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A COMPARISON BETWEEN THE SELBERG AND THE BRUGGEMAN-KUZNETSOV TRACE FORMULAS II

C.J. Mozzochi
P.O. Box 1424, Princeton, NJ 08542, USA
e-mail: cjm@ix.netcom.com

Abstract: Determining the n-level correlation of the eigenvalues of the hyperbolic Laplacian for the modular surface is a notoriously difficult problem, and very little of substance is known.

In Part I we showed that with regard to one aspect of this problem one gets a dramatically better result by the Bruggeman-Kuznetsov trace formula. In this Part II we show that with regard to another aspect of the problem one gets a dramatically better result by the Selberg trace formula.

In this paper, in addition to the Selberg trace formula, we employ the Rudnick-Sarnak construction, which was created to analyze the n-level correlation of the zeros of the Riemann zeta function and principal L-functions, together with the weighted Bruggeman-Kuznetsov trace formula, as formulated by Sarnak and Iwaniec.

To the memory of Atle Selberg.

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1. Introduction

We consider $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$.

Hence we have that

$$\lambda_0 = 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \quad \text{are the eigenvalues}$$

(i.e. the point spectrum) of the hyperbolic Laplacian associated with Γ and

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$$\lambda_j = s_j(1-s_j) \text{ with } s_j = \frac{1}{2} + it_j,$$

so

$$\lambda_j = \frac{1}{4} + t_j^2 \quad \text{with } t_j > 0,\ j = 1,2,3,\cdots\ .$$

The Weyl-Selberg formula states that for $\Gamma = \mathrm{PSL}(2,\mathbf{Z})$,

$$M(T) = \sum_{0<t_j\le T} m(t_j) = \frac{1}{12}T^2 + c_1 T \log T + O(T)\,,$$

where $m(t_j)$ denotes the multiplicity of λ_j, which is equal to the dimension of the eigenspace of λ_j (cf. [13], p. 142).

This estimate follows from the Selberg trace formula by an appropriate use of test functions, say $h(t), g(x)$, where (as in every such application of the trace formula)

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{-ixt}dt,$$

$$h(t) \text{ is holomorphic in the strip } |\mathrm{Im}\ t| \le \frac{1}{2} + \varepsilon,$$

$$h(t) \ll (|t|+1)^{-2-\varepsilon} \text{ in the strip } |\mathrm{Im}\ t| \le \frac{1}{2} + \varepsilon,$$

and

$$h(t) \text{ is even.}$$

Then for the group Γ the Selberg trace formula states

$$h\left(\frac{i}{2}\right) + \sum_{t_j} m(t_j)h(t_j) = I + E + P + H\,,$$

where I is the contribution of the identity and E, P and H, respectively, are the contributions of the conjugacy classes of the elliptic, parabolic, and hyperbolic elements of Γ. Specifically,

$$I = \frac{1}{12}\int_{-\infty}^{\infty} t\tanh(\pi t)\ h(t)dt,$$

$$E = \sum_{\{R\}}\sum_{k=1}^{m-1} \frac{1}{2m\sin\left(\frac{\pi k}{m}\right)} \int_{-\infty}^{\infty} \frac{e^{-2\pi tk/m}}{1+e^{-2\pi t}} h(t)dt,$$

where $\{R\}$ denotes an elliptic class in Γ, and $m = \mathrm{ord}\{R\} = 2,3$,

$$P = P_{\mathrm{PSL}(2,\mathbf{Z})} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}(\frac{1}{2}+it) + \frac{\Gamma'}{\Gamma}(1+it)\right\} h(t)dt$$

$$+ g(0)\log\frac{\pi}{2} + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log\, n),$$

$$H = \sum_{\{P\}}\sum_{k=1}^{\infty}\frac{\log\, NP}{(NP)^{k/2} - (NP)^{-k/2}}g(k\log\, NP),$$

where $\{P\}$ denotes a primitive hyperbolic class in Γ and NP denotes its norm.

All series and integrals converge absolutely.

H. Iwaniec gives a nice presentation of the Selberg trace formula in [4].

The notation in this paper closely follows that in [11].

Throughout, c denotes an easily computed, absolute constant, but not necessarily the same constant in each occurrence.

Let

$$f(x_1,\ldots,x_n) = \int_{R^n}\phi(\alpha)\delta(\alpha_1+\cdots+\alpha_n)e(-x\cdot\alpha)d\alpha, \tag{1.1}$$

where ϕ is symmetric and compactly supported C^2 function on R^n and δ is the Dirac delta function.

It is not difficult to see that:

$$f(x_1,\ldots,x_n) \text{ is symmetric.} \tag{1.2}$$

$$f(x+t(1,\ldots,1)) = f(x) \text{ for } t\in R. \tag{1.3}$$

$$f(x)\to 0 \text{ rapidly as } |x|\to\infty \text{ in the hyperplane } \sum_{j=1}^{n}x_i = 0. \tag{1.4}$$

Let

$$C_n(f,h,T) = \sum_{t_1}\cdots\sum_{t_n}h_1\left(\frac{t_1}{T}\right)\cdots h_n\left(\frac{t_n}{T}\right)f\left(\frac{L}{2\pi}t_1,\ldots,\frac{L}{2\pi}t_n\right), \tag{1.5}$$

where $L=\log T$ except in Part 5, where $L = 2\pi cT$.

$$c = \frac{\mathrm{Vol}\,(\Gamma/N)}{4\pi} = \frac{1}{12},$$

$$h_j(r) = \int_{-\infty}^{\infty}g_j(u)e^{iru}du \text{ with even } g_j\in C_c^{\infty}(R). \tag{1.6}$$

Let

$$N(T) = |\{t_j \mid 0 < t_j \le N\}| \sim cT^2. \tag{1.7}$$

Let

$$\bar{t}_j = ct_j^2. \tag{1.8}$$

Let

$$C_n(f,T) = \sum_{0<t_1\le T} \cdots \sum_{0<t_n\le T} f(\bar{t}_1,\ldots,\bar{t}_n). \tag{1.9}$$

We note the obvious abuse of notation with regard to the fact that in the definition of $C_n(f,h,T)$ and $C_n(f,T)$ we drop the j subscript on the t's.

