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A NOTE ON OPTIMIZATION PROBLEMS WITH ADDITIVELY SEPARABLE OBJECTIVE FUNCTION AND MAX-SEPARABLE CONSTRAINTS

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Abstract: The aim of this note is to point out the possibility of extending and unifying the results of Shu-Cherng Fang et al [2], Xiao-bing Qu et al [5] and the citations therein to a wider class of optimization problems with additively separable objective function and max-separable constraints. Some ideas of Jajou [3], Jajou et al [4] are used throughout the paper. Besides, we use some results of Cuninghame-Green [1], Vorobjov [6], in which problems with a less general set of feasible solutions and max-separable objective functions were solved.

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1. Introduction

The aim of this note is to point out the possibility of extending and unifying the results from Shu-Cherng Fang et al [2], Xiao-bing Qu et al [5] and the citations

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therein to a wider class of optimization problems with additively separable objective function and max-separable constraints. Some ideas of Jajou [3], Jajou et al [4] are used throughout the paper. Besides, we use some results of Cuninghame-Green [1], Vorobjov [6], in which problems with a less general set of feasible solutions and max-separable objective functions were solved. These results as well as results from Shu-Cherng Fang et al [2], Xiao-bing Qu et al [5] can be obtained as special cases from the results contained in this paper. The paper is organized as follows. Section 2 contains the formulation of the problem, in Section 3 we investigate some properties of the set of feasible solutions of the formulated problem and a transformation of the problem to a 0-1 optimization problem, which is known from the literature as the set covering problem, and results concerning the reduction of the size of the problem are contained in Section 4.

2. Formulation of the Problem

Let us introduce the following notations:

$$J = \{1, \ldots, n\},\ I = \{1, \ldots, m\},\ R = (-\infty, \infty),\ R_+ \equiv (0, \infty)\,,$$

$$R^n = R \times \cdots \times R\ (n\text{-times}),\quad x = (x_1, \ldots, x_n) \in R^n,$$

$r_{ij} : R \longrightarrow Q\ \forall i \in I,\ j \in J$ are strictly increasing continuous functions such that $r_{ij}(R) = Q,\ Q \subseteq R$, (i.e. the range of r_{ij} is equal to $Q,\ Q \subseteq R$), $x = (x_1, \ldots, x_n) \in R^n$,

$$r_i(x) \equiv \max_{j \in J}(r_{ij}(x_j)) \quad \text{for all} \quad i \in I$$

where $f_j :\ R \longrightarrow R$ are continuous functions.

We will consider the following optimization problem:

Problem (P).

$$\text{Minimize } \textstyle\sum_{j \in J} f_j(x_j)$$

subject to

$$r_i(x) \;=\; b_i, \quad i \in I,\ \ \underline{x} \leq\ x,$$

where $\underline{x} \in R^n$, $b_i \in Q,\ i \in I$ are given.

The set of feasible solutions of Problem (P) will be denoted by M.

3. Transformation to a Set Covering Problem

Let us introduce the following notations for all $i \in I,\ j \in J$:

$$x_j^{(i)} \equiv r_{ij}^{-1}(b_i)\,, \quad S_j(x_j) \equiv \{i \in I; r_{ij}(x_j) = b_i\}$$
$$\overline{x}_j \equiv \min_{i\in I} x_j^{(i)}\,, \quad M_j^{(*)} \equiv \{x_j^*\ ;\ f_j(x_j^*) = \min_{\underline{x}_j \le x_j \le \overline{x}_j} f_j(x_j)\}\,.$$

Let us note that under our assumptions about functions $f_j,\ j \in J$, the set $M_j^{(*)}$ is a nonempty and closed subset of R. Let us note further that we can assume w.l.o.g. that $f_j(x_j^*) = 0\ \forall x_j^* \in M_j^{(*)}$ for any $j \in J$ since if it were $f_j(x_j^*) \neq 0$, we could replace function $f_j(x_j)$ with the shifted function $f_j(x_j) - f_j(x_j^*)$, which would have the same properties as $f_j(x_j)$ and the minimal value equal to 0. In what follows we shall accept this assumption and assume from the beginning that $f_j(x_j^*) = 0\ \forall j \in J,\ x_j^* \in M_j^{(*)}$.

