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# APPELL'S EQUATIONS AS A COVARIANT FORM OF THE MOTION EQUATIONS IN A SUB-SPACE OF RIEMANN'S SPACE

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**Abstract:** In this paper a geometrical interpretation of Appell's equations is provided. For this purpose a 'mass-based' Euclid's multi-dimensional space. This space was arrived at by using the Cartesian product of the sets of the coordinates of points of a material system. In this space, by applying the equations of holonomic constraints, Riemann's multi-dimensional space was introduced. Then, making use of the equations of non-holonomic constraints, a sub-space of Riemann's space was introduced. While Appell's equations are the equations of coordinates, in the covariant form, of the motion equations of a material system in the sub-space of Riemann's space.

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**Key Words:** Appell's equations, Riemann's space

## 1. Introduction

As we know [1], from a certain point of view, the most general form of the motion equations of a material system are Appell's equations in the form of:

$$\frac{\partial S}{\partial \ddot{e}_\nu} = \Phi_\nu. \tag{1.1}$$

In the papers [10], [11] it is proved that should Appell's equations be applied to a perfectly rigid body, then arriving at them for a tree dimensional space has to be based on a transformation of the general equation of analytic dynamics:

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$$\sum_{i=1}^{n}\left(\vec{W}_i - m_i \ddot{\vec{r}}_i\right)\cdot \delta\vec{r}_i = 0\,, \tag{1.2}$$

which, in turn, has to be formed based upon d'Alembert's supplemented and generalised law for a single perfectly rigid body, replaced by a system of material points interconnected by weightless and perfectly rigid rods of the formula:

$$U\left[\delta\vec{A},\delta\vec{F},\vec{P}_j;\vec{r}\,(t)\,,\vec{r}\,(t)\,,\vec{r}_j\,(t)\right] \overset{!}{\equiv} U\left[\vec{A}_i,\vec{W}_i;\vec{r}_i\,(t)\right] = 0. \tag{1.3}$$

In interpreting dynamics of a system of $n$ material points, tightened by ideal holonomic constraints, as dynamics of a single point with a unit mass, in Euclid's 'mass-based' $3n$-dimensional space $E_{3n}$ and in Riemann's $k$-dimensional space $V_k$, it will be proved that for material systems, additionally tightened with non-holonomic constraints, Appell's equations represent a covariant form of the motion equations in an l-dimensional sub-space $V_l$ of Riemann's space $V_k$.

## 2. Dynamics of a Holonomic System

At first, interpretation of the dynamics of a free material point will be specified as the dynamics of a point with a unit mass in the so-called Euclid's "mass-based" $3n$-dimensional space. Multi-dimensionality of that space will be obtained as the Cartesian product of the sets of all co-ordinates of points in a three-dimensional space. Then, in that space, Newton's $3n$-dimensional law for a single point with a unit mass will be defined.

Let in Euclid's three-dimensional space $E_{3n}$ there be an ortho-Cartesian system of co-ordinates and of $n$ material points with the masses $\overset{(i)}{m}$, co-ordinates $x^i$ and forces $X^i$, acting thereon, designated as follows:

$$\overset{(3\nu-2)}{m} = \overset{(3\nu-1)}{m} = \overset{(3\nu)}{m}, x^{3\nu-2}, x^{3\nu-1}, x^{3\nu}, X^{3\nu-2}, X^{3\nu-1}, X^{3\nu}, \tag{2.1}$$

$$\nu = 1,2,\ldots,n, \qquad i = 1,2,\ldots,3n\,.$$

Newton's second law for each material point, supplemented [5], [6], [10], [11], [12], [13], [14] by the motion equation or by preliminary conditions, is in the following form:

$$\overset{(i)}{m}\,\ddot{x}^i = X^i\,, \tag{2.2}$$

$$x^i = x^i(t)\,,$$

and

$$\overset{(i)}{m}\,\ddot{x}^i = X^i\,, \tag{2.3}$$

$$\text{for} \quad t = t_0 \quad x^i = x_0^i, \quad \dot{x}^i = \dot{x}_0^i, \quad i = 1, 2, \ldots, 3n\,.$$

The following transformation will be introduced:

$$y^i = \sqrt{\overset{(i)}{m}} x^i, \quad i = 1, 2, \ldots, 3n \qquad \text{(see [3], [4], [15])}, \tag{2.4}$$

where it is assumed that $\overset{(i)}{m} = const$. The set of the co-ordinates $y^i$ is designated as $\epsilon_{y^i}$ and the Cartesian product of those sets is created and designated with the letter $E_{3n}$ [3], [4]:

$$E_{3n} = \epsilon_{y^1} x \epsilon_{y^2} x \ldots x \epsilon_{y^i} x \ldots x \epsilon_{y3n}. \tag{2.5}$$

The so derived product of the Cartesian multiplication is Euclid's $3n$-dimensional space. The elements of that space are points with the co-ordinates:

$$(y^1, y^2, \ldots, y^i, \ldots, y^{3n}) \in E_{3n}, \tag{2.6}$$

which unambiguously determine the configuration of a material system. In that space, the picture of the changing locations of the material points is a curve, according to which a single material point with a unit mass moves, and its equation is as follows:

$$y^i = y^i(t), \quad i = 1, 2, \ldots, 3n. \tag{2.7}$$

By deriving once and twice the transforming formula (2.4) with respect to time, the following is obtained:

$$\dot{y}^i = \sqrt{\overset{(i)}{m}} \dot{x}^i, \qquad \ddot{y}^i = \sqrt{\overset{(i)}{m}} \ddot{x}^i, \qquad i = 1, 2, \ldots, 3n. \tag{2.8}$$

Afterwards, by substituting the acceleration co-ordinates $\ddot{x}^i$ from the motion equations (2.2) to the second formula (2.8) the following is derived:

$$\ddot{y}^i = \sqrt{\overset{(i)}{m}} \frac{X^i}{\overset{(i)}{m}} = \frac{X^i}{\sqrt{\overset{(i)}{m}}}, \qquad i = 1, 2, \ldots, 3n. \tag{2.9}$$

The formula, as below, is introduced:

$$Y^i = \frac{X^i}{\sqrt{\overset{(i)}{m}}}, \qquad i = 1, 2, \ldots, 3n, \tag{2.10}$$

where $Y^i$ is the co-ordinate of the force acting upon the mass $\overset{(i)}{m}$ in the space $E_{3n}$. By comparing the formulae (2.9) and (2.10) the following is arrived at:

$$1 \cdot \ddot{y}^i = Y^i, \quad 1, 2, \ldots, 3n, \tag{2.11}$$

$$\text{for} \quad t = t_0 \quad y^i = \sqrt{\overset{(i)}{m}} x^i = \sqrt{\overset{(i)}{m}} x_0^i = y_0^i, \quad \dot{y}^i = \sqrt{\overset{(i)}{m}} \dot{x}^i = \sqrt{\overset{(i)}{m}} \dot{x}_0^i = \dot{y}_0^i \,.$$

So the dynamics of a free set of material points may be interpreted as dynamics of a single point with a unit mass in Euclid's "mass-based" configurative space $E_{3n}$.