Let

$$k(\bar{h}) = \int_{-\infty}^{\infty} |r|^n h_1(r)\ldots h_n(r)dr. \tag{1.10}$$

In the first parts of this paper we establish by the Bruggeman-Kuznetsov trace formula under the assumption that the support of ϕ is included in

$$\left\{(\alpha_1,\ldots,\alpha_n)\,\middle|\, \sum_{j=1}^{n} |\alpha_j| \le (4-\varepsilon_0)\right\}$$
$$\cap \sim \left\{(\alpha_1,\ldots,\alpha_n)\,\middle|\, |\alpha_i| \ge \epsilon_1 \quad \text{for any fixed arbitrarily small } \epsilon_1 > 0\right\}.$$

Theorem 1.1. *For each* $n \ge 2$

$$C_n(f,h,T) = O(T^{n(2-\varepsilon)}).$$

Theorem 1.2. *For each* $n \ge 2$

$$C_n(f,T) = O\left(\frac{T^{n(2-\varepsilon)+1}}{\log T}\right),$$

where in both theorems $0 < \varepsilon < \frac{1}{Q}$, *and* Q *is large and computable.*

Then in the last part of the paper we establish by means of the Selberg trace formula with a weaker assumption on the support of ϕ, namely, just that it is compact and with the assumption that $L = 2\pi cT$.

Theorem 1.3. *For each* $n \ge 2$

$$C_n(f,h,T) = O(T^{n+1}).$$

Theorem 1.4. *For each* $n \ge 2$ *and for each* $\varepsilon > 0$

$$C_n(f,T) = O(T^{n+1+\varepsilon}).$$

2. Proof of Theorem 1.2 and Theorem 1.2

By straightforward manipulation we have

$$C_n(f,h,T) = \int\limits_{R^n} \prod_{i=1}^{n} \left\{ \sum_{t_i} h_i \left(\frac{t_i}{T} \right) e^{-iLt_i\alpha_i} \right\} \phi(\alpha)\delta(\alpha_1 + \cdots + \alpha_n) d\alpha \,.$$

Clearly,

$$\sum_{t_j} h \left(\frac{t_j}{T} \right) e^{-iLt_j\alpha} = \sum_{0<t_j} h(t_j, \alpha, T)\,,$$

where

$$h(t,\alpha,T) = h\left(\frac{t}{T}\right) e^{-iLt\alpha} + h\left(\frac{t}{T}\right) e^{iLt\alpha} = h_+(t,\alpha,T) + h_-(t,\alpha,T)\,.$$

In Section 4 we will establish by means of the weighted Bruggeman-Kuznetsov trace formula

Lemma 2.1.

$$\sum_{0<t_j} h(t_j,\alpha,T) = T^2 g(\alpha T) + O(T^{2-\varepsilon}), \quad \text{if} \quad |\alpha| \leq (4-\varepsilon_0),$$

where

$$g(\theta) = \frac{1}{12} \int\limits_0^\infty x h(x) \cos(\theta \log Tx) dx.$$

Lemma 2.2. *For any fixed arbitrarily small* $\epsilon_1 > 0$

$$g(\theta) \ll \begin{cases} 1 & \text{for all } \theta\,, \\ \frac{1}{|\theta|^2} & \epsilon_1 \leq |\theta|\,. \end{cases}$$

Proof. Immediate by partial integration twice. □

Hence by Lemma 2.1 we have

$$\begin{aligned} C(f,h,T) &= \int\limits_{R^n} \prod_{j=1}^{n} (T^2 g_j(\alpha_j T) + O(T^{2-\varepsilon}))\phi(\alpha)\delta(\alpha_1 + \cdots + \alpha_n) d\alpha \\ &= T^{2n} \int\limits_{R^n} \prod_{j=1}^{n} g_j(\alpha_j T)\phi(\alpha)\delta(\alpha_1 + \cdots + \alpha_n) d\alpha + O(T^{n(2-\epsilon)}). \end{aligned}$$

This is established by unfolding the integral and then applying Lemma 2.2 in the obvious way to each $g_j(\alpha_j T)$ in each resulting product term, where the $\epsilon_1 > 0$ in that lemma is the same $\epsilon_1 > 0$ in the definition of the set, in which the support of ϕ is included.

Let $y_j = T\alpha_j$. Since $y_1 + \cdots + y_n = 1$, by change of variables we have

$$C(f,h,T) = T^{n+1} \int\limits_V \prod_{j=1}^{n} g_j(y_j)\phi\left(\frac{y}{T^2}\right)\delta(y_1 + \cdots + y_n)dy + O(T^{n(2-\varepsilon)}),$$

where

$$V = \{(y_1, \ldots, y_n) \in R^n \mid y_k \ll T\}.$$

By expanding ϕ in a Taylor series about 0 and truncating the series after the first term, we have

$$\phi\left(\frac{y}{T^2}\right) = \phi(0) + O(1),$$

so that by Lemma 2.2

$$C_n(f,h,T) = T^{n+1}\phi(0) \int\limits_V \prod_{j=1}^{n} g_j(y_j)\delta(y_1 + \cdots + y_n)dy + O(T^{n(2-\varepsilon)}).$$

