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INTERPOLATION THEOREMS

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Abstract: In this paper, we discuss the interpolation theorems related the ending points between weak $(1, 1)$ and strong (L^∞, BMO).

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1. Introduction

We know BMO, the space of functions of bounded mean oscillation, was first introduced and used in different contexts by John and Nirenberg [6]. Fefferman and Stein proved that the dual of H^1 is BMO [5]. Since then, in several instances, BMO has served as a substitution for L^∞ [3], [5], [7], [8], [9], [10]. It allows us to use BMO as ending point for a variety of interpolation theorems.

Now, let us start with some basic definitions.

Definition 1.1. T is a *sublinear operator*, if T is an operator from a linear space of functions on $(\mathbf{R^d}, \mu)$ to another linear space of functions on $(\mathbf{R^d}, \mu)$ and satisfies the following two conditions:

1. $T(f + g) \le T(f) + T(g)$,
2. $T(\lambda f) = |\lambda| T(f)$,

where f and g are functions on $(\mathbf{R^d}, \mu)$ in the domain of T, where $\mathbf{R^d}$ is $d-$ dimensional Euclidean space and μ is a measure on $\mathbf{R^d}$.

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Definition 1.2. Let $\mathcal{M}_0(\mathbf{R^d}, \mu)$ be the family of all μ measurable functions with finite values almost everywhere.

Definition 1.3. *The distribution function of* f is defined as follows: for $\lambda \geq 0$ and $f(x) \in \mathcal{M}_0(\mathbf{R^d}, \mu)$,

$$\mu_f(\lambda) = \mu\{\mathbf{x} \in \mathbf{R^d} : |f(\mathbf{x})| > \lambda\}.$$

Definition 1.4. *The decreasing rearrangement* of f is denoted as f^*. So f^* is defined on $[0, \infty)$ and if f belongs to $\mathcal{M}_0(\mathbf{R^d}, \mu)$, then

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\} \quad t \geq 0 = \sup\{\lambda : \mu_f(\lambda) > t\}.$$

Note. $f^*(t)$ is a decreasing function.

Definition 1.5. If $f \in \mathcal{M}_0(\mathbf{R^d}, \mu)$, $f^{**}(t)$ will denote the maximal function of f^* defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \qquad t > 0.$$

2. Restricted Weak Operators

In this section, we are going to prove that T is restricted weak type $(1, 1)$ if T is a sublinear and strong bounded from L^∞ to BMO. We use the notations and definitions as in [2], [1], and [3].

Proposition 2.1. *If* $f \in \mathcal{M}_0(\mathbf{R^d}, \mu)$ *and* $0 < p < \infty$, *then*

$$\int_{\mathbf{R^d}} |f|^p d\mu = p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) d\lambda = \int_0^\infty |f^*(t)|^p dt. \tag{1}$$

Furthermore, in the case $p = \infty$

$$\operatorname{ess}\sup_{\mathbf{x} \in \mathbf{R^d}} |f(\mathbf{x})| = \inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0) \geq f^*(t) \quad t \geq 0. \tag{2}$$

Proof. This proof is on p. 43 of [2]. □

Definition 2.2. *The Lorentz space* $L^{p,q} = L^{p,q}(\mathbf{R^d}, \mu)$ consists of all f in $\mathcal{M}_0(\mathbf{R^d}, \mu)$ for which the quantity

$$\|f\|_{pq} = \begin{cases} \{\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\}^{\frac{1}{q}} & \text{if } 0 < q < \infty\,, \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty \end{cases}$$

is finite.

Note. $L^{p,p} = L^p$ since $\|f\|_{p,p} = \|f\|_p$ by Proposition 2.1.

Definition 2.3. If there exists an M such that

$$t(T\chi_E)^*(t) \leq M\mu(E) \quad \text{for } t > 0,$$

we say that T is of *restricted weak type* $(1,1)$.

Proposition 2.4. *If T is weak type (1,1), then T is of restricted weak type* $(1,1)$.

Proof. Let E be a measurable set in $\mathbf{R^d}$ and χ_E be the characterizing function of E. By the definition weak type (1,1), we have

$$\lambda\mu\{\mathbf{x} : |(T\chi_E)(\mathbf{x})| > \lambda\} \leq M\|\chi_E\|_1. \tag{3}$$

According to the definition of the decreasing rearrangement

$$(T\chi_E)^*(t) = \sup\{\lambda : \mu_{T\chi_E}(\lambda) > t\}.$$

In (3) replace λ by $T\chi_E^*(t)$ and $\mu_{T\chi_E}(\lambda)$ by t. We have

$$t(T\chi_E)^*(t) \leq M\|\chi_E\|_1.$$

But

$$\|\chi_E\|_1 = \int_E 1 d\mu = \mu(E).$$

So, we have

$$t(T\chi_E)^*(t) \leq M\mu(E).$$

Therefore, T is of restricted weak type (1,1). □

Definition 2.5. The definition of $\wedge f_{Q_0}$.

$$\wedge f_{Q_0}(\mathbf{x}) = \sup_{Q \subset Q_0, x \in Q} \frac{1}{|Q|} \int |f(\mathbf{y}) - f_Q| d\mathbf{y} \qquad x \in Q_0,$$

where Q_0 and Q are cubes with sides parallel to the axes in $\mathbf{R^d}$ and $f_Q = \frac{1}{|Q|} \int\limits_Q f(\mathbf{x}) d\mathbf{x}$.

Definition 2.6. *Bounded mean oscillation on* Q_0 are the collection of functions on Q_0 with $\wedge f_{Q_0} \in L^\infty(Q_0)$. We write as $BMO(Q_0)$.