Designating the independent versors of the co-ordinate axis $y^i$ in space $E_{3n}$ with the letters $\vec{e}_i, \quad i = 1, 2, \ldots, 3n$, a $3n$-dimensional representation of the motion equations, velocity vector, acceleration vector, and force vector which acts on the material system, is taken into consideration. Those quantities are specified by the formulae:

$$\begin{aligned} \vec{r} &= \sum_{i=1}^{3n} y^i \vec{e}_i = \vec{r}(t), \quad \vec{v} = \sum_{i=1}^{3n} \dot{y}^i \vec{e}_i = \vec{v}(t), \\ \vec{a} &= \sum_{i=1}^{3n} \ddot{y}^i \vec{e}_i = \vec{a}(t), \quad \vec{Q} = \sum_{i=1}^{3n} Y^i \vec{e}_i \,. \end{aligned} \tag{2.12}$$

Whereas Newton's $3n$-dimensional second law, supplemented by the preliminary conditions, in the vector form, assumes the form:

$$1 \cdot \vec{a} = \vec{Q}, \tag{2.13}$$

$$\text{for} \quad t = t_0 \quad \vec{r} = \vec{r}_0, \quad \vec{v} = \vec{v}_0 \,.$$

Kinetic energy of a material system and Appell's function are expressed by the formulae, as appropriate:

$$E_k = \frac{1}{2} \sum_{i=1}^{3n} \overset{(i)}{m} (\dot{x}^i)^2 = \frac{1}{2} \sum_{i=1}^{3n} (\dot{y}^i)^2 = \frac{1}{2} \vec{v} \cdot \vec{v}, \tag{2.14}$$

$$S = \frac{1}{2} \sum_{i=1}^{3n} \overset{(i)}{m} (\ddot{x}^i)^2 = \frac{1}{2} \sum_{i=1}^{3n} (\ddot{y}^i)^2 = \frac{1}{2} \vec{a} \cdot \vec{a}. \tag{2.15}$$

Then, by scalar multiplication of both sides of the motion equation (2.13) by the versor of the axis of the co-ordination system, equal to the derivative of the acceleration vector $\vec{a}$ with respect to the second derivative of the co-ordinate $\ddot{y}^i$, that is $\vec{e}_i = \frac{\partial \vec{a}}{\partial \ddot{y}^i}$ and using the term of Appell's function, as set out in (2.15), i.e.:

$$1 \cdot \vec{a} = \vec{Q} / \cdot \vec{e}_i = \frac{\partial \vec{a}}{\partial \ddot{y}^i}, \tag{2.17}$$

the following is derived:

$$\vec{a} \cdot \vec{e}_i = \ddot{y}^i = \vec{a} \cdot \frac{\partial \vec{a}}{\partial \ddot{y}^i} = \frac{\partial}{\partial \ddot{y}^i} (\frac{1}{2} \vec{a} \cdot \vec{a}) = \frac{\partial S}{\partial \ddot{y}^i} = \vec{Q} \cdot \vec{e}_i = Y^i \tag{2.18}$$

or

$$\ddot{y}^i = \frac{\partial S}{\partial \ddot{y}^i} = Y^i, \qquad i = 1, 2, \ldots, 3n. \tag{2.19}$$

So, in that case, Appell's equations are the equations of co-ordinates with respect to the axis $y^i$, in Euclid's space $E_{3n}$, see [11].

Thereafter it is assumed that the set of material points is tightened with ideal holonomic constraints in the following form:

$$f_p(x^i, t) = 0, \qquad i = 1, 2, \ldots, 3n, \qquad p = 1, 2, \ldots, a. \tag{2.20}$$

This denotes, that the then tightened system of materials points has $k = 3n - a$ degrees of freedom $k = 3n - a$ of curvilinear co-ordinates are introduced to describe the system's motion:

$$u^\alpha = u^\alpha(t), \qquad \alpha = 1, 2, \ldots, k = 3n - a. \tag{2.21}$$

Thanks to the equations of constraints (2.20), Cartesian co-ordinates in the function of the curvilinear co-ordinates are expressed by the formulae:

$$y^i = \sqrt{\overset{(i)}{m}} x^i = \sqrt{\overset{(i)}{m}} x^i(u^\alpha, t) = y^i(u^\alpha, t), \;\; i = 1, 2, \ldots, 3n, \;\; \alpha = 1, 2, \ldots, k. \tag{2.22}$$

Whereas the location of a point, in the vector form, is defined by the formula:

$$\vec{r} = \sum_{i=1}^{3n} y^i(u^\alpha, t)\vec{e}_i = \vec{r}(u^\alpha, t), \quad \alpha = 1, 2, \ldots, k = 3n - a. \tag{2.23}$$

The above equations, in the parametrical form (2.22) and vector form (2.23), represent a $k$-dimensional hyperspace, submerged in Euclid's space $E_{3n}$. That hyperspace is Riemann's space which will be designated as $V_k$. The basis vector and metrical tensor of that space are expressed by the formulae:

$$\underset{\alpha}{\vec{r}} = \frac{\partial \vec{r}}{\partial u^\alpha} \tag{2.24}$$

and

$$g_{\alpha\beta} = \underset{\alpha}{\vec{r}} \cdot \underset{\beta}{\vec{r}} = \sum_{i=1}^{3n} \frac{\partial y^i}{\partial u^\alpha} \frac{\partial y^i}{\partial u^\beta} = g_{\alpha\beta}(u^\gamma, t), \quad \alpha, \beta, \gamma = 1, 2, \ldots, k. \tag{2.25}$$