Lemma 2.3.

$$\int\limits_V \prod_{j=1}^{n} g_j(y_j)\delta(y_1 + \cdots + y_n)dy = \int\limits_{R^n} \prod_{j=1}^{n} g_j(y_j)\delta(y_1 + \cdots + y_n)dy + O(1).$$

Proof. Indeed, the difference between the two integrals is an integral over a set contained in the union

$$\bigcup_{k=1}^{n} V_k, \text{ where } V_k = \{(y_1, \ldots, y_n) \in R^n \mid |y_k| \gg T\}. \qquad \square$$

It suffices to estimate the integral over each such V_k, say $k = n$. Then by Lemma 2.2

$$\int\limits_{V_n} \prod_{j=1}^{n-1} g_j(y_j)g_n(-(y_1, \ldots, y_{n-1}))dy_1, \ldots, y_{n-1} \ll 1.$$

But since h is even, $|x|h(x) = \int_{-\infty}^{\infty} g(\theta) \in^{i\theta x} d\theta$ so that by Parseval's equation (cf. [14], Volume 2, p. 248) we have

$$\int\limits_{R^n} \prod_{j=1}^{n} g_j(y_j)\delta(y_1 + \cdots + y_n)dy = \frac{1}{2\pi} \int |r|^n h_1(r) \cdots h_n(r)dr,$$

and Theorem 1.1 is established.

We now establish Theorem 1.2 by first showing that in Theorem 1.1 we can take $h_j = \chi_E$, the characteristic function of the set $E = [-1, 1] - [-\varepsilon, \varepsilon]$.

In Theorem 1.1 we established that for h_j as in (1.6) and f satisfying (1.1)

$$\frac{1}{T^{n(2-\varepsilon_1)}}\sum h_1\left(\frac{t_1}{T}\right)\cdots h_n\left(\frac{t_n}{T}\right)f\left(\frac{L}{2\pi}t_1,\ldots,\frac{L}{2\pi}t_n\right)-k(\bar{h})u(f,T)\to 0, \quad (2.1)$$

where $\mu(f,T)=cT^{2(1-n)+n\varepsilon_1}\phi(0)$.

By taking linear combinations, we obtain

$$\frac{1}{T^{n(2-\varepsilon_1)}}\sum h\left(\frac{\vec{t}}{T}\right)f\left(\frac{L}{2\pi}\vec{t}\right)-k(\bar{h})u(f,T)\to 0\,, \quad (2.2)$$

where $(\vec{r})$ is a finite linear combination of functions of the form $|r|h_1(r_1)\cdots|r|h_n(r_n)$

$$k(\bar{h})=\int_{-\infty}^{\infty}|r^n|h(r,\ldots,r)dr.$$

We extend the validity of (2.2) by noting that if

$$H(\bar{r})=\chi_E(x_1)\cdots\chi_E(x_n),$$

then given $\varepsilon>0$ there are finite linear combinations h_1,h_2 as above so that $h_1\le H\le h_2$ and $\int_{-\infty}^{\infty}|r^n|(h_1-h_1)(r,\ldots,r)dr<\varepsilon$.

We use these to show that (2.2) is valid for H. Indeed, given f as above, we can find an $f_+>0$ with $|f|<f_+$ and f_+ admissible for (2.2) (since we assume ϕ is C^2).

If we set

$$D(H,f;T)=\frac{1}{T^{n(2-\varepsilon_1)}}\sum H\left(\frac{\vec{t}}{T}\right)f\left(\frac{L}{2\pi}\vec{t}\right)-k(H)u(f,T),$$

then

$$\begin{aligned}
|D(H,f;T)|&\le|D(h_1,f;T)|+|D(H-h_1,f;T)|\\
&\le|D(h_1,f;T)\Big|+\frac{1}{T^2}\sum(h_2-h_1)\left(\frac{\vec{t}}{T}\right)f_+\left(\frac{L}{2\pi}\vec{t}\right)+K(H-h_1)|u(f,T)|\\
&\le|D(h_1,f;T)|+|D(h_2-h_1,f_+;T)|+k(h_2-h_1)|u(f_+,T)|+k(H-h_1)|u(f,T)|,
\end{aligned}$$

and since (2.2) is valid for the first two terms, we find

$$\begin{aligned}
\limsup_T|D(H,f;T)|&\le k(h_1-h_1)|u(f_+,T)|+k(H-h_1)|u(f,T)|\\
&<\varepsilon(|u(f_+,T)|+|u(f,T)|)\,.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have $D(H,f;T)\to 0$.

Next, we need to discuss the passage from the normalizations $\left(\frac{L}{2\pi}\right)t$ appearing in the definition of the smooth sums $C_n(f,h,T)$ in (1.5), and the normalizations $\bar{t} = ct^2$, initially used to define $C_n(f,T)$ in (1.9). We explain it for the pair correlation (i.e. $n = 2$). Consider for $\psi(x) = f(x+y,y)$,

$$\widetilde{C}(\psi,T) = \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T}} \psi(ct_1^2 - ct_2^2)\,,$$

$$C(\psi,T) = \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T}} \psi\left(\frac{L}{2\pi}(t_1 - t_2)\right)\,.$$

Then we claim that

$$S(\psi,T) = \widetilde{C}(\psi,T) - C(\psi,T) = O(T^{n(2-\epsilon)}).$$

This will show that the different normalizations lead to the same main term.