Theorem 3.1. *(a) $x \in M$ if and only if $\underline{x} \le x \le \overline{x}$ & $\bigcup_{j\in J} S_j(x_j = I$.*

(b) If $x_j < \overline{x}_j$, then $S_j(x_j) = \emptyset$.

(c) $M \neq \emptyset$ if and only if $\underline{x} \le \overline{x}$ & $\bigcup_{j\in J} S_j(\overline{x}_j) = I$.

(d) If $x \in M$, then $x \le \overline{x}$, (i.e. $\overline{x}$ is the maximum element of M).

Proof. (a) Let for some x the conditions $\underline{x} \le x \le \overline{x}$ & $\bigcup_{j\in J} S_j(x_j) = I$ be fulfilled and let k be an arbitrary index from I. We have to prove that $r_k(x) = b_k$. Since $x_j \le \overline{x_j}\ \forall\ j \in J$, and functions r_{ij} are increasing, we obtain for all $i \in I,\ j \in J$:

$$r_{ij}(x_j) \le r_{ij}(\overline{x}_j) \le r_{ij}(x_j^{(i)}) = b_i.$$

Since $\bigcup_{j\in J} S_j(x_j) = I$, there exists an index $j(k) \in J$ such that $k \in S_{j(k)}(x_{j(k)})$ and thus $r_{kj(k)}(x_{j(k)}) = b_k$. Therefore we have:

$$r_k(x) = \max_{j\in J} r_{kj}(x_j) = r_{kj(k)}(x_{j(k)}) = b_k,$$

what was to be proved.

To prove the opposite assertion, let us assume that the conditions are not fulfilled, i.e. that either $\underline{x} \le x \le \overline{x}$ or $\bigcup_{j\in J} S_j(x_j) = I$ does not hold. If the former condition does not hold, then x is evidently infeasible (either $\underline{x} \not\le x$ or $x_j > x_j^{(i)}$ for some $i \in I$ so that $r_{ij}(x_j) > b_i$ and therefore $r_i(x) > b_i$). If the latter condition is not fulfilled, there exists an index $i \in I$ such that $i \notin S_j(x_j)\ \forall j \in J$ and therefore $r_{ij}(x_j) \neq b_i\ \forall j \in J$ and therefore $r_i(x) \neq b_i$ and $x \notin M$. This completes the proof of assertion (a).

(b) If $x_j < \overline{x}_j$, we have $x_j < x_j^{(i)}$ $\forall i \in I$ so that $r_{ij}(x_j) < b_i$ $\forall i \in I$ and therefore $i \notin S_j(x_j)$ $\forall i \in I$. It follows that $S_j(x_j) = \emptyset$, what we had to prove.

(c) Let us assume that $\underline{x} \leq \overline{x}$ & $\bigcup_{j\in J} S_j(\overline{x}_j) = I$. Then we obtain according to assertion (a) that $\overline{x} \in M$ so that $M \neq \emptyset$.

To prove the opposite assertion, let us assume that the conditions are not fulfilled. If $\underline{x} \not\leq \overline{x}$, then set M is evidently empty. Assume now that $\bigcup_{j\in J} S_j(\overline{x}_j) \neq I$ (i.e. $\bigcup_{j\in J} S_j(\overline{x}_j) \subset I$) and let $x \in M$. It must be $\underline{x} \leq x \leq \overline{x}$. Since $x_j \leq \overline{x}_j$ $\forall j \in J$, it must be $S_j(x_j) \subseteq S_j(\overline{x}_j)$ $\forall j \in J$ so that $\bigcup_{j\in J} S_j(x_j) \subseteq \bigcup_{j\in J} S_j(\overline{x}_j) \subset I$ and therefore according to (a) $x \notin M$. This contradiction completes the proof of assertion (c).

(d) The assertion follows immediately form assertion (a).

The proof of the theorem is completed. □

It follows from Theorem 3.1 that the optimal solution of Problem (P) can be found among points $x = (x_1, \ldots, x_n)$, for which there exists an index set K, $K \subseteq J$ such that $x_j = \overline{x}_j$ for $j \in K$, and $x_j = x_j^*$ for $j \in J \setminus K$, where $\bigcup_{j\in K} S_j(\overline{x}_j) = I$, i.e. the sets $S_j(\overline{x}_j)$, $j \in K$ must cover the set I. This fact leads to the idea to reformulate Problem (P) as the following set covering problem.