The velocity vector of a point is defined by the formula:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\vec{r}[u^\alpha(t), t] = \frac{\partial \vec{r}}{\partial u^\alpha} v^\alpha + \frac{\partial \vec{r}}{\partial t} = \underset{\alpha}{\vec{r}}\, \dot{u}^\alpha + \frac{\partial \vec{r}}{\partial t}, \tag{2.26}$$

where: $v^\alpha = \dot{u}^\alpha$, $\quad \alpha = 1, 2, \ldots, k = 3n - a$, is a counter-variant co-ordinate of the velocity in space $V_k$. Whereas the acceleration vector of a point assumes the form:

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{u}^\gamma \underset{\gamma}{\vec{r}} + \underset{\alpha\beta}{\vec{r}}\, \dot{u}^\alpha \dot{u}^\beta + \frac{\partial^2 \vec{r}}{\partial t^2}, \quad \alpha, \beta, \gamma = 1, 2, \ldots, k = 3n - a. \tag{2.27}$$

Derivative of the acceleration vector $\vec{a}$ with respect to the second derivative of the curvilinear co-ordinate $\ddot{u}^\gamma$, is expressed by the formula:

$$\underset{\gamma}{\vec{r}} = \frac{\partial \vec{a}}{\partial \ddot{u}^\gamma}. \tag{2.28}$$

On a material system, apart of the active force $\vec{Q}$, there will also act the reaction of constraints, which will be designated as $\vec{R}_h$. On assumption that those constraints are ideal, the reaction is perpendicular, in every point, to Riemann's space $V_k$. This denotes, the reaction $\vec{R}_h$ fulfils the following conditions:

$$\vec{R}_h \cdot \underset{\alpha}{\vec{r}} = 0, \qquad \alpha = 1, 2, \ldots, k = 3n - a. \tag{2.29}$$

Newton's second law for a material system, tightened with holonomic constraints, interpreted as a dynamic equation of a single point with a unit mass, in Euclid's "mass-based" configurative space $E_{3n}$, supplemented by the equation of the constraints and preliminary conditions, has the form:

$$1 \cdot \vec{a} = \vec{Q} + \vec{R}_h / \cdot \underset{\gamma}{\vec{r}} = \frac{\partial \vec{a}}{\partial \ddot{u}^\gamma}, \tag{2.30}$$

$$\vec{r} = \vec{(}u^\gamma, t), \quad \gamma = 1, 2, \ldots, k = 3n - a, \quad \text{for} \quad t = t_0 \quad \vec{r} = \vec{r}_0, \quad \vec{v} = \vec{v}_0\,.$$

Then, by scalar multiplication of both sides of the motion equation (2.30) by the basis vector $\underset{\gamma}{\vec{r}}$ which, according to formula (2.28), is equal to the derivative of the acceleration vector $\vec{a}$ versus the second derivative of the curvilinear coordinate $\ddot{u}^\gamma$ and, making use of the term of Appell's function, as laid down in (2.15), the following is arrived at:

$$\vec{a} \cdot \underset{\gamma}{\vec{r}} = a_\gamma = \vec{a} \cdot \frac{\partial \vec{a}}{\partial \ddot{u}^\gamma} = \frac{\partial}{\partial \ddot{u}^\gamma}(\frac{1}{2}\vec{a} \cdot \vec{a}) = \frac{\partial S}{\partial \ddot{u}^\gamma}$$
$$= \frac{d}{dt}(\frac{\partial E_k}{\partial \dot{u}^\gamma}) - \frac{\partial E_k}{\partial u^\gamma} = \vec{Q} \cdot \underset{\gamma}{\vec{r}} + \vec{R}_h \cdot \underset{\gamma}{\vec{r}} = Q_\gamma, \tag{2.31}$$

hence

$$a_\gamma = \frac{\partial S}{\partial \ddot{u}^\gamma} = \frac{d}{dt}(\frac{\partial E_k}{\partial \dot{u}^\gamma}) - \frac{\partial E_k}{\partial u^\gamma} = Q_\gamma, \tag{2.32}$$

for, as it is known, while applying the proved identities:

$$\frac{\partial \vec{v}}{\partial v^\gamma} = \underset{\gamma}{\vec{r}}, \qquad \frac{d}{dt}(\underset{\gamma}{\vec{r}}) = \frac{\partial \vec{v}}{\partial u^\gamma}, \tag{2.33}$$

the following is right:

$$\frac{d\vec{v}}{dt} \cdot \underset{\gamma}{\vec{r}} = \frac{d}{dt}(\vec{v} \cdot \underset{\gamma}{\vec{r}}) - \vec{v}\frac{d}{dt}(\underset{\gamma}{\vec{r}}) = \frac{d}{dt}(\vec{v} \cdot \frac{\partial \vec{v}}{\partial v^\gamma}) - \vec{v}\frac{\partial \vec{v}}{\partial u^\gamma} = \frac{d}{dt}(\frac{\partial E_k}{\partial \dot{u}^\gamma}) - \frac{\partial E_k}{\partial u^\gamma}\,, \tag{2.34}$$
$$\gamma = 1, 2, \ldots, k = 3n - a\,.$$

Thus, in the case of a holonomic system, Appell's equations being the equations of co-ordinates in the covariant form, in Riemann's space $V_k$, are also Lagrange's equations of the second kind [11].

### 3. Dynamics of a Non-Holonomic System

At first, it is assumed that the set of material points is tightened with ideal holonomic and scleronomic constraints. Equation of Riemann's space $V_k$, identified by those constraints, is expressed by the formula:

$$\vec{r} = \vec{r}(u^\alpha), \qquad \alpha = 1, 2, \ldots, k = 3n - a. \tag{3.1}$$

It is also assumed that the set of material points is additionally tightened with non-holonomic constraints, linear to velocity, where free terms are left out:

$$a^s_\gamma v^\gamma = 0, \qquad s = 1, 2, \ldots, b, \quad \gamma = 1, 2, \ldots, k = 3n - a. \tag{3.2}$$

Then, additionally, apart from the active force $\vec{Q}$ and the reaction $\vec{R}_h$ of holonomic constraints, on a material system the reaction $\vec{R}_{nh}$ of non-holonomic constraints will act.