$$S(\psi,T) = S_1(\psi,T) + S_2(\psi,T) + S_3(\psi,T)\,,$$

where

$$S_1(\psi,T) = \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T \\ t_1 = t_2}} \left(\psi(ct_1^2 - ct_2^2) - \psi\left(\frac{L}{2\pi}(t_1 - t_2)\right)\right)\,,$$

$$S_2(\psi,T) = \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T \\ t_2 < t_1}} \left(\psi(ct_1^2 - ct_2^2) - \psi\left(\frac{L}{2\pi}(t_1 - t_2)\right)\right)\,,$$

$$S_3(\psi,T) = \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T \\ t_1 < t_2}} \left(\psi(ct_1^2 - ct_2^2) - \psi\left(\frac{L}{2\pi}(t_1 - t_2)\right)\right)\,.$$

By the Weyl-Selberg law and partial integration we have

$$S_1(\psi,T) \ll T^2.$$

By the mean value theorem it is easy to see that

$$S_2(\psi,T) \ll T \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T \\ t_2 < t_1}} \psi'(\delta(t_1,t_2,T)_{|}t_1 - t_2|\,, \tag{2.3}$$

where $0 < \frac{L}{2\pi}(t_1 - t_2) < \delta(t_1,t_2,T) < c(t_1^2 - t_2^2)$ and

$$S_3(\psi,T) \ll T \sum_{\substack{T \le t_1 \le 2T \\ T \le t_2 \le 2T \\ t_1 < t_2}} |\psi'(\delta(t_1,t_2,T)|\ |t_1 - t_2|\,, \tag{2.4}$$

where

$$c(t_1^2 - t_2^2) < \delta(t_1, t_2, T) < \frac{L}{2\pi}(t_1 - t_2) < 0.$$

For this fixed ψ let $\psi_1 \geq |\psi'|$ be a rapidly decreasing function on R, which is even and monotone on $[0, \infty]$. Then by (2.3)

$$S_2(\psi, T) \ll \frac{T}{\log T} \sum_{\substack{T \leq t_1 \leq 2T \\ t \leq t_2 \leq 2T \\ t_2 < t_1}} \psi_1\left(\frac{L}{2\pi}(t_1 - t_2)\right) |\frac{L}{2\pi}(t_1 - t_2)|\,,$$

so that

$$S_2(\psi, T) \ll \frac{T}{\log T} \sum_{\substack{T \leq t_1 \leq 2T \\ T \leq t_2 \leq 2T}} \psi_1\left(\frac{L}{2\pi}(t_1 - t_2)\right) \left|\frac{L}{2\pi}(t_1 - t_2)\right|.$$

We can find a majorant ψ_1 and h which are admissible in Theorem 1.1 satisfying $\psi_+(x) \geq |x|\psi_1(x)$ and $h \geq \chi_{[1,2]}$. Then with these choices, we have

$$S_2(\psi, T) \ll \frac{T}{\log T} \sum_{t_1} \sum_{t_2} h\left(\frac{t_1}{T}\right) h\left(\frac{t_2}{T}\right) \psi_+\left(\frac{L}{2\pi}(t_1 - t_2)\right)\,,$$

so that

$$S_2(\psi, T) \ll \frac{T^{n(2-\varepsilon)+1}}{\log T}$$

by Theorem 1.1. By a similar argument, we have $S_3(\psi, T) \ll \frac{T^{n(2-\varepsilon)+1}}{\log T}$.

3. The Weighted Bruggeman-Kuznetsov Trace Formula

Theorem 3.1. (Bruggeman-Kuznetsov) *Let h satisfy the Selberg trace formula conditions* (cf. [4]). *Then*

$$\sum_{1 \leq j} h(t_j)\bar{\mathcal{V}}_j(m)\mathcal{V}_j(n) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\bar{\mathcal{N}}(m,t)\mathcal{N}(n,t)dt$$
$$= \delta_{mn}\frac{1}{\pi}\int_{-\infty}^{\infty} t\,\tanh(\pi t)h(t)dt + \sum_{c=1}^{\infty}\frac{s(m,n,c)}{c}h^+\left(\frac{4\pi\sqrt{|mn|}}{c}\right),$$

where $h^+(x) = 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{h(t)t}{\cosh \pi t}dt$ *and where* $\mathcal{V}_j(n)$, $\mathcal{V}_j(m)$, $\mathcal{N}(n,t)$, $\mathcal{N}(m,t)$ *are Fourier coefficients of any arithmetical system of Maass forms, and the eigenpacket of Eisenstein series.*

For a nice proof of this theorem, see [4].

Now choose the unique Hecke basis $\{\phi_j\}$. Then

$$\mathcal{V}_j(n) = \lambda_j(n)\gamma_{\phi_j}$$

and

$$\mathcal{V}_j(m) = \lambda_j(m)\gamma_{\phi_j}\,,$$

where

$$\gamma_{\phi_j} = \left(\frac{L(1, \operatorname{sym}^2 \phi_j)}{2\pi}\right)^{1/2}, \quad \text{and} \quad \lambda_j(1) = 1.$$

Letting $n = m$ we get the so-called weighted Bruggeman-Kuznetsov trace formula.

Theorem 3.2.

$$\sum_{1\leq j} h(t_j)|\lambda_j(n)|^2\gamma_{\phi_j}^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)|\mathcal{N}(n,t)|^2 dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} t\,\tanh(\pi t)h(t)dt + \sum_{c=1}^{\infty}\frac{s(n,n,c)}{c}h^+\left(\frac{4\pi n}{c}\right).$$

4. Proof of Lemma 2.1

The techniques used in this section were first used by Iwaniec in [2]. They were refined and elaborated by Sarnak in [12]. See also [5] and [9].

