solution y of (1).

Our first lemma gives conditions under which the function Z is bounded from below away from zero. We will need the condition that

$$\int_0^\infty \frac{R'_+(t)}{R(t)}\,dt<\infty\,. \tag{6}$$

Notice that this implies that the function R is bounded from above, i.e., there exists a constant R_1 such that

$$0\le R(t)\le R_1<\infty \quad \text{for} \quad t\in\mathbb{R}_+\,. \tag{7}$$

Lemma 2.1. *Assume that condition* (6) *holds. For any nontrivial solution* y *of equation* (1), *there exists a positive constant* Z_0 *such that*

$$0<Z_0\le Z(t) \quad \textit{for} \quad t\in\mathbb{R}_+\,. \tag{8}$$

Proof. Let y be a nontrivial solution of (1). Then,

$$Z'(t)=-\frac{R'(t)}{R^2(t)}\left|y^{[1]}(t)\right|^\delta$$

and so

$$-\frac{R'_+(t)}{R(t)}\le-\frac{R'(t)}{R(t)}\,\frac{\left|y^{[1]}(t)^\delta\right|}{R(t)\,Z(t)}=\frac{Z'(t)}{Z(t)}$$

for $t\in\mathbb{R}_+$. Integrating, we obtain

$$\exp\Big\{-\int_0^t\frac{R'_+(s)}{R(s)}\,ds\Big\}\le\frac{Z(t)}{Z(0)},$$

and so (6) implies (8) holds. $\square$

Our next two lemmas are modeled after results in [2].

Lemma 2.2. *The equation*

$$\left(|\omega'|^{p-1}\omega'\right)' + f(\omega) = 0 \tag{9}$$

has a periodic solution satisfying $\omega(0) = d$ *and* $\omega'(0) = 0$ *with* $d > 0$*. If* $2T$ *is the smallest positive period of* ω*, then* $\omega(0) = -\omega(T)$*. Moreover,*

$$\omega \quad \text{is decreasing on} \quad [0, T] \quad \text{and increasing on} \quad [T, 2T]\,.$$

Proof. In view of Corollary 6.1 in [15], the solution ω is oscillatory. The proof is then the same as the proof of Lemma 1 in [2] without any changes. □

Lemma 2.3. *For every nontrivial solution* y *of* (1)*, there exists a continuous positive function* φ *such that*

$$\int_0^{y(t)} f(s)\,ds = Z(t)\int_0^{\omega(\varphi(t))} f(s)\,ds \tag{10}$$

and

$$y^{[1]}(t) = \left(R(t)\,Z(t)\right)^{1/\delta}\left|\omega'\big(\varphi(t)\big)\right|^{p-1}\omega'\big(\varphi(t)\big) \tag{11}$$

for $t \geq 0$*, where* ω *is the periodic solution of* (9) *with* $d > 0$ *satisfying*

$$\int_0^d f(s)\,ds = \frac{1}{\delta}\,. \tag{12}$$

Moreover, y *is nonoscillatory if and only if* φ *is bounded on* $\mathbb{R}_+$*, and it is oscillatory if and only if* $\lim_{t\to\infty}\varphi(t) = \infty$*.*

Proof. The proof is similar to that of Lemma 2 in [2] (where $f(x) = |x|^\lambda \operatorname{sgn} x$). Define

$$\omega^{[1]}(t) = \left|\omega'(t)\right|^{p-1}\omega'(t) \quad \text{for} \quad t \in \mathbb{R}_+.$$

Let y be a nontrivial solution of (1) and let $\{\tau_n\}_{n=1}^\infty$ be the sequence (finite or infinite) of all zeros of y' on $\mathbb{R}_+$ if such zeros of y' exist. According to Corollary 1 in [1], y has only a finite number of zeros on any finite interval and it has no double zeros. Hence, from (1) we see that the sequence $\{\tau_n\}$ exists and the zeros of y and $y^{[1]}$ separate each other.