Assuming that the rank of the matrix $[a^s_\gamma]$ is equal to b, that is:

$$\mathrm{Rank}\,[a^s_\gamma] = b, \tag{3.3}$$

b of the first co-ordinates of the velocity vector may be expressed by $k - b$ of its other co-ordinates. So the following is in place:

$$v^t = b^t_\nu v^{b+\nu}, \quad t = 1, 2, \ldots, b, \quad \nu = 1, 2, \ldots, l = k - b. \tag{3.4}$$

Then, by designating the other co-ordinates of the velocity vector by $\dot{e}^\nu$ and referring to them as kinematical parameters, i.e.:

$$v^{b+\nu} = \dot{e}^\nu, \qquad \nu = 1, 2, \ldots, l = k - b, \tag{3.5}$$

all co-ordinates of the velocity vector may be expressed in such a manner by the kinematical parameters $\dot{e}^\nu$. So the equation of non-holonomic constraints assumes the following form:

$$v^\alpha = b^\alpha_\nu \dot{e}^\nu, \quad \alpha = 1, 2, \ldots, b, b+1, b+2, \ldots, k, \quad \nu = 1, 2, \ldots, l = k - b. \tag{3.6}$$

It may be noticed that in the equations (3.4) one side of them may be shifted to the other side:

$$-\delta^s_t v^t + B^s_{b+\sigma} v^{b+\sigma} = 0, \qquad s, t = 1, 2, \ldots, b, \qquad \sigma = 1, 2, \ldots l, \tag{3.7}$$

and that the left sides of those equations represent a scalar product of the two vectors: the velocity vector $\vec{v}$ whose counter-variant co-ordinates may be expressed by the formula:

$$v^\beta (B^t_{b+\nu} v^{b+\nu}, \delta^{b+\sigma}_{b+\nu} v^{b+\nu}) = v^\beta (B^t_{b+\nu} \dot{e}^\nu, \delta^{b+\sigma}_{b+\nu} \dot{e}^\nu), \tag{3.8}$$

$$t = 1,2,\ldots,b, \qquad \nu,\sigma = 1,2,\ldots,l, \qquad \beta = 1,2,\ldots,k,$$

and of some new vectors $\vec{B}^s$, which will permit determining the reaction of the non-holonomic constraints $\vec{R}_{nh}$ and whose co-variant co-ordinates may be represented as follows:

$$B^s_\beta(-\delta^s_t, B^s_{b+\sigma}), \quad s,t = 1,2,\ldots,b, \quad \sigma = 1,2,\ldots,l, \quad \beta = 1,2,\ldots,k. \tag{3.9}$$

Then, the equations of the non-holonomic constraints may also be expressed by the formula:

$$\vec{B}^s \cdot \vec{v} = B^s_\beta v^\beta = 0, \quad s = 1,2,\ldots,b, \quad \beta = 1,2,\ldots,b,b+1,b+2,\ldots,k. \tag{3.10}$$

Now that the formula (3.6) may be written as follows:

$$\frac{du^\alpha}{dt} = v^\alpha = b^\alpha_\nu \dot{e}^\nu = b^\alpha_\nu(u^\delta)\frac{de^\nu}{dt}, \tag{3.11}$$

$$\alpha,\delta = 1,2,\ldots,b,b+1,b+2,\ldots,k = 3n-a, \quad \nu = 1,2,\ldots,l = k-b,$$

as well as

$$du^\alpha = b^\alpha_\nu de^\nu, \tag{3.12}$$

so

$$\frac{\partial u^\alpha}{\partial e^\nu} = b^\alpha_\nu(u^\delta). \tag{3.13}$$

Therefore it is assumed that in Riemann's space $V_k$ there is given the field of an intermediary tensor [2], [11], [17]:

$$b^\alpha_\nu = b^\alpha_\nu(u^\delta), \quad \alpha,\delta = 1,2,\ldots,k = 3n-a, \quad \nu = 1,2,\ldots,l = k-b, \tag{3.14}$$

between Riemann's space $V_k$ and a subspace of that space, designated with $V_l$ which connects the co-ordinates of the velocity vector $\vec{v}$ in those both spaces.

The equation of motion of a point with a unit mass in Euclid's "mass-based" configurative space $E_{3n}$ and Riemann's space $V_k$ with the equation (3.1) has the following form:

$$\vec{r} = \vec{r}[u^\alpha(t)], \quad \alpha = 1,2,\ldots,k = 3n-a. \tag{3.15}$$

The velocity vector $\vec{v}$ tangential to Riemann's space $V_k$ is expressed by the formula:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{\partial\vec{r}}{\partial u^\alpha}\dot{u}^\alpha = \underset{\alpha}{\vec{r}}\, v^\alpha, \quad \alpha = 1,2,\ldots,k = 3n-a, \tag{3.16}$$

where: $v^\alpha$ is a counter-variant co-ordinate of the velocity and the vectors $\underset{\alpha}{\vec{r}}$ form the basis of Riemann's space $V_k$. Now that the counter-variant co-ordinates of velocity are expressed by the formula (3.6) so the velocity vector $\vec{v}$ adopts the form:

$$\vec{v} = \underset{\alpha}{\vec{r}}\, v^\alpha = \underset{\alpha}{\vec{r}}\, b^\alpha_\nu \dot{e}^\nu, \tag{3.17}$$

$$\alpha = 1, 2, \ldots, k = 3n - a, \quad \nu = 1, 2, \ldots, l = k - b,$$

as well as

$$\vec{v} = \underset{\nu}{\vec{R}} \dot{e}^{\nu} = \vec{v}(\dot{e}^{\nu}), \tag{3.18}$$

where the vectors

$$\underset{\nu}{\vec{R}} = \underset{\alpha}{\vec{r}} b^{\alpha}_{\nu} = \underset{\nu}{\vec{R}}(u^{\beta}), \tag{3.19}$$

$$\alpha, \beta = 1, 2, \ldots, k = 3n - a, \quad \nu = 1, 2, \ldots, l = k - b,$$

create a new basis, the basis of the subspace $V_l$ of Riemann's space $V_k$. As the velocity vector $\vec{v}$ is expressed by the formula (3.18) so this denotes, pursuant to the definition that the kinematical parameters $\dot{e}^{\nu}$, are the counter-variant co-ordinates of the velocity vector on that subspace $V_l$ whereas this formula also represents a vector form of the equation of non-holonomic constraints. The metric tensor of that new subspace $V_l$ is determined out of the formula:

$$G_{\nu\rho} = \underset{\nu}{\vec{R}} \cdot \underset{\rho}{\vec{R}} = (\underset{\alpha}{\vec{r}} b^{\alpha}_{\nu}) \cdot (\underset{\beta}{\vec{r}} b^{\beta}_{\rho}) = g_{\alpha\beta} b^{\alpha}_{\nu} b^{\beta}_{\rho}, \tag{3.20}$$

$$\alpha, \beta = 1, 2, \ldots, k = 3n - a, \quad \nu, \rho = 1, 2, \ldots, l = k - b\,.$$