Problem (P1).

$$\text{Minimize } \sum_{j\in J} c_j y_j$$

subject to

$$\sum_{j\in J} a_{ij} y_j \geq 1, \ \forall\, i \in I,$$

$$y_j \in \{0,1\}\ , \ \forall\, j \in J,$$

where $c_j = f_j(\overline{x}_j)$, $a_{ij} = 1$ if $i \in S_j(\overline{x}_j)$, and $a_{ij} = 0$ otherwise.

The optimal solution of this problem represents the “cheapest” covering of set I with subsets $S_j(\overline{x}_j)$ under the assumption that the cost of $S_j(\overline{x}_j)$ is equal to $c_j = f_j(\overline{x}_j)$. To include $S_j(\overline{x}_j)$ into the covering of I means to set $y_j = 1$ and $x_j = \overline{x}_j$. If $S_j(\overline{x}_j)$ is not included into the covering, we set $y_j = 0$ and can choose any $x_j \in [\underline{x}_j, \overline{x}_j]$. Since we want to minimize the objective function, it is natural in this case to choose x_j equal to any element of $M_j^{(*)}$ so that the value of f_j is as small as possible (i.e. it is under our assumptions equal to zero). Vector x chosen in this way will be called “connected with the feasible solution y of Problem (P1)”. Let us remark that the optimal solution $x^{(opt)}$ of Problem (P) must be among the vectors x connected with one of the optimal solutions of Problem (P1) $y^{(opt)}$. Note that $y!^{(opt)}$ represents the cheapest covering of the index set I.

Therefore if $y^{(opt)} = (y_1^{(opt)}, \dots, y_n^{(opt)})$ is the optimal solution of Problem (P1), we can obtain the optimal solution $x^{(opt)} = (x_1^{(opt)}, \dots, x_n^{(opt)})$ of Problem (P) as follows:

$$x_j^{(opt)} = \overline{x}_j, \text{ if } y_j^{(opt)} = 1, \; x_j^{(opt)} = x_j^* \in M_j^{(*)} \text{ otherwise.}$$

Let us note that in this case we will obtain:

$$\sum_{j \in J} c_j y_j^{(opt)} = \sum_{j \in J} f_j(x_j^{(opt)}).$$

Remark 3.1. Problem (P1) can be solved by making use of some methods from the literature appropriate for solving set covering problems. Another transformation to a 0-1 optimization problem and the corresponding branch-and-bound algorithm proposed in Shu-Cherng Fang et al [2] can be also applied to transforming and solving Problem (P).

Remark 3.2. To reduce the necessary computational effort connected with solving Problem (P1), it is desirable to reduce the size of our original Problem (P), if possible. In the next section we will present a reduction procedure, which makes possible to reduce the size of Problem (P) and indirectly the size of the associated set covering Problem (P1).

4. Reduction of the Problem Size

Before starting to solve Problem (P), we can apply to it a reduction procedure similar to that proposed for more special problems in Shu-Cherng Fang et al [2]. We will describe the successive steps of the reduction procedure adjusted to Problem (P).

Reduction Procedure: (0) Input index sets I, J;

(1) If either $J = \emptyset$ or $I = \emptyset$, then go to (5). Otherwise set $J^{(0)} := J$, $I^{(0)} := I$ and determine sets $S_j(\overline{x}_j)$ for $j \in J$.