Lemma 4.0.

$$h(z,\alpha,T) \text{ is even},$$
$$h(z,\alpha,T) \text{ is entire},$$
$$h(z,\alpha,T) \ll_k \frac{T^{|\alpha|\,|y|}\varepsilon^{\frac{c|y|}{T}}}{\left(1+\frac{|z|}{T}\right)^k}.$$

Proof. The proof follows from partial summation and the elementary properties of the Fourier transform. Note that the lemma is true if $h(z,\alpha,T)$ is replaced with $h_+(z,\alpha,T)$ or $h_-(z,\alpha,T)$. □

Lemma 4.1. $t_j^{-\epsilon} \ll_\epsilon |\gamma_{\phi_j}|^2 \ll \log t_j$.

Proof. This is established in [11]. □

Lemma 4.2. $\sum_{0<t_j\le T^2} |h(t_j,\alpha,T)| = O(T^2)$.

Proof. The proof follows by partial summation. □

Let

$$S(\phi_j) = \sum_{n=1}^{\infty} \gamma_{\phi_j}^2 |\lambda_j(n)|^2 g\left(\frac{n}{M}\right), \tag{4.1}$$

where $M = T^{\Delta}$ for $0 < \Delta < \frac{1}{100}$.

Lemma 4.3. *$S(\phi_j) = E_1(t_j)$, where $E_1(t_j) = O(M^{\frac{s}{4}}|t_j^{\epsilon}|)$.*

Proof. This is immediate from the estimate due to Kim and Sarnak (cf. [4])

$$|\lambda_j(n)| \ll n^{\frac{7}{64}+\epsilon}$$

and Lemma 4.1. □

In Lemma 1 in [6] let $\sigma = 1-\delta$, where $\delta > 0$ is chosen so that $(\delta b+\epsilon) < \frac{1}{10}$. Consider $0 < t_j \le T^2$. Define $R = \{\rho = \beta + i\gamma \mid 0 < \delta \le 1/2, (1-\delta) \le \beta < 1, |\gamma| \le \log^3 T^2\}$. Define $B = \{t_j \mid 0 < t_j \le T^2$ and $L(\rho, \mathrm{sym}^2\phi_j) = 0$ if $\rho \in R\}$. By Lemma 1 in [6] we have

$$\sum_{t_j\in B} m(t_j) \ll T^{\frac{1}{10}}. \tag{4.2}$$

Define $G = \{t_j | 0 < t_j \le T^2\} - B$. Clearly, if $t_j \in G$, then $L(s, \mathrm{sym}^2\phi_j)$ has no zero in the domain $(1-\delta) < \sigma < 1$, $|t| \le \log^3 T^2$. Hence by Lemma 2 in [6] in the domain $(1-\delta/2) < \sigma < 1$, $|t| \le \log^2 T$ we have

$$L(s, \mathrm{sym}^2\phi_j) \ll_\epsilon T^{\epsilon} \qquad \text{for any } \epsilon > 0. \tag{4.3}$$

Lemma 4.4. *If $t_j \in G$,*

$$\sum_{n=1}^{\infty} |\lambda_j(n)|^2 g\left(\frac{n}{M}\right) = \frac{ML(1,\mathrm{sym}^2\phi_j)}{\zeta(2s)} + O\left(M^{1-\delta}T^{\epsilon}\right).$$

Proof.

$$S = \sum_{n=1}^{\infty} |\lambda_j(n)|^2 g\left(\frac{n}{M}\right) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} \frac{L(s,\mathrm{sym}^2\phi_j)}{\zeta(s)}\tilde{g}(s)M^s ds,$$

where $\tilde{g}(s) = \int_0^{\infty} g(x)x^{s-1}dx$. By moving the line of integration to $(1-\delta)$ we get

$$S = \frac{ML(1,\mathrm{sym}^2\phi_j)}{\zeta(2)} + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=1-\delta} \frac{L(s,\mathrm{sym}^2\phi_j)}{\zeta(2s)}\tilde{g}(s)M^s ds,$$

and the result follows from (4.3). □

Lemma 4.5. *If $t_j \in G$, then*

$$S(\phi_j) = \frac{12}{\pi} M + E_2(T)\,, \quad \text{where} \quad E_2(T) = O\left(M^{1-\delta} T^{\epsilon}\right).$$

Proof. This follows immediately from Lemma 4.4 and the fact that $\zeta(2) = \frac{\pi^2}{6}$, $\gamma_{\phi_j}^2 = \frac{2\pi}{L(1,\mathrm{sym}^2\phi_j)}$ and, by Lemma 4.1, $\gamma_{\phi_j}^2 \ll t_j^{\epsilon}$. □

Lemma 4.6. $\sum\limits_{T^2<t_j} h(t_j, \alpha, T) = O(1)$.

Proof. The proof follows from partial summation. □

Lemma 4.7. $\sum\limits_{T^2<t_j} S(\phi_j) h(t_j, \alpha, T) = O(M^{5/4} T^{\varepsilon})$.

Proof. The proof follows from Lemma 4.3 and partial summation. □

Lemma 4.8. $\sum\limits_{n=1}^{\infty} g\left(\frac{n}{M}\right) = M + O(1)$.

Proof. Immediate by partial summation, and the trivial observation $\Delta(x) = \sum\limits_{n=1}^{x} 1 = x + O(1)$. □

Lemma 4.9.