Now let us proceed to determine the acceleration vector $\vec{a}$. By computing the derivative of the velocity vector versus time the following is obtained:

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}\{\dot{e}^{\nu}(t)\, \underset{\nu}{\vec{R}}[u^{\beta}(t)]\} = \ddot{e}^{\nu}\, \underset{\nu}{\vec{R}} + \underset{\nu\beta}{\vec{R}}\, \dot{e}^{\nu} v^{\beta} \\ &= (\ddot{e}^{\sigma} + \Gamma^{\sigma}_{\nu\beta} \dot{e}^{\nu} v^{\beta})\, \underset{\sigma}{\vec{R}} + \underset{\nu\beta}{\vec{R}^{n}}\, \dot{e}^{\nu} v^{\beta} = a^{\nu}\, \underset{\nu}{\vec{R}} + \underset{\nu\beta}{\vec{R}^{n}}\, \dot{e}^{\nu} v^{\beta},\end{aligned} \tag{3.21}$$

where

$$a^{\sigma} = \ddot{e}^{\sigma} + \Gamma^{\sigma}_{\nu\beta} \dot{e}^{\nu} v^{\beta}, \quad \nu, \sigma = 1, 2, \ldots, l, \quad \beta = 1, 2, \ldots, k, \tag{3.22}$$

is the counter-variant co-ordinate of the acceleration in the subspace $V_l$ of Riemann's space $V_k$. Then Appell's function will be found out. As the acceleration is expressed by the formula (3.21), so Appell's functions adopts the following form:

$$\begin{aligned}S = \frac{1}{2}\vec{a} \cdot \vec{a} &= \frac{1}{2} a^{\nu} a^{\pi}\, \underset{\nu}{\vec{R}} \cdot \underset{\pi}{\vec{R}} + a^{\nu}\, \underset{\nu}{\vec{R}} \cdot \underset{\rho\omega}{\vec{R}^{n}}\, \dot{e}^{\rho} v^{\omega} + \underset{\rho\omega}{\vec{R}^{n}} \cdot \underset{\sigma\beta}{\vec{R}^{n}}\, \dot{e}^{\rho} v^{\omega} \dot{e}^{\sigma} v^{\beta} \\ &= \frac{1}{2} a^{\nu} a^{\pi} G_{\nu\pi} + \underset{\rho\omega}{\vec{R}^{n}} \cdot \underset{\sigma\beta}{\vec{R}^{n}}\, \dot{e}^{\rho} v^{\omega} \dot{e}^{\sigma} v^{\beta} = S[a^{\sigma}(\ddot{e}^{\nu})],\end{aligned} \tag{3.23}$$

because

$$\underset{\nu}{\vec{R}} \cdot \underset{\rho\omega}{\vec{R}^{n}} = b^{\alpha}_{\nu}\, \underset{\alpha}{\vec{r}} \cdot \underset{\rho\omega}{\vec{R}^{n}} = 0, \tag{3.24}$$

now that the vectors $\underset{\rho\omega}{\vec{R}^{n}}$ are perpendicular to Riemann's space $V_l$ and $G_{\nu\pi}$ is the metric tensor of the subspace $V_l$ and is expressed by the formula (3.20). Then, by computing the derivative of Appell's function $S$ with respect to the derivative of the kinematical parameter $\ddot{e}^\nu$ the following is obtained:

$$\frac{\partial S}{\partial \ddot{e}^\nu} = \frac{\partial S}{\partial a^\sigma}\frac{\partial a^\sigma}{\partial \ddot{e}^\nu} = \frac{\partial S}{\partial a^\sigma}\delta^\sigma_\nu = \frac{\partial S}{\partial a^\nu} = a^\pi G_{\pi\nu} = a_\nu, \quad \nu,\sigma,\pi = 1,2,\dots,l, \tag{3.25}$$

because computing the derivative of the counter-variant co-ordinate of the acceleration $a^\sigma$ versus the derivative of the kinematical parameter $\ddot{e}^\nu$, in accordance with the formula (3.22), one obtains: $\frac{\partial a^\sigma}{\partial \ddot{e}^\nu} = \delta^\sigma_\nu, \quad \sigma,\nu = 1,2,\dots,l$.

Newton's second law for a non-free material system, interpreted as the equation of motion of a single point with a unit mass, in Euclid's "mass-based" configurative space $E_{3n}$, as the equation of motion of that point in Riemann's space $V_k$, as well as the equation of motion of that point in the subspace $V_l$ in Riemann's space $V_k$, supplemented by the equation of the holonomic constraints, equation of non-holonomic constraints and preliminary conditions, has the form:

$$\begin{gathered} 1\cdot\vec{a} = \vec{Q} + \vec{R}_h + \vec{R}_{nh}, \\ \vec{r} = \vec{r}(u^\alpha), \quad \alpha = 1,2,\dots,k = 3n-a, \\ \vec{v} = \vec{v}(\dot{e}^\nu), \quad \nu = 1,2,\dots,l = k-b = 3n-a-b, \\ \text{for} \qquad t = t_0 \quad \vec{r} = \vec{r}_0, \quad \vec{v} = \vec{v}_0\,. \end{gathered} \tag{3.26}$$

In that equation, $\vec{Q}$ denotes the sum of all active forces acting upon the system, and is expressed by the formula:

$$\vec{Q} = \sum_{i=1}^{3n} Y^i \vec{e}_i = Q^\alpha \underset{\alpha}{\vec{r}} = \Phi^\nu \underset{\nu}{\vec{R}}, \quad \alpha = 1,2,\dots,k, \quad \nu = 1,2,\dots,l. \tag{3.27}$$

The vector $\vec{R}_h$ denotes the reaction of ideal holonomic constraints that fulfil the conditions:

$$\vec{R}_h \cdot \underset{\alpha}{\vec{r}} = 0, \qquad \alpha = 1,2,\dots,k = 3n-a. \tag{3.28}$$

Whereas the vector $\vec{R}_{nh}$ denotes the reaction of non-holonomic constraints orthogonal to the subspace $V_l$ of space $V_k$. That reaction is expressed as a linear combination of the vectors $\vec{B}^s$, $s = 1,2,\dots,b$, that is

$$\vec{R}_{nh} = \sum_{s=1}^{b} \lambda_s \vec{B}^s. \tag{3.29}$$

Orthogonality of the reaction of non-holonomic constraints $\vec{R}_{nh}$ to the subspace $V_l$, i.e. orthogonality of that reaction to each of the basis vectors $\underset{\nu}{\vec{R}}$ will be proved by computing the scalar product, equal to zero, of the reaction vector