(2) Let $K^{(1)} \equiv \{j \in J \; ; \; \overline{x}_j \in M_j^{(*)}\}$. If $K^{(1)} = \emptyset$, go to (3). Otherwise set $L^{(1)} \equiv \bigcup_{j \in K^{(1)}} S_j(\overline{x}_j)$ and $x_j^{(opt)} := \overline{x}_j \; \forall j \in K^{(1)}$ and eliminate $j \in K^{(1)}$ from J and $i \in L^{(1)}$ from I, (i.e. set $J := J \setminus K^{(1)}, \; I := I \setminus L^{(1)}$) and go to (1)

(3) Let $K^{(2)} \equiv \{j \in J \; ; \exists \; i \in S_j(\overline{x}_j) \; \& \; i \notin \bigcup_{k \in (J \setminus \{j\})} S_k(\overline{x}_k)\}$. If $K^{(2)} = \emptyset$, go to (4). Otherwise set $L^{(2)} \equiv \; \bigcup_{j \in K^{(2)}} S_j(\overline{x}_j)$; then set $x_j^{(opt)} = \overline{x}_j \; \forall j \in K^{(2)}$

and eliminate indexes $j \in K^{(2)}$ from J, and $i \in L^{(2)}$ from I, (i.e. set $J := J \setminus K^{(2)}$, $I := I \setminus L^{(2)}$), and go to (1).

(4) Let $K^{(3)} \equiv \{j \in J \; ; \; \exists k(j) \in J$ such that $S_j(\overline{x}_j) \subseteq S_{k(j)}(\overline{x}_{k(j)})$ & $f(\overline{x}_j) \geq f_{k(j)}(\overline{x}_{k(j)})\}$. If $K^{(3)} = \emptyset$, go to (5). Otherwise set $x_j^{opt} = x_j^* \in M_j^{(*)}$ $\forall j \in K^{(3)}$ and eliminate $K^{(3)}$ from J, (i.e. set $J := J \setminus K^{(3)}$).

(5) The reduction process is finished. Establish $x^{(opt)}$ and stop.

Remark 4.1. (a) Establishing of $x^{(opt)}$ means either to solve the reduced $(0,1)-$ problem for the reduced sets J, I (if in step (5) both J and I are nonempty) or set $x_j^{(opt)} = x_j^* \in M_j^{(*)}$ (if $J \neq \emptyset$ and $I = \emptyset$). If in step (5) $J = \emptyset$, all coordinates of $x^{(opt)}$ are already determined (if in this case $I \neq \emptyset$, $x^{(opt)}$ satisfies automatically the remaining equations with $i \in I$).

(b) Since on each iteration of the reduction process at least one index of set $I \cup J$ is eliminated, the reduction process terminates after maximally $m+n$ iterations.

Example 4.1. Let $r_{ij}(x_j) = a_{ij}x_j$, $\underline{x}_j = 0$, $a_{ij} \in [0,1]$, $f_j(x_j) = c_j x_j$, $c_j \geq 0$, $b_i \in [0,1]$, $\forall i$, j. Then we obtain a problem similar to the problem considered in [5]. Note that in [5] it is additionally required $x_j \in [0,1]$. We can include the example from [5] into our general scheme as follows:

(1) If in problems from [5] some coefficient a_{ij} is equal to zero, we can set formally $x_j(i) = \infty$.

(2) We define index set $J^* \equiv \{j \in J; a_{ij} > b_i \; \forall \; i \in I\}$. If $J^* = \emptyset$, it is for any feasible $x = (x_1, \ldots, x_n)^T$ $x_j \leq 1$ $\forall j \in J$. If $J^* \neq \emptyset$, we incorporate formally the inequalities $x_j \leq 1$ by introducing an additional equation of the form $\max_{j \in J}(a_{m+1j}x_j) = b_{m+1}$, where $b_{m+1} = 1$, $a_{m+1j} = 1$ for all $j \in J^*$, and $a_{m+1j} = 0$ for $j \in J \setminus J^*$. After that, we calculate $\overline{x}_j$ as above.

The following numerical example illustrates the algorithm proposed above. It shows that the reduction procedure may in some cases even eliminate the solving the set covering problem (P1) at all.