Set

$$h(t) = \operatorname{sgn} y(t)\, Z^{-1}(t)\int_0^{y(t)} f(s)\,ds \quad \text{for} \quad t \in \mathbb{R}_+\,.$$

From (5), we have

$$-\frac{1}{\delta} < h(t) < \frac{1}{\delta} \quad \text{for} \quad t \neq \tau_k, \quad k = 1, 2, \ldots, \tag{13}$$

and

$$h(\tau_k) = \frac{1}{\delta}\operatorname{sgn} y(\tau_k) \quad \text{for} \quad k = 1, 2, \ldots. \tag{14}$$

Now define the function $h_1(t)$ by

$$\operatorname{sgn} h_1(t) \int_0^{h_1(t)} f(s)\, ds = h(t) \quad \text{for} \quad t \in \mathbb{R}_+ .$$

From this and (12)–(14), we have

$$-d < h_1(t) < d \quad \text{for} \quad t \neq \tau_k \tag{15}$$

and

$$h_1(\tau_k) = d \operatorname{sgn} y(\tau_k) \quad \text{for} \quad k = 1, 2, \dots \tag{16}$$

Next, we construct the function φ satisfying (10).

Consider three cases:

(i) y' does not have zeros on $\mathbb{R}_+$;

(ii) y' has a finite number N of zeros on $\mathbb{R}_+$;

(iii) y' has infinitely many zeros on $\mathbb{R}_+$.

In case (i), using (15), we define

$$\varphi(t) = \omega_0\big(h_1(t)\big) \quad \text{for} \quad t \in \mathbb{R}_+,$$

where ω_0 is the inverse function of ω on $[0, T]$ (on $[T, 2T]$) if $y'(0) < 0$ (if $y'(0) > 0$).

Let either case (ii) or (iii) hold and suppose $y(\tau_1) < 0$; the proof if $y(\tau_1) > 0$ is similar. If $\tau_1 > 0$, define

$$\varphi(t) = \omega_0\big(h_1(t)\big) \quad \text{for} \quad t \in [0, \tau_1) .$$

Then, for $t \geq \tau_1 \geq 0$, define

$$\varphi(t) = \omega_n\big(h_1(t)\big) \quad \text{on} \quad [\tau_n, \tau_{n+1}),$$

where ω_n is the inverse function of ω on the interval $[nT, (n+1)T)$; it is possible to do this in view of (15) and (16). In case (ii), define

$$\varphi(t) = \omega_{N+1}\big(h_1(t)\big) \quad \text{on} \quad [\tau_N, \infty),$$

where ω_{N+1} is the inverse function of ω on the interval $\big[(N+1)T, (N+2)T\big)$. Hence, φ is defined on $\mathbb{R}_+$ and it is continuous there since (16) implies

$$\lim_{t \to \tau_n -} \varphi(t) = \lim_{t \to \tau_n -} \omega_n\big(h_1(t)\big) = nT = \varphi(\tau_n) .$$

Note that we also have

$$nT < \varphi(t) < (n+1)T \quad \text{on} \quad (\tau_n, \tau_{n+1}) .$$

Thus, (10) holds, and y is nonoscillatory (oscillatory) if and only if φ is bounded $\big(\lim_{t\to\infty} \varphi(t) = \infty\big)$. Moreover,

$$\operatorname{sgn} y^{[1]}(t) = \operatorname{sgn} \omega'\big(\varphi(t)\big) . \tag{17}$$

In order to prove (11), note that

$$\left(|\omega^{[1]}(t)| + \delta \int_0^{\omega(t)} f(s)\,ds\right)' = 0,$$

and so integrating and using (12) and the initial conditions for ω at $t = 0$, we have

$$\left|\omega^{[1]}(t)\right|^{\delta} + \delta \int_0^{\omega(t)} f(s)\,ds = 1 \quad \text{for} \quad t \in \mathbb{R}_+\,. \tag{18}$$