$\vec{R}_{nh}$ and basis vectors $\underset{\nu}{\vec{R}}$ that is:

$$\vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = \sum_{s=1}^{b} \lambda_s \vec{B}^s \cdot \underset{\nu}{\vec{R}} = 0. \tag{3.30}$$

And namely, by differentiating the equation of non-holonomic constraints in the form (3.10) versus the kinematical parameter $\dot{e}^\nu$ the following is immediately obtained:

$$\frac{\partial}{\partial \dot{e}^\nu}(\vec{B}^s \cdot \vec{v}) = \vec{B}^s \cdot \frac{\partial \vec{v}}{\partial \dot{e}^\nu} = \vec{B}^s \cdot \underset{\nu}{\vec{R}} = 0, \quad s = 1, 2, \ldots, b, \quad \nu = 1, 2, \ldots, l, \tag{3.31}$$

as, according to the equation (3.18) for the velocity vector in the new basis, the derivatives of the velocity vector $\vec{v}$ with respect to the kinematical parameter $\dot{e}^\nu$ are expressed by the formulae:

$$\frac{\partial \vec{v}}{\partial \dot{e}^\nu} = \underset{\nu}{\vec{R}}, \quad \nu = 1, 2, \ldots, l. \tag{3.32}$$

Now one will proceed from the equation of motion of a system in the vector form to the equation of co-ordinates in the covariant form, in the subspace $V_l$ of Riemann's space $V_k$. In this case, as we know, both sides of the motion equation in the vector form (3.26) need to be multiplied by the basis vectors $\underset{\nu}{\vec{R}}$ in the subspace $V_l$, equal to the derivative of the acceleration vector with respect to the derivative of the kinematical parameter $\ddot{e}^\nu$, that is

$$\underset{\nu}{\vec{R}} = \frac{\partial \vec{a}}{\partial \ddot{e}^\nu}, \quad \nu = 1, 2, \ldots, l. \tag{3.33}$$

So the following is in place:

$$\vec{a} = \vec{Q} + \vec{R}_h + \vec{R}_{nh} \quad / \cdot \underset{\nu}{\vec{R}} = \frac{\partial \vec{a}}{\partial \ddot{e}^\nu}, \tag{3.34}$$

hence

$$\vec{a} \cdot \underset{\nu}{\vec{R}} = \vec{a} \cdot \frac{\partial \vec{a}}{\partial \ddot{e}^\nu} = \vec{Q} \cdot \underset{\nu}{\vec{R}} + \vec{R}_h \cdot \underset{\nu}{\vec{R}} + \vec{R}_{nh} \cdot \underset{\nu}{\vec{R}}, \tag{3.35}$$
$$\nu = 1, 2, \ldots, l = k - b = 3n - a - b\,.$$

As

$$\vec{a} \cdot \underset{\nu}{\vec{R}} = a_\nu = \vec{a} \cdot \frac{\partial \vec{a}}{\partial \ddot{e}^\nu} = \frac{\partial}{\partial \ddot{e}^\nu}(\frac{1}{2}\vec{a} \cdot \vec{a}) = \frac{\partial S}{\partial \ddot{e}^\nu}, \tag{3.36}$$
$$\nu = 1, 2, \ldots, l = k - b = 3n - a - b\,,$$

where $S = \frac{1}{2}\vec{a} \cdot \vec{a}$ is Appell's function, for which (3.25) is true, that is:

$$\frac{\partial S}{\partial \ddot{e}^\nu} = \frac{\partial S}{\partial a^\sigma} \frac{\partial a^\sigma}{\partial \ddot{e}^\nu} = \frac{\partial S}{\partial a^\sigma} \delta^\sigma_\nu = \frac{\partial S}{\partial a^\nu} = a^\pi G_{\pi\nu} = a_\nu, \tag{3.37}$$

and

$$\vec{Q} \cdot \underset{\nu}{\vec{R}} = \Phi_\nu = \vec{Q} \cdot \underset{\alpha}{\vec{r}} b^\alpha_\nu = Q_\alpha b^\alpha_\nu, \tag{3.38}$$

as well as

$$\vec{R}_h \cdot \underset{\nu}{\vec{R}} = \vec{R}_h \cdot \underset{\alpha}{\vec{r}} b^\alpha_\nu = 0, \qquad \vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = 0\,. \tag{3.39}$$

So the equation of motion, in its covariant form, is reduced to the form:

$$a_\nu = \vec{a} \cdot \frac{\partial \vec{a}}{\partial \ddot{e}^\nu} = \frac{\partial}{\partial \ddot{e}^\nu}(\frac{\vec{a} \cdot \vec{a}}{2}) = \frac{\partial S}{\partial \ddot{e}^\nu} = \Phi_\nu, \tag{3.40}$$

$$\nu = 1, 2, \ldots, l = k - b = 3n - a - b\,.$$

So it has been proved that Appell's equations in the form:

$$a_\nu = \frac{\partial S}{\partial \ddot{e}^\nu} = \Phi_\nu,$$

represent the equations of co-ordinates, expressed by the covariant co-ordinates in the subspace $V_l$ of Riemann's space $V_k$. Geometrical interpretation of Appell's equations is provided in [7], [8], [9], [10], [11], [12], [13], [14].

It is to be noted that the non-holonomic constraints, being limitations put on the velocity vector $\vec{v}$ only, enable a point to move around Riemann's entire space $V_k$. However, true motion of that point (the whole of a material system), as specified by the preliminary conditions, by which the equation of motion has been supplemented, will explicitly be defined by curvilinear co-ordinates in the form of: $u^\alpha = u^\alpha(t)$ and the kinematical parameters $\dot{e}^\nu = \dot{e}^\nu(t)$. Those equations explicitly determine the curve representing true motion of a material system in Riemann's space $V_k$ and space $V_l$. This denotes that at all times of true motion of the system, the metric tensor $g_{\alpha\beta}$ and $G_{\nu\delta}$, the basis vectors $\underset{\alpha}{\vec{r}}$ and $\underset{\nu}{\vec{R}}$, $\alpha, \beta = 1, 2, \ldots, k = 3n - a$, $\nu, \delta = 1, 2, \ldots, l = k - b$, are – thanks to the preliminary conditions – unambiguously determined so is the entire geometry of those spaces.