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} t \tanh(\pi t) h(t, \alpha, T) dt\right)$$
$$= \frac{4MT^2}{\pi} \int_0^{\infty} x W(x) \cos(\alpha T \log Tx) dx + O(T^2).$$

Proof. The proof follows in a straightforward manner from the easily established decomposition

$$\frac{2}{\pi} \int_0^{\infty} t\, h_+(t, \alpha, T) dt + \frac{2}{\pi} \int_0^{\infty} t\, h_-(t, \alpha, T) dt + O(1)$$

and Lemma 4.8. □

Lemma 4.10. *Assume* $0 < x$ *and* $0 < t$. *Let* $z = (\frac{x^2}{4} + t^2)^{1/2}$ *Then*

$$J_{2it}(x) = \pi^{-1/2} z^{-1/2} e^{\frac{-\pi i}{4}} e^{\pi t} e\left(\frac{z}{\pi} - \frac{t}{\pi} \log\left(\frac{2(z+t)}{x}\right)\right) \cdot \left\{1 + O\left(\frac{1}{t}\right)\right\}.$$

Proof. This is established in [1]. □

Clearly, $h_T^+(x) = 2i \int\limits_{-\infty}^{\infty} J_{2it}(x) \left(\frac{h_+(t,\alpha,T)+h_-(t,\alpha,T)}{\cosh \pi t}\right) t\, dt$.

Using the fact that $J_{2it}(x) = \overline{J_{-2it}(x)}$, we have

$$h_T^+(x) = 4i^2 \mathrm{Im} \int_{-\infty}^{\infty} J_{2it}(x) \frac{h_+(t,\alpha,T)t}{\cosh \pi t}\, dt.$$

Lemma 4.11. *For every $0 < \varepsilon < \frac{1}{100}$ there exists $0 < \Delta(\varepsilon) < \frac{1}{100}$ such that if $0 < \Delta < \Delta(\varepsilon)$*

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=1}^{T^{1-\varepsilon}} \frac{S(n,n,c)}{c} h_T^+(x) \ll T^2\,.$$

Proof.

$$h_T^*(x) = 4i^2 \mathrm{Im} \int_0^B \frac{J_{2it}(x) h_+(t,\alpha,T) t\, dt}{\cosh \pi t} - 4i^2 \mathrm{Im} \int_0^A \bar{J}_{2it}(x) h_+(-t,\alpha,T) t\, dt$$

$$\ll \int_0^B |J_{2it}(x)| \frac{|h_+(t,\alpha,T)| t\, dt}{\cosh \pi t}$$

$$\ll \int_0^1 \frac{|J_{2it}(x)| t\, dt}{\cosh \pi t} + \int_1^B \frac{|J_{2it}(x)| h_+(t,\alpha,T)| t\, dt}{\cosh \pi t} = I_a + I_b$$

We first consider I_a

$$I_a \ll \int_0^1 \frac{\left(\frac{x^2}{4} + t^2\right)^{-1/4} \varepsilon^{\pi t} t\, dt}{\cosh \pi t} + \int_0^1 \frac{\left(\frac{x^2}{4} + t^2\right)^{-1/4} \varepsilon^{\pi t} dt}{\cosh \pi t}.$$

But it is easy to see that

$$\left(\frac{x^2}{4} + t^2\right)^{-1/4} \ll T^{1/2-\varepsilon/2}.$$

Hence $I_a \ll T^{1/2-\frac{\varepsilon}{2}}$. We now consider I_b.

$$I_b \ll \int_1^B \frac{t^{1/2} \varepsilon^{\pi t} |h_+(t,\alpha,T)|}{\left(1 + \frac{x^2}{4t^2}\right)^{1/4} \cosh \pi t} dt \ll \int_1^B t^{1/2} |h_+(t,\alpha,T)| dt.$$

By Lemma 4.0 and a simple change of variable we have

$$I_b \ll T^{3/2}.$$

The proof is completed by straightforward application of the Weil bound for Kloosterman sums, namely

$$S(n,n,c) \ll n^{1/2} c^{1/2+\varepsilon_2}. \qquad \square \tag{4.4}$$

The Taylor series for $J_{2it}(x)$ is

$$J_{2it}(x) = \left(\frac{x}{2}\right)^{2it} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+1+2it)} \left(\frac{x}{2}\right)^{k}$$
$$= \frac{\left(\frac{x}{2}\right)^{2it}}{\Gamma(1+2it)} + \left(\frac{x}{2}\right)^{2it} \sum_{k=1}^{\infty} \frac{1}{k!\Gamma(k+1+2it)} \left(\frac{x}{2}\right)^{2k} = J^a_{2it}(x) + J^b_{2it}(x).$$

And if $0 < x < 1$ we have
$$J^b_{2it}(x) \ll \frac{x^2}{|\Gamma(1+2it)|},$$
since $|\Gamma(1+v)| \leq |\Gamma(k+1+v)|$.

Lemma 4.12. *For every $0 < \varepsilon_0 < \frac{1}{100}$ there exists $0 < \varepsilon_0' < \frac{1}{100}$ and $0 < \Delta(\varepsilon_0) < \frac{1}{100}$ such that for all $0 < \Delta \leq \Delta(\varepsilon_0)$ for all $0 < \varepsilon \leq \varepsilon_0'/2$ (where ε_0 is as specified in Lemma 2.1)*
$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=T^{1-\varepsilon}}^{\infty} \frac{S(n,n,c)}{c} \left(4i^2 \mathrm{Im} \int_{-\infty}^{\infty} J^a_{2it}(x) \frac{h_+(t,\alpha,T)}{\cosh \pi t} \right) \ll T^2.$$

Proof. Consider
$$I_a = \int_{-\infty}^{\infty} \frac{\left(\frac{x}{2}\right)^{2it} h_+(t,\alpha,T)}{\Gamma(1+2it)} \, dt\,.$$
It is immediate from Stirling's formula that:
$$\text{For} \quad 0 < t_0 \leq t: \qquad \frac{1}{\Gamma(\sigma+2it)} \ll_\sigma e^{\pi t} t^{1/2-\sigma}. \tag{3.5}$$