From this, (5), and (10), we obtain

$$\begin{aligned} Z(t) - \frac{|y^{[1]}(t)|^{\delta}}{R(t)} &= \delta \int_0^{y(t)} f(s)\,ds \\ &= \delta Z(t) \int_0^{\omega(\varphi(t))} f(s)\,ds = Z(t)\big(1 - \left|\omega^{[1]}(\varphi(t))\right|^{\delta}\big), \end{aligned}$$

or

$$\frac{|y^{[1]}(t)|^{\delta}}{R(t)} = Z(t)\left|\omega^{[1]}(\varphi(t))\right|^{\delta} \quad \text{for} \quad t \in \mathbb{R}_+\,.$$

From this and (17) we see that (11) holds. □

Remark 2.4. Notice that Lemmas 2.2 and 2.3 hold without assuming that f is nondecreasing or that (3), (4), and (6) hold.

We will let N denote the set of all nonoscillatory solutions of (1) and let

$$N_0 = \left\{y \in N : \lim_{t\to\infty} y(t) = 0 \;\text{ and }\; \lim_{t\to\infty} |y^{[1]}(t)| \in (0,\infty)\right\},$$

$$N_1 = \left\{y \in N : \lim_{t\to\infty} |y(t)| \in (0,\infty) \;\text{ and }\; \lim_{t\to\infty} |y^{[1]}(t)| \in (0,\infty)\right\},$$

and

$$N_2 = \left\{y \in N : \lim_{t\to\infty} |y(t)| \in (0,\infty) \;\text{ and }\; \lim_{t\to\infty} y^{[1]}(t) = 0\right\}.$$

The next lemma shows that nonoscillatory solutions of each of these types exist.

Lemma 2.5. *Assume that conditions* (3), (4), *and* (6) *hold. Then* $N = N_0 \cup N_1 \cup N_2$ *and* $N_i \neq \emptyset$ *for* $i = 0, 1, 2$.

Proof. Since f is nondecreasing on $\mathbb{R}$ and (3)–(4) hold, Lemma 1.1 in [14] implies that any nonoscillatory solution of equation (1) satisfies $y^{[1]}(t) \neq 0$ for all large t. Moreover, Theorems 1 and 4 in [10] imply $N = N_0 \cup N_1 \cup N_2 \cup N_3$, where

$$N_3 = \left\{y \in N : \lim_{t\to\infty} y(t) = \lim_{t\to\infty} y^{[1]}(t) = 0\right\}.$$

If $y \in N_3$, then, by Lemma 2.1, (8) holds. Since y is nonoscillatory, Lemma

2.3 implies φ is bounded, and so (8) and (10) imply $\lim_{t\to\infty} \omega(\varphi(t)) = 0$. Hence, $\lim_{t\to\infty} \varphi(t) = T \in (0,\infty)$ exists. Moreover, $\omega(T) = 0$ and it follows from (18) that $\lim_{t\to\infty} |\omega^{[1]}(\varphi(t))| = 1$. From (11), we have

$$\lim_{t\to\infty} R(t)Z(t) = 0.$$

Since y is nonoscillatory, assume that $y(t) > 0$ for large t; the case $y(t) < 0$ is similar. Then, $y'(t) < 0$ eventually and

$$-y'(t) = a^{-\frac{1}{p}}(t)\big(R(t)\,Z(t)\big)^{\frac{1}{p+1}}\left|\omega'\big(\varphi(t)\big)\right|.$$

Since $\lim_{t\to\infty} \omega^{[1]}(\varphi(t)) = -1$, an integration yields

$$y(t) \le \int_t^\infty a^{-\frac{1}{p}}(\sigma)\big(R(\sigma)\,Z(\sigma)\big)^{\frac{1}{p+1}}\omega'\big(\varphi(\sigma)\big)\,d\sigma \tag{19}$$

with

$$\lim_{t\to\infty} \big(R(t)\,Z(t)\big)^{\frac{1}{p+1}}\omega'\big(\varphi(t)\big) = 0.$$

On the other hand, Lemma 3 in [10] ensures the existence of a constant $B > 0$ such that

$$B\int_t^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma \le y(t) \quad \text{for} \quad t \ge t_2.$$

This contradicts (19) and proves that $N_3 = \emptyset$.