At this point, the orthogonality of the non-holonomic constraints, as defined by the formula (3.29) to the subspace $V_l$ determined by the basis vectors $\underset{\nu}{\vec{R}}$, will otherwise be proved. Fro this purpose the equation of the non-holonomic constraints as defined by the formula (3.6), may be converted to the form of:

$$\vec{B}^s \cdot \vec{v} \equiv (-v^\alpha + b^\alpha_\nu \dot{e}^\nu) \equiv -\delta^s_t v^t + B^s_{b+\sigma} v^{b+\sigma} = 0, \tag{3.41}$$

where: $\dot{e}^\sigma = v^{b+\sigma}$, $s, t = 1, 2, \ldots, b$, $\nu, \sigma = 1, 2, \ldots, l$. It follows, therefore, that the covariant co-ordinates of the vectors $\vec{B}^s$ assume the form (3.9), whereas the co-ordinates of the velocity vector $\vec{v}$ are expressed by the formulae (3.8).

On the other hand, the velocity vector in the form of:

$$\vec{v} = v^{\beta}\underset{\beta}{\vec{r}} = b^{\beta}_{\nu}\dot{e}^{\nu}\underset{\beta}{\vec{r}} = \dot{e}^{\nu}\underset{\nu}{\vec{R}} = B^{t}_{b+\nu}\dot{e}^{\nu}\underset{t}{\vec{r}} + \delta^{b+\sigma}_{b+\nu}\dot{e}^{\nu}\underset{b+\sigma}{\vec{r}} = (B^{t}_{b+\nu}\underset{t}{\vec{r}} + \delta^{b+\sigma}_{b+\nu}\underset{b+\sigma}{\vec{r}})\dot{e}^{\nu}, \quad (3.42)$$

permits specifying the basis vector $\underset{\nu}{\vec{R}}$ of the subspace $V_l$:

$$\underset{\nu}{\vec{R}} = b^{\beta}_{\nu}\underset{\beta}{\vec{r}} = B^{t}_{b+\nu}\underset{t}{\vec{r}} + \delta^{b+\sigma}_{b+\nu}\underset{b+\sigma}{\vec{r}} \quad (3.43)$$

and the covariant co-ordinates of those vectors:

$$b^{\beta}_{\nu}(B^{t}_{b+\nu}, \delta^{b+\sigma}_{b+\nu}). \quad (3.44)$$

As the reaction of the non-holonomic constraints is expressed by the formula (3.29), so the scalar product of the vectors $\vec{R}_{nh}$ and $\underset{\nu}{\vec{R}}$ is equal zero, that is $\vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = 0$ because, by employing the formulae (3.9) and (3.44) one obtains:

$$\vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = \sum_{s=1}^{b} \lambda_s \vec{B}^s \cdot \underset{\nu}{\vec{R}} = \sum_{s=1}^{b} \lambda_s B^s_{\beta} b^{\beta}_{\nu} = \sum_{s=1}^{b} \lambda_s (-\delta^s_t B^t_{b+\nu} + B^s_{b+\sigma}\delta^{b+\sigma}_{b+\nu})$$

$$= \sum_{s=1}^{b} \lambda_s [(-1)B^s_{b+\nu} + B^s_{b+\nu}(1)] = 0, \quad (3.45)$$

which signifies that the reaction $\vec{R}_{nh}$ is perpendicular to Riemann's space $V_l$ as it should be [8], [10], [11].

Now it is assumed that a material system, tied with holonomic and rheonomic for which the equation of Riemann's space is expressed by the formula:

$$\vec{r} = \vec{r}(u^{\alpha}, t), \quad \alpha = 1, 2, \ldots, k = 3n - a, \quad (3.46)$$

is additionally tied with non-holonomic constraints, with free terms, in the form of:

$$a^s_{\gamma} v^{\gamma} + a^s_0 = 0, \quad s = 1, 2, \ldots, b, \quad \gamma = 1, 2, \ldots, k = 3n - a. \quad (3.47)$$

By proceeding similarly as it was the case for the constraints determined by the formula (3.2), these equations may be transformed to the form of:

$$v^{\alpha} = b^{\alpha}_{\nu}\dot{e}^{\nu} + b^{\alpha}_0, \quad \alpha = 1, 2, \ldots, k = 3n - a, \quad \nu = 1, 2, \ldots, l, \quad (3.48)$$

and

$$\vec{B}^s \cdot \vec{v} + B^s_0 = 0, \quad s = 1, 2, \ldots, b. \quad (3.49)$$

In Riemann's space, the equation of motion assumed the following form:

$$\vec{r} = \vec{r}[u^{\alpha}(t), t], \quad \alpha = 1, 2, \ldots, k = 3n - a. \quad (3.50)$$

Therefore, the velocity vector will be expressed as follows:

$$\vec{v} = \frac{d\vec{r}}{dt} = \underset{\alpha}{\vec{r}} v^{\alpha} + \frac{\partial \vec{r}}{\partial t}. \quad (3.51)$$

Substituting the formula (3.48) to the above formula (3.51) one arrives at:

$$\vec{v} = \underset{\alpha}{\vec{r}} b^{\alpha}_{\nu} \dot{e}^{\nu} + \underset{\alpha}{\vec{r}} b^{\alpha}_{0} + \frac{\partial \vec{r}}{\partial t} = \dot{e}^{\nu} \underset{\nu}{\vec{R}} + \underset{\alpha}{\vec{r}} b^{\alpha}_{0} + \frac{\partial \vec{r}}{\partial t} = \vec{v}(\dot{e}^{\nu}, t), \tag{3.52}$$

where

$$\underset{\nu}{\vec{R}} = \underset{\alpha}{\vec{r}} b^{\alpha}_{\nu}, \tag{3.53}$$

is the basis vector determining the subspace $V_l$ of Riemann's space $V_k$. The metric tensor of that subspace is determined by the formula:

$$G_{\nu\pi} = \underset{\nu}{\vec{R}} \cdot \underset{\pi}{\vec{R}} = g_{\alpha\beta} b^{\alpha}_{\nu} b^{\beta}_{\pi}, \quad \alpha, \beta = 1, 2, \ldots, k, \quad \nu, \pi = 1, 2, \ldots, l. \tag{3.54}$$

It is to be remarked that the formula (3.52) also represents a vector form of the equation of non-holonomic constraints.