In I_a we move the line of integration to $-i\theta$, where $\theta = \left(\frac{1}{4} + \varepsilon_2\right)$, where ε_2 is that of (3.4) obtaining
$$I_a = \left(\frac{x}{2}\right)^{2\theta} \int_{-A}^{B} \frac{\left(\frac{x}{2}\right)^{2it} h_+(t-i\theta,\alpha,T) t \, dt}{\Gamma(1+2\theta+2it)\cosh(\pi(t-i\theta))}$$
$$- i\theta \left(\frac{x}{2}\right)^{2\theta} \int_{-A}^{B} \frac{\left(\frac{x}{2}\right)^{2it} h_+(t-i\theta,\alpha,T) dt}{\Gamma(1+2\theta+2it)\cosh(\pi(t-i\theta)}.$$

By a simple change of variable, using the fact that $\frac{1}{\Gamma(z)}$ is entire and $\Gamma(z) = \overline{\Gamma(\bar{z})}$ together with (3.5) we obtain
$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{B} |h_+(t-i\theta,\alpha,T)| t^{(1/2-2\theta} dt$$

$$+ \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{A} |h_+(-t - i\theta, \alpha, T)| t^{(1/2-2\theta)} dt$$

$$+ \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{B} |h_+(t - i\theta, \alpha, T)| t^{(-1/2-2\theta)} dt$$

$$+ \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{A} |h_+(-t - i\theta, \alpha, T)| t^{(-1/2-2\theta)} dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right).$$

Then by Lemma 4.0 we have

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{B} t^{(1/2-2\theta)} \frac{1}{\left(1 + |\frac{t}{T} - \frac{i\theta}{T}|\right)^k} dt$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{A} t^{(1/2-2\theta)} \frac{1}{\left(1 + |\frac{t}{T} + \frac{i\theta}{T}|\right)^k} dt$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{B} t^{(-1/2-2\theta)} \frac{1}{\left(1 + |\frac{t}{T} - \frac{i\theta}{T}|\right)^k} dt$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{A} t^{(-1/2-2\theta)} \frac{1}{\left(1 + |\frac{t}{T} + \frac{i\theta}{T}|\right)^k} dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right).$$

So that

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{A} \frac{1}{\left(1 + \frac{|t|}{T}\right)^k} dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right),$$

and by a simple change of variable

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta+1}.$$

The above moving of the line of integration is valid if

$$I_B = \int_{0}^{\theta} J_{2iz}^{a}(x) \frac{h_+(z, \alpha, T) z dz}{\cosh \pi z} \to 0 \quad \text{for each } x \text{ and } T,$$

as $B \to \infty$, where $z(t) = B - it$ for $0 \le t \le \theta$, and

$$I_A = \int_0^\theta J^a_{2iz}(x) \frac{h_+(z,\alpha,T) z dz}{\cosh \pi z} \to 0 \quad \text{for each } x \text{ and } T,$$

as $A \to \infty$, where $z(t) = -A - it$ for $0 \le t \le \theta$.

By direct substitution and using the fact that

$$\frac{1}{|\Gamma(\sigma+it)|} \le e^{\frac{\pi t}{2}} (ct)^{1/2-\sigma}$$

we have

$$I_B \ll \int_0^\theta \left(\frac{x}{2}\right)^{2t} |h_+(B-it,\alpha,T)| B (cB)^{-1/2-2t} dt$$
$$+ \int_0^\theta \left(\frac{x}{2}\right)^{2t} |h_+(B-it,\alpha,T)| t (cB)^{-1/2-2t} dt$$

so that by Lemma 4.0 we have that $I_B \to 0$ for each x, T as $B \to \infty$. In a similar way we show that $I_A \to 0$ for each x, T as $A \to \infty$. The proof of the lemma is completed by straightforward application of the Weil bound for Kloosterman sums. □

Lemma 4.13. *For every $0 < \varepsilon_0 < \frac{1}{100}$ there exists $0 < \varepsilon_0' < \frac{1}{100}$ and $0 < \Delta(\varepsilon_0) < \frac{1}{100}$ such that for all $0 < \Delta \le \Delta(\varepsilon_0)$ for all $0 < \varepsilon < \varepsilon_0'/2$ (where ε_0 is as specified in Lemma 2.1)*

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=T^{1-\varepsilon}}^{\infty} \frac{S(n,n,c)}{c} \left(4i^2 \mathrm{Im} \int_{-\infty}^{\infty} J^b_{2it}(x) \frac{h_+(L(t,\alpha,T)) t\, dt}{\cosh \pi t} \right) \ll T^2.$$

Proof. Consider

$$I_b = \int_{-\infty}^{\infty} J^b_{2it}(x) \frac{h_+(L(t,\alpha,T)) t dt}{\cosh \pi t}.$$

By a simple change of variable using the fact that $\frac{1}{\Gamma(z)}$ is entire and $\Gamma(z) = \overline{\Gamma(\bar{z})}$ together with (3.5) we obtain

$$I_b \ll x^2 \int_{t_0}^{B} |h_+(L(t,\alpha,T))| t^{1/2} dt + x^2 \int_{t_0}^{A} |h_+(L(-t,\alpha,T))| t^{1/2} dt + O(x^2).$$

Then by Lemma 4.0 and by a simple change of variable we have

$$I_b \ll x^2 T^{3/2}.$$

The proof of the lemma is completed by straightforward application of the Weil bound for Kloosterman sums. □

Lemma 4.14.