The conclusion $N_0 \ne 0$ follows from Theorem 2.2 in [14], while Theorem 3 in [10] implies $N_1 \ne \emptyset$ and $N_2 \ne \emptyset$. □

For our next lemma, we will assume that there exist $\varepsilon > 0$, $M > 0$, and $q > 0$ such that

$$|f(x)| \le |x|^q \quad \text{either for} \quad |x| \le \varepsilon \quad \text{or for} \quad |x| \ge M\,. \tag{20}$$

Lemma 2.6. *Let* (3), (4), (6), *and* (20) *hold. Then any nontrivial solution of* (1) *is nonoscillatory. In particular, any nontrivial solution of* (2) *is nonoscillatory.*

Proof. If the first part of (20) holds, then it follows from Corollary 7.1 in [15] (with $k = 1$ and 2 and $\varphi_2(t,u) = r(t)\max_{0\le s\le u}\big(f(s) + |f(-s)|\big)$ there) that every nontrivial solution y of (1) is either nonoscillatory or is oscillatory and satisfies $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y^{[1]}(t) = 0$. If the second part of (20) holds, this is also true by [15, Corollary 7.3]2 (with $k = 1$ and $k = 2$).

We will show that there are no oscillatory solutions. Suppose, for the sake of obtaining a contradiction, that y is an oscillatory solution of (1). Then Lemma

2.1 implies (8) holds, and by Lemma 2.3, $\lim_{t\to\infty} \varphi(t) = \infty$. Now,

$$\lim_{t\to\infty} \int_0^{y(t)} f(s)\,ds = 0, \quad \text{but} \quad \lim_{t\to\infty} Z(t) \int_0^{\omega(\varphi(t))} f(s)\,ds$$

does not exist. In view of (10), this is a contradiction, and so all nontrivial solution of (1) are nonoscillatory. □

The final lemma in this section is of a more technical nature.

Lemma 2.7. *If* (3) *and* (4) *hold, then* $\int_0^\infty a^{-\frac{1}{p}}(s) R^\alpha(s)\,ds < \infty$.

Proof. By Hölder's inequality, (3), and (4), we have

$$\int_0^\infty a^{-\frac{1}{p}}(s) R^\alpha(s)\,ds = \int_0^\infty a^{-\frac{\beta}{p}}(s) r^\alpha(s)\,ds$$
$$\le \Big[\int_0^\infty \big(a^{-\frac{\beta}{p}}(t)\big)^{\frac{1}{\beta}}\,dt\Big]^\beta \Big[\int_0^\infty \big(r^\alpha(t)\big)^{\frac{1}{\alpha}}\,dt\Big]^\alpha = \Big[\int_0^\infty a^{-\frac{1}{p}}(t)\,dt\Big]^\beta \Big[\int_0^\infty r(t)\,dt\Big]^\alpha$$
$$< \infty\,. \quad \square$$

3. Main Results

Our first theorem reveals a good deal of information about the behavior of solutions of equation (1) especially as it relates to the limit-point/limit-circle properties.

Theorem 3.1. *Assume that* (3), (4), *and* (6) *hold.*

(i) $N = N_0 \cup N_1 \cup N_2$ *and* $N_i \ne \emptyset$, $i = 0, 1, 2$.

(ii) *Let* $y \in N_0$, *and assume that for* $\operatorname{sgn} c_1 = \operatorname{sgn} c \ne 0$, $\lim_{u\to 0} f(c_1 u)/f(cu)$ *exists as a finite positive number. Then*

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = \infty.$$

Moreover,

$$\int_0^\infty y(\sigma)\, f\big(y(\sigma)\big)\,d\sigma < \infty$$

if and only if

$$\int_0^\infty \Big| f\Big(\int_s^\infty a^{-\frac{1}{p}}(\tau)\,d\tau\Big)\Big| \int_s^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma\,ds < \infty\,.$$

(iii) *If $y \in N_1$, then y is of the strong nonlinear limit-point type, i.e.,*

$$\int_0^\infty y(\sigma)\, f\big(y(\sigma)\big)\, d\sigma = \infty \quad \text{and} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty\,.$$