Whereas the acceleration vector may be presented in the form of:

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{e}^{\nu} \underset{\nu}{\vec{R}} + \vec{a}_r, \tag{3.55}$$

where $\vec{a}_r$ is a reminder that evolves when the velocity vector undergoes differentiation:

$$\vec{a}_r = \underset{\nu\beta}{\vec{R}} \dot{e}^{\nu} v^{\beta} + \dot{e}^{\nu} \frac{\partial \underset{\nu}{\vec{R}}}{\partial t} + \underset{\alpha\beta}{\vec{r}} v^{\beta} b^{\alpha}_{0} + b^{\alpha}_{0} \frac{\partial \underset{\alpha}{\vec{r}}}{\partial t} + \underset{\alpha}{\vec{r}} v^{\beta} \frac{\partial b^{\alpha}_{0}}{\partial u^{\beta}} + \underset{\alpha}{\vec{r}} \frac{\partial b^{\alpha}_{0}}{\partial t} + v^{\beta} \frac{\partial \underset{\beta}{\vec{r}}}{\partial t} + \frac{\partial^2 \vec{r}}{\partial t^2}. \tag{3.56}$$

Applying the formulae (3.52) and (3.55), the derivatives of the velocity vector $\vec{v}$ with respect the kinematical parameters $\dot{e}^{\nu}$, as well as derivatives of the acceleration vector $\vec{a}$ with respect to the derivative of the kinematical parameter $\ddot{e}^{\nu}$, will be expressed by the formulae:

$$\frac{\partial \vec{v}}{\partial \dot{e}^{\nu}} = \underset{\nu}{\vec{R}}, \quad \nu = 1, 2, \ldots, l = k - b, \tag{3.57}$$

and

$$\frac{\partial \vec{a}}{\partial \ddot{e}^{\nu}} = \underset{\nu}{\vec{R}}, \quad \nu = 1, 2, \ldots, l = k - b. \tag{3.58}$$

Newton's second law, supplemented by the equations of holonomic and non-holonomic constraints, respectively, the latter being represented by the formula (3.52), and preliminary conditions, is expressed by the formulae:

$$1 \cdot \vec{a} = \vec{Q} + \vec{R}_h + \vec{R}_{nh} \quad / \cdot \underset{\nu}{\vec{R}} = \frac{\partial \vec{a}}{\partial \ddot{e}^{\nu}}, \tag{3.59}$$

$$\vec{r} = \vec{r}(u^{\alpha}, t), \quad \alpha = 1, 2, \ldots, k = 3n - a,$$

$$\vec{v} = \vec{v}(\dot{e}^{\nu}, t), \quad \nu = 1, 2, \ldots, l = k - b = 3n - a - b,$$

$$\text{for} \quad t = t_0 \quad \vec{r} = \vec{r}_0, \quad \vec{v} = \vec{v}_0,$$

where $\vec{R}_h$ is the reaction of holonomic constraints, and is orthogonal to Riemann's space $V_k$, whereas $\vec{R}_{nh}$ is the reaction of non-holonomic constraints, and is orthogonal to the subspace $V_l$ of Riemann's space $V_k$. Further, by a scalar multiplication of Newton's law of motion, as determined by the formula (3.59) by the basis vectors $\underset{\nu}{\vec{R}}, \quad \nu = 1, 2, \ldots, l$, as laid down in the formula (3.58), the following is obtained:

$$\vec{a} \cdot \underset{\nu}{\vec{R}} = a_\nu = \vec{a} \cdot \frac{\partial \vec{a}}{\partial \ddot{e}^\nu} = \frac{\partial}{\partial \ddot{e}^\nu}(\frac{1}{2}\vec{a} \cdot \vec{a}) = \frac{\partial S}{\partial \ddot{e}^\nu}$$
$$= \vec{Q} \cdot \underset{\nu}{\vec{R}} + \vec{R}_h \cdot \underset{\nu}{\vec{R}} + \vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = \Phi_\nu, \quad (3.60)$$

where $S = \frac{1}{2}\vec{a} \cdot \vec{a}$ is Appell's function, and additionally

$$\vec{Q} \cdot \underset{\nu}{\vec{R}} = Q_\alpha b^\alpha_\nu = \Phi_\nu, \quad (3.61)$$

and

$$\vec{R}_h \cdot \underset{\nu}{\vec{R}} = \vec{R}_h \cdot \underset{\alpha}{\vec{r}} b^\alpha_\nu = 0, \qquad \vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = 0, \quad (3.62)$$

as, for the reaction of non-holonomic constraints as represented by the formula (3.29), as well as for the derivative of the equation of constraints in the form (3.49) versus the kinematical parameter $\dot{e}^\nu$ and by applying the formula (3.57):

$$\frac{\partial}{\partial \dot{e}^\nu}(\vec{B}^s \cdot \vec{v} + B^s_0) = \vec{B}^s \cdot \frac{\partial \vec{v}}{\partial \dot{e}^\nu} = \vec{B}^s \cdot \underset{\nu}{\vec{R}} = 0, \quad (3.63)$$

the following is immediately arrived at:

$$\vec{R}_{nh} \cdot \underset{\nu}{\vec{R}} = \sum_{s=1}^{b} \lambda_s \vec{B}^s \cdot \underset{\nu}{\vec{R}} = 0\,. \quad (3.64)$$

So the equation of motion in the vector form, as determined by the aforesaid formula (3.59), for rheonomic systems, too, in which equations of non-holonomic constraints include free terms, assumes the form of the equations of the coordinates of the covariant character in the subspace $V_l$ of Riemann's space $V_k$, which, in turn, are embodied by Appell's equations, i.e.

$$a_\nu = \frac{\partial S}{\partial \ddot{e}^\nu} = \Phi_\nu, \quad \nu = 1, 2, \ldots, l. \quad (3.65)$$

In this manner, to Appell's equations a very simple geometrical interpretation has been assigned. Those equations, for non-holonomic systems, are just the equations of motion in the covariant form, in a certain subspace of Riemann's space, similarly as Lagrange's equations of the second kind – for holonomic systems – are the equations of motion, also in the covariant form, in Riemann's space, cf. the formula (2.32).

So the foregoing interpretation of dynamics of a material system, seen as dynamics of a point with a unit mass, in Euclid's "mass-based" configurative space $E_{3n}$, as dynamics of that point in Riemann's space $V_k$ determined by holonomic constraints, and the subspace $V_l$ of Riemann's space $V_k$, determined by non-holonomic constraints, permit assigning the very simple geometric interpretation to Appell's equations, for a free system, for a holonomic system and for any – holonomic and non-holonomic – material system [7], [8], [9], [10], [11], [12], [13], [14].

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