Example 4.2. Let us consider the following optimization problem:

Minimize

$$\sum_{j \in J} f_j(x_j) = x_1 + 2x_2 + x_3^2 + (-x_4 + 2/3) + (|x_5 - 1/2|) + |x_6|$$

subject to

$$r_1(x) = \max\{x_1 - 2, 2x_2 - 6, x_3 - 1, (x_4 - 2)^3, x_5 - 10, (x_6 - 11)^3\} = 0,$$

$r_2(x) = \max\{x_1 - 5, x_2 - 3, x_3 - 1, x_4 - 1, x_5 - 1, x_6 - 20\} = 0,$

$r_3(x) = \max\{(x_1 - 3)^3, x_2 - 3, x_3 - 8, 2x_4 - 8, 3x_5 - 12, x_6 - 5\} = 0,$

$r_4(x) = \max\{(2x_1 - 14)^3, x_2^2 - 25, x_3^3 - 27, 3x_4 - 2, x_5 - 3, x_6 - 8\} = 0,$

$r_5(x) = \max\{(x_1 - 3)^3, x_2 - 18, 2x_3 - 18, x_4 - 1, x_5 - 2, x_6 - 1\} = 0,$

$x_j \geq 0, \quad \forall j \in J.$

We have therefore in this example:

$m = 5,\ n = 6,\ I = \{1, \ldots, 5\},\ J = \{1, \ldots, 6\},\ \underline{x} = 0 \in R^6,\ b_i = 0\ \forall\ i \in I,$

$\overline{x} = (2, 3, 1, 2/3, 1, 1)^T,\ x^* = (0, 0, 0, 2/3, 1/2, 0)^T,$

$S_1(\overline{x}_1) = \{1\},\ S_2(\overline{x}_2) = \{1, 2, 3\},\ S_3(\overline{x}_3) = \{1, 2\},$

$S_4(\overline{x}_4) = \{4\},\ S_5(\overline{x}_5) = \{2\},\ S_6(\overline{x}_6) = \{5\},$

$(f_1(\overline{x}_1),\ \ldots,\ f_6(\overline{x}_6)) = (2, 6, 1, 0, 1/2, 1),$

$(f_j(x_j^*) = 0\ \forall j \in J.$

The corresponding Problem (P1) without applying he reduction procedure is the following:

Minimize

$$\sum_{j \in J} c_j y_j = 2y_1 + 6y_2 + y_3 + 0y_4 + (1/2)y_5 + y_6\,,$$

subject to

$y_1 + y_2 + y_3 \geq 1 \quad y_2 + y_3 + y_4 \geq 1$

$y_2 \geq 1 \quad y_4 \geq 1 \quad y_6 \geq 1 \quad y_j \in \{0, 1\} \quad \forall j \in J\,.$

We will apply now the reduction procedure proposed in Section 4 above. On the first iteration of the procedure we obtain:

Iteration 1. (0) $J := \{1, \ldots, 6\}$, $I := \{1, \ldots, 5\}$;

(1) $I \neq \emptyset, \quad J \neq \emptyset$, go to (2);

(2) $K^{(1)} = \{4\}, \quad L^{(1)} = \{4\}, x_4^{(opt)} = \overline{x}_4 = x_4^* = 2/3,$

$J := J \setminus K^{(1)} = \{1, 2, 3, 5, 6\}, \quad I := I \setminus L^{(1)} = \{1, 2, 3, 5\}$; go to (1).

Iteration 2. (1) $I \neq \emptyset, \quad J \neq \emptyset$;

(2) $K^{(1)} = \emptyset$, go to (3);

(3) $K^{(2)} = \{2, 6\}, \quad L^{(2)} = \{1, 2, 3, 5\}, \quad x_2^{(opt)} := \overline{x}_2 = 3, \quad x_6^{(opt)} := \overline{x}_6 = 1$, $J := J \setminus K^{(2)} = \{1, 3, 5\}, \quad I := I \setminus L^{(2)} = \emptyset$; go to (1).

Iteration 3. (1) $I = \emptyset$, go to (5);

(5) $J = \{1, 3, 5\}$, $x_1^{(opt)} = x_1^* = 0, \quad x_3^{(opt)} = x_3^* = 0, \quad x_5^{(opt)} = x_5^* = 1/2$

$x^{(opt)} = (0, 3, 0, 2/3, 1/2, 1)$,

the optimal value of the objective function is

$$\sum_{j \in J} f_j(x_j^{(opt)}) = 0 + 6 + 0 + 0 + 0 + 1 = 7,$$

STOP.

Note that in this case the reduction process leads us directly to the optimal solution of Problem (P), i.e. we eliminated completely the necessity to solve any set covering problem.

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