(iv) *If $y \in N_2$, then*

$$\int_0^\infty y(\sigma)\, f\big(y(\sigma)\big)\, d\sigma = \infty.$$

Moreover,

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma < \infty$$

if and only if

$$\int_0^\infty \frac{1}{R(s)}\Big(\int_s^\infty r(\sigma)\, d\sigma\Big)^\delta ds < \infty\,.$$

(v) *If* (20) *holds, then every nontrivial solution of* (1) *is nonoscillatory.*

Proof. Part (i) follows from Lemma 2.5 and part (v) follows from Lemma 2.6.

If $y \in N_1$, then (7), (8), and the fact that f is nondecreasing clearly shows that y is of the strong nonlinear limit-point type, so (iii) is proved.

If $y \in N_2$, then $\int\limits_0^\infty y(t)\, f\big(y(t)\big)\, dt = \infty$ and there exists $t_0 \in \mathbb{R}_+$ and $B > 0$ such that

$$\frac{B}{2} \le y(t) \le B \quad \text{for} \quad t \in [t_0, \infty)$$

(the case $y(t) < 0$ is similar). By an integration of (1) on $[t, \infty)$ and the fact that $\lim\limits_{t\to\infty} y^{[1]}(t) = 0$, we have

$$f\Big(\frac{B}{2}\Big)\int_t^\infty r(s)\, ds \le y^{[1]}(t) = \int_t^\infty r(s) f\big(y(s)\big)\, ds \le f(B)\int_t^\infty r(s)\, ds$$

for $t \ge t_0$. Raising this inequality to the power δ, dividing by $R(t)$, and integrating again, we obtain

$$\begin{aligned} f^\delta\Big(\frac{B}{2}\Big)\int_{t_0}^\infty R^{-1}(\sigma)\Big(\int_\sigma^\infty r(s)\, ds\Big)^\delta d\sigma &\le \int_{t_0}^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma \\ &\le f^\delta(B)\int_{t_0}^\infty R^{-1}(\sigma)\Big(\int_\sigma^\infty r(s)\, ds\Big)^\delta d\sigma \end{aligned}$$

and so (iv) is proved.

If $y \in N_0$, then (6) implies $\int\limits_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty$. Since $y^{[1]}$ does not oscillate,

y' does not oscillate as well. Suppose that $y(t) > 0$ for large t; the case $y(t) < 0$ is similar. Then there exists $t_1 \in \mathbb{R}_+$ such that

$$y'(t) < 0 \quad \text{and} \quad y^{[1]}(t) < 0 \quad \text{on} \quad [t_1, \infty)\,.$$

Since y is nonoscillatory, the function φ defined by Lemma 2.3 is bounded, so (8) and (10) imply

$$\lim_{t\to\infty} \omega(\varphi(t)) = 0.$$

Thus, there exists $T \in (0, \infty)$ such that $\lim\limits_{t\to\infty} \varphi(t) = T$ and $\omega(T) = 0$. Hence, $\omega^{[1]}(T) = 1$, so (18) and (12) imply there exists $t_2 \ge t_1$ such that

$$\frac{1}{2} \le \left|\omega^{[1]}\big(\varphi(t)\big)\right| \le 1 \quad \text{for} \quad t \ge t_2\,. \tag{21}$$

Since $y \in N_0$, (11) and (21) imply there exist $t_3 \ge t_2$, $M_1 > 0$, and $M_2 > 0$ such that

$$0 < M_1 \le R(t)Z(t) \le M_2 \quad \text{for} \quad t \ge t_3.$$

From (11), we have

$$-y'(t) = a^{-\frac{1}{p}}(t)\big(R(t)\,Z(t)\big)^{\frac{1}{p+1}}\left|\omega'\big(\varphi(t)\big)\right|$$

or

$$C_0\, a^{-\frac{1}{p}}(t) \le -y'(t) \le C_1\, a^{-\frac{1}{p}}(t) \quad \text{for} \quad t \ge t_2$$