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right)\left(\frac{1}{\pi}\int_{-\infty}^{\infty} H_T(t)|\mathcal{N}(n,t)|^2 dt\right) = O(TM^{1+\epsilon}).$$

Proof. The result follows by straightforward calculation from the well-known facts and definitions:

$$\mathcal{N}(n,t) \mathrel{\mathring{=}} \left(\frac{4\pi|n|}{\cosh \pi t}\right)^{1/2} \varphi(n, 1/2+it),$$

where

$$\varphi(n,s) = \pi^s \Gamma^{-1}(s)\zeta(2s)^{-1}|n|^{-1/2} \sum_{ab=|n|} \left(\frac{a}{b}\right)^{s-1/2},$$

$$\frac{1}{\zeta(s)} = O(\log^{\Delta}(t)),$$

uniformly for $1 \le \sigma$, and Sterling's asymptotic formula in the form

$$\Gamma(\sigma + at) = (2\pi)^{1/2} t^{\sigma-1/2} e^{\frac{-\pi t}{2}} \left(\frac{t}{e}\right)^{it} (1 + O(t^{-1}))$$

if $t > 0$, where the implied constant depends on σ. □

We now prove Lemma 2.1.

Let $S = \sum_{0<t_j} S(\phi_j) h(t_j, \alpha, T)$,

$$S = \sum_{0<t_j\le T^2} S(\phi_j)h(t_j,\alpha,T) + \sum_{T^2<t_j} S(\phi_j)h(t_j,\alpha,T).$$

By Lemma 4.5 we have

$$S = \frac{12M}{\pi} \sum_{\substack{0<t_j\le T^2 \\ t_j\in G}} h(t_j,\alpha,T) + \sum_{\substack{0<t_j\le T^2 \\ t_j\in G}} E_2(T)h(t_j,\alpha,T)$$

$$+ \sum_{\substack{0<t_j\le T^2 \\ t_j\in B}} S(\phi_j)h(t_j,\alpha,T) + \sum_{T^2<t_j} S(\phi_j)h(t_j,\alpha,T)$$

$$\pm \frac{12M}{\pi} \sum_{\substack{0<t_j\le T^2 \\ t_j\in B}} h(t_j,\alpha,T) \pm \frac{12M}{\pi} \sum_{T^2\le t_j} h(t_j,\alpha,T).$$

Hence

$$\begin{aligned} S = \frac{12M}{\pi} \sum_{0<t_j} h(t_j,\alpha,T) &+ \sum_{\substack{0<t_j\le T^2 \\ t_j\in G}} E_2(T) h(t_j,\alpha,T) \\ &+ \sum_{\substack{0<t_j\le T^2 \\ t_j\in B}} S(\phi_j) h(t_j,\alpha,T) + \sum_{T^2<t_j} S(\phi_j) h(t_j,\alpha,T) \\ &- \frac{12M}{\pi} \sum_{\substack{0<t_j\le T^2 \\ t_j\in B}} h(t_j,\alpha,T) - \frac{12M}{\pi} \sum_{T^2<t_j} h(t_j,\alpha,T). \end{aligned}$$

Hence by Lemma 4.2, Lemma 4.3, Lemma 4.5, Lemma 4.6 and Lemma 4.7 we have

$$\begin{aligned} S = \frac{12M}{\pi} \sum_{0<t_j} h(t_j,\alpha,T) + O(M^{1-\delta} T^{2+\epsilon}) + O\left(M^{5/4} T^{\frac{1}{10}+\epsilon}\right) \\ + O(M^{5/4} T^{\varepsilon}) + O(M T^{1/10}) + O(M); \end{aligned}$$

so that

$$S = \frac{12M}{\pi} \sum_{0<t_j} h(t_j,\alpha,T) + O(M^{1-\delta} T^{2+\epsilon}) + O\left(M^{5/4} T^{\frac{1}{10}+\epsilon}\right).$$

But by interchanging the order of summation we have

$$S = \sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \left(\sum_{0<t_j} h(t_j,\alpha,T) |\lambda_j(n)|^2 \gamma_{\phi_j}^2 \right).$$

Then Lemma 2.1 follows from Theorem 3.2, Lemma 4.9, Lemma 4.11, Lemma 4.12, Lemma 4.13 and Lemma 4.14.

5. A Proof of Theorem 1.3 and Theorem 1.4

The following well known, important lemma can be established by the Selberg trace formula in a straightforward manner.

Lemma 5.1. *For the modular surface and for $T > 0$*

$$N(T+1) - N(T) = O(T).$$

We first establish Theorem 1.3. Since the h functions are rapidly decreasing, we may assume

$$C_n(f,h,T) = \sum_{t_1}^{T} \cdots \sum_{t_n}^{T} h_1\left(\frac{t_1}{T}\right) \cdots h\left(\frac{t_n}{T}\right) f\left(\frac{cT}{2\pi}\, t_1, \cdots \frac{cT}{2\pi}\, t_n\right)$$

and by (1.3)

$$C_n(f,h,T) = \sum_{t_1}^{T} \cdots \sum_{t_n}^{T} h\left(\frac{t_1}{T}\right) \cdots h\left(\frac{t_n}{T}\right)$$
$$f\left(0, \frac{cT}{2\pi}(t_2 - t_1), \ldots, \frac{cT}{2\pi}(t_n - t_1)\right).$$

Without loss of generality we may assume f is non-negative, and since f is rapidly decreasing we may further assume

$$t_1 - t_2 \ll \frac{1}{T}$$
$$\vdots$$
$$t_1 - t_n \ll \frac{1}{T},$$

and Theorem 1.3 follows from Lemma 5.1 and the Weyl-Selberg formula.

Theorem 1.4 follows from Theorem 1.3 in essentially, but not exactly, the same way that Theorem 1.2 follows from Theorem 1.1.

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