Definition 2.7. The bounded mean oscillation on whole space $\mathbf{R^d}$, we write as BMO. We also can find very detailed definitions and results about BMO in [4] and [5].

Definition 2.8. If there is a constant C such that

$$\|T\chi_E\|_W = \sup_{0<t}\{(T\chi_E)^{**}(t) - (T\chi_E)^*(t)\} \leq C\|\chi_E\|_\infty$$

on $\mathbf{R^d}$, we say that T is of restricted type (∞, ∞).

Proposition 2.9. *If f is an integrable function in Q_0, then*

$$f^{**}(t) - f^*(t) \leq C(\wedge f_{Q_0})^*(t), \quad 0 < t < \frac{|q_0|}{6}.$$

Proof. The proof of this proposition is in [2], p. 377. □

Proposition 2.10. *If T is a bounded operator from L^∞ to BMO, then T is of restricted weak type (∞, ∞).*

Proof. For any measurable set $E \subset \mathbf{R}^d$ with $\mu(E) < \infty$, we obviously have $\chi_E \in L^\infty$. Since T is a bounded operator from L^∞ to BMO, we have

$$\|T\chi_E\|_{BMO} \leq C\|\chi_E\|_\infty.$$

For any cube Q with sides parallel to the axes in $\mathbf{R^d}$

$$\|T\chi_E\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |T\chi_E(\mathbf{x}) - T\chi_{EQ}|d\mathbf{x}.$$

For any $0 < t < \infty$, choose Q_0 with sides parallel to the axes such that $0 < t < \frac{|Q_0|}{6}$.χ_E is an integrable function with support on E. By Proposition 2.9

$$T\chi_E^{**}(t) - T\chi_E^*(t) \leq C(T\chi_{EQ_0})^*(t).$$

Since the decreasing rearrangement is decreasing, we have $(T\chi_{EQ_0})^*(t) \leq (T\chi_{EQ_0})^*(0)$ and the following estimate

$$\begin{aligned} T\chi_E^{**}(t) - T\chi_E^*(t) &\leq C(T\chi_{EQ_0})^*(0) = C \operatorname{ess\,sup}_{\mathbf{R}^n} |\wedge T\chi_{EQ_0}| \\ &\leq C\|\wedge T\chi_{EQ_0}\|_\infty \leq C\|T\chi_E\|_{BMO} \leq C\|\chi_E\|_\infty . \end{aligned}$$

Therefore, T is of restricted weak type (∞, ∞). □

3. Interpolation Theorems

In this section, we are going to prove an interpolation result that shows sublinear operators are strong type (p, p) for $1 < p < \infty$, if these operators are weak type $(1, 1)$ and bounded operators from L^∞ to BMO. We use the notations and definitions as in [2], [1], and [3].

Theorem 3.1. *If T is a sublinear operator and is of restricted weak type*

$(1,1)$ *and of restricted weak type* (∞,∞)*, for any simple function f, we have*

$$\|Tf\|_p \le C_p\|f\|_p \quad 1<p<\infty,$$

where C_p *are dependent only on* T *and* p*. In particular,* T *has a unique extension to a bounded sublinear operator on* L^p *for* $1<p<\infty$*.*

Proof. The proof of this theorem is on p. 386 of [2] and in [1]. □

Theorem 3.2. *If* T *is a sublinear operator and weak type* $(1,1)$*, and a bounded operator from* L^∞ *to* BMO*, then* T *is strong type* (p,p) *for* $1<p<\infty$

$$\|Tf\|_p \le C_p\|f\|_p\,.$$

Proof. Since T is weak type $(1,1)$ and a bounded operator from L^∞ to BMO, by Proposition 2.4 and Proposition 2.10, we have that T is of restricted weak type $(1,1)$ and of restricted weak type (∞,∞). We know that T is also a sublinear operator. By Theorem 3.1, we have that for all simple functions f,

$$\|Tf\|_p \le C_p\|f\|_p \quad 1<p<\infty\,.$$

By Theorem 3.1 again, T has a unique extension to a bounded sublinear operator on L^p. Therefore, we have

$$\|Tf\|_p \le C_p\|f\|_p \quad f\in L^p\,.$$ □

4. Applications

We can use the sublinear interpolation theorem in previous section to prove the following linear operator interpolation theorems.

Definition 4.1. T is a linear operator from a linear space of functions on $(\mathbf{R^d},\mu)$ to another linear space of functions on $(\mathbf{R^d},\mu)$ and satisfies the following two conditions:

1. $T(f+g)=T(f)+T(g)$,
2. $T(\lambda f)=\lambda T(f)$,

where f and g are functions on $(\mathbf{R^d},\mu)$ in the domain of T, where $\mathbf{R^d}$ is $d-$ dimensional Euclidean space and μ is a measure on $\mathbf{R^d}$.

Theorem 4.2. *T is a linear operators from* $L^2(\mathbf{R^d})$ *to* $L^1(\mathbf{R^d})$ *and strong type* $(2,1)$*. T is also a strong bounded linear operator from* L^∞ *to* BMO*. We have T is strong type* (p,p) *for* $1<p<\infty$*.*

Proof. Since $\|f\|_1 \le \|f\|_2$, we have $\|T(f)\|_1 \le \|T(f)\|_2$. We know that T strong $(2,1)$, we have $\|T(f)\|_1 \le \|T(f)\|_2 \le M\|T(f)\|_1$. Then, T is strong type

$(1,1)$. Therefore, T is weak type $(1,1)$. Because T is a linear operator, T is a sublinear operator, too. By Theorem 3.2, we have T is strong type (p,p) for $1 < p < \infty$. □

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