with $C_0 = \frac{1}{2}(M_1)^{\frac{1}{p+1}}$ and $C_1 = (M_2)^{\frac{1}{p+1}}$. Integrating and using the fact that $\lim\limits_{t\to\infty} y(t) = 0$, we obtain

$$C_0 \int_t^\infty a^{-\frac{1}{p}}(t)\,dt \le y(t) \le C_1 \int_t^\infty a^{-\frac{1}{p}}(t)\,dt,$$

or

$$\begin{aligned} C_0 \int_t^\infty f\Big(C_0 \int_s^\infty a^{-\frac{1}{p}}(\tau)\,d\tau\Big) \int_s^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma\,ds &\le \int_t^\infty y(s)\, f\big(y(s)\big)\,ds \\ \le C_1 \int_t^\infty f\Big(C_1 \int_s^\infty a^{-\frac{1}{p}}(\tau)\,d\tau\Big) \int_s^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma\,ds& \end{aligned} \tag{22}$$

for $t \ge t_2$. Using l'Hôpital's Rule and the assumptions in (ii), we have

$$1 \le \lim_{t\to\infty} \frac{\displaystyle\int_t^\infty f\Big(C_3 \int_s^\infty a^{-\frac{1}{p}}(\tau)\,d\tau\Big) \int_s^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma\,ds}{\displaystyle\int_t^\infty f\Big(C_2 \int_s^\infty a^{-\frac{1}{p}}(\tau)\,d\tau\Big) \int_s^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma\,ds}$$

$$= \lim_{t\to\infty} \frac{\int_t^\infty f(C_3 \int_s^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma)\, ds}{\int_t^\infty f(C_2 \int_s^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma)\, ds} = M < \infty$$

for $C_3 > C_2 > 0$. Thus, part (ii) is proved, and this completes the proof of the theorem. □

Remark 3.2. Condition (6) is not needed in part (iv) of Theorem 3.1. In part (iii), condition (6) can be replaced by

$$\int_0^\infty \frac{1}{R(\sigma)} d\sigma = \infty. \tag{23}$$

Among other things, Theorem 3.1 gives conditions under which equation (1) always has a strong nonlinear limit-point type solution, and with some additional conditions holding, gives conditions under which all solutions of (1) are of the strong nonlinear limit-point type.

The following corollary gives necessary and sufficient conditions for any solution $y \in N_0$ to be of the nonlinear limit-circle type. Notice that in view of Theorem 3.1 (ii), such a solution could not be of the strong nonlinear limit-circle type, but there could be strong nonlinear limit-point type solutions.

Corollary 3.3. *Let* (3), (4), *and* (6) *hold and let* $y \in N_0$. *Then*

$$\int_0^\infty y(t)\, f\big(y(t)\big)\, dt < \infty$$

if and only if

$$\int_0^\infty \Big| f\Big(C \int_s^\infty a^{-\frac{1}{p}}(\tau)\, d\tau\Big)\Big| \int_s^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma\, ds < \infty$$

for every $0 < |C| < \infty$.

Proof. The result follows from (22) and the fact that f is nondecreasing. □

We can drop conditions (4) and (6) obtain a somewhat better nonlinear limit-point type result for the solutions in N_0.

Corollary 3.4. *Let* (3) *hold and let* $y \in N_0$. *If*

$$\int_0^\infty \Big| f\Big(C \int_s^\infty a^{-\frac{1}{p}}(\tau)\, d\tau\Big)\Big| \int_s^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma\, ds = \infty \tag{24}$$

for all $C \neq 0$, *then*

$$\int_0^\infty y(t)\, f\big(y(t)\big)\, dt = \infty.$$

Proof. Let $y \in N_0$, say $y(t) > 0$ for $t \geq T$. Then Lemma 3 in [10] implies there exists $C > 0$ such that

$$C \int_t^\infty a^{-\frac{1}{p}}(s)\, ds \leq y(t) \quad \text{for} \quad t \geq T.$$

Now f is nondecreasing, so

$$C \int_t^\infty f\big(C \int_s^\infty a^{-\frac{1}{p}}(\tau)\, d\tau\big) \int_s^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma\, ds \leq \int_t^\infty y(s)\, f\big(y(s)\big)\, ds. \tag{25}$$

The conclusion then follows from (24) and (25). □

Remark 3.5. If $a^{\frac{1}{p}}(t) = t^s$, then (24) holds for equation (2) if and only if

$$1 < s \leq 1 + \frac{1}{\lambda + 1}.$$

Observe here that we do not obtain the conclusion that y is of the strong nonlinear limit-point type since we are not requiring that (6) or (23) hold.

The next theorem formulates our results for equation (2).

Theorem 3.6. *Assume that* (3), (4), *and* (6) *hold. Then every nontrivial solution of* (2) *belongs to* $N_0 \cup N_1 \cup N_2$ *and* $N_i \neq \emptyset$ *for* $i = 0, 1, 2$.

(i) If $y \in N_0$, *then* $\int\limits_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty$. *Moreover,*

$$\int_0^\infty \big|y(\sigma)\big|^{\lambda+1} d\sigma < \infty$$

if and only if

$$\int_0^\infty \Big(\int_s^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma\Big)^{\lambda+1} ds < \infty\, .$$

(ii) If $y \in N_1$, *then*

$$\int_0^\infty \big|y(\sigma)\big|^{\lambda+1} d\sigma = \infty \quad \text{and} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty\, .$$

(iii) If $y \in N_2$, *then* $\int\limits_0^\infty \big|y(\sigma)\big|^{\lambda+1} d\sigma = \infty$. *Moreover,*

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma < \infty$$

if and only if

$$\int_0^\infty \frac{1}{R(s)} \Big(\int_s^\infty r(\sigma)\, d\sigma\Big)^\delta ds < \infty\, .$$

Proof. The conclusions follow immediately from Lemma 2.6 and Theorem

3.1 since

$$\lim_{u\to\infty}\frac{f(C_1u)}{f(C_2u)}=\left(\frac{C_1}{C_2}\right)^{\lambda}<\infty \quad\text{for}\quad \operatorname{sgn}C_1=\operatorname{sgn}C_2\neq 0\,. \qquad \square$$

Remark 3.7. For the case $\lambda > p$ in Theorem 3.6, the fact that all solutions are nonoscillatory can be obtained from Theorem 14.2 of Mirzov [15], but his result does not apply if $\lambda < p$ without additional hypotheses.

4. Examples and Concluding Remarks

We first note that condition (6) may be replaced by

$$g_+(t)=\left(\frac{a^{\frac{1}{p}}(t)R'(t)}{R^{1+\alpha}(t)}\right)_+ \quad\text{is bounded on } \mathbb{R}_+. \tag{26}$$

This can easily be seen from the fact that

$$\frac{R'_+(t)}{R(t)}=g_+(t)a^{-\frac{1}{p}}(t)R^{\alpha}(t)$$

and then applying (26) and Lemma 2.7. The condition

$$R\in AC^1_{\text{loc}}(R_+),\quad \lim_{t\to\infty}g(t)=0,\quad\text{and}\quad \int_0^\infty |g'(\sigma)|\,d\sigma<\infty, \tag{27}$$

was used by the authors [4], [5] in the study of asymptotic properties of solutions of equations of the form (1). In those papers, however, conditions (3), (4), and (6) were not assumed to hold.

Remark 4.1. It follows from [2, Theorem 2] that for $\lambda\neq p$ and (27) holding, all solutions of (2) are nonoscillatory if and only if (3) and (4) hold. Our result that all solutions of (2) are nonoscillatory for all p and λ if (3), (4), and (6) hold is somewhat better.

The following example shows that there is an equation that has a nonlinear limit-circle type solution that is not of the strong nonlinear limit-circle type, and at the same time, it has a nonlinear limit-point type solution that is not of the strong nonlinear limit-point type.

Example 4.2. Consider the equation

$$\big((t+1)^2y'\big)'+(t+1)^{-2}|y|^{\lambda}\operatorname{sgn}y=0\quad t\geq 0, \tag{28}$$

where $\lambda > 0$. The hypotheses of Theorem 3.6 are satisfied,

$$\int_0^\infty \Big(\int_t^\infty a^{-\frac{1}{p}}(\sigma)\,d\sigma \Big)^{\lambda+1} dt = \int_0^\infty (t+1)^{-1-\lambda} dt < \infty$$

and

$$\int_0^\infty \frac{1}{R(t)} \Big(\int_t^\infty r(\sigma)\,d\sigma \Big)^{\delta} dt = \int_0^\infty (t+1)^{-2}\,dt < \infty\,.$$

Thus, there is a solution y_1 of (28) such that

$$\int_0^\infty \big|y_1(\sigma)\big|^{\lambda+1} d\sigma < \infty \quad \text{and} \quad \int_0^\infty \frac{|y_1^{[1]}(\sigma)|^{\delta}}{R(\sigma)}\,d\sigma = \infty$$

and there is a solution y_2 such that

$$\int_0^\infty \big|y_2(\sigma)\big|^{\lambda+1} d\sigma = \infty \quad \text{and} \quad \int_0^\infty \frac{|y_2^{[1]}(\sigma)|^{\delta}}{R(\sigma)}\,d\sigma < \infty.$$

Moreover, $y_1 \in N_0$ and $y_2 \in N_2$. Theorem 3.6 (ii) also shows that this equation has a solution in N_1 that is of the strong nonlinear limit-point type. We want to emphasize here that Theorem 3.6 ensures that all nontrivial solutions of (28) are nonoscillatory.

Next, we present some examples of equations that are of the strong nonlinear limit-point type.

Example 4.3. By Theorem 3.6, both of the equations

$$\big(t^{\frac{3}{2}} y'\big)' + t^{-\frac{3}{2}} y = 0 \quad t \ge 0,$$

and

$$\big(t^4 (y')^3\big)' + t^{-\frac{4}{3}} |y|^2 \operatorname{sgn} y = 0 \quad t \ge 0,$$

are of the strong nonlinear limit-point type. This follows from the fact that all solutions are nonoscillatory and

$$\int_0^\infty \Big(\int_t^\infty a^{-\frac{1}{p}}(s)\,ds \Big)^{\lambda+1} dt = \int_0^\infty \frac{1}{R(t)} \Big(\int_t^\infty r(s)\,ds \Big)^{\delta} dt = \infty$$

for both of these equations.

Our final example is for the general case of the coefficients being powers of t.

Example 4.4. Consider the equation

$$\big(t^a |y'|^{p-1} y'\big)' + t^b |y|^{\lambda} \operatorname{sgn} y = 0 \quad t \ge 1, \tag{29}$$

where $p > 0$, $\lambda > 0$, and $a/p \le -b$. If

$$p < a \le \frac{\lambda+2}{\lambda+1}p \quad \text{and} \quad a - 2p - 1 \le b < -1,$$

then, by Theorem 3.6, equation (29) is of the strong nonlinear limit-point type.

As a final remark, we wish to point out the interesting fact that the results here do not explicitly depend on a relationship between p and λ. This can be seen from Example 4.2, where it does not matter if $\lambda = 2 > 1 = p$ or $\lambda = \frac{1}{2} < 1 = p$. This was not the case in the papers [4] and [5] where it was necessary to consider the cases $\lambda \ge p$ (*super-half-linear*) and $\lambda < p$ (*sub-half-linear*) separately. While there were some similarities and parallelisms between the results in those two papers, there were some striking differences as well. As noted earlier, in both of those papers it was assumed that

$$\int_0^\infty (a^{-\frac{1}{p}}(t) + r(t))\,dt = \infty,$$

which is explicitly excluded here.

If $p = 1$, then condition (3) becomes

$$\int_0^\infty \frac{d\sigma}{a(\sigma)} < \infty. \tag{30}$$

Other nonlinear limit-point/limit-circle results for equations with (30) holding can be found in [9], but they do not overlap those in this paper.

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