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CLASSIFICATION OF SOLUTIONS TO SYSTEMS OF TWO-SIDED EQUATIONS WITH INTERVAL COEFFICIENTS

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Abstract: The main purpose of the paper is to extend some results about one-sided systems of $(\max, +)$-linear equations with interval coefficients (i.e. systems with variables on one side of the equations only) from Cechlárová [2], Cechlárová et al [3] to a general class of two-sided systems (i.e. systems with variables on both sides of the equations). Basic concepts and definitions were motivated by Cechlárová [2], Cechlárová et al [3] and Rohn [7].

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Key Words: (max, plus) linear equations, two-sided equations, interval coefficients

1. Introduction

The main purpose of the paper is to propose a general theoretical model, which makes possible to extend some results from Cechlárová [2], Cechlárová et al [3] concerning the one-sided interval systems of $(\max, +)$-linear equations to two-sided systems considered in Butkovič et al [1] as well as to some further systems

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with a similar structure, which satisfy assumptions of the general model. Basic concepts and their definitions were motivated by Cechlárová [2], Cechlárová et al [3], Rohn [7]. The first results concerning one-sided systems of $(\max, +)$- or $(\max, .)$-equations can be found in Cuninghame-Green [4], Vorobjov [8]. Two-sided systems in a more general context were studied in Cuninghame-Green et al [5]. The following motivating example shows situations, where the results of this paper can be applied.

Example 1.1. Let two groups of transport means (e.g. trains and buses) be considered. Let x_j, y_k denote departure times of train j or bus k for $j \in J \equiv \{1, \ldots, n\}$, $k \in K \equiv \{1, \ldots, r\}$ respectively. Let us assume that we have m places (villages, towns, stations) $i \in I \equiv \{1, \ldots, m\}$. Let a_{ij}, b_{ik} denote the traveling time of train j or bus k to place i, respectively for all $i \in I$, $j \in J$, $k \in K$. Therefore, $x_j + a_{ij}$, $y_k + b_{ik}$ are arrival times of train j or bus k to i, respectively.

The last train arrives to place i at a time $a_i(x) \equiv \max_{j\in J}(a_{ij} + x_j)$ and the last bus arrives to i at a time $b_i(y) \equiv \max_{k\in K}(b_{ik} + y_k)$, where we set $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_r)^T$. A synchronization will mean to determine departure times x_j, y_k for all $j \in J$, $k \in K$ (i.e. finding values x_j, y_k) in such a way that $a_i(x)$, $b_i(y)$ satisfy some relation $R_i \in \{=, \leq, \geq\}$. Since we cannot choose x_j, y_k quite arbitrarily, it is additionally required that $x_j \in \left[\underline{x}_j, \overline{x}_j\right]$, $y_k \in \left[\underline{y}_k, \overline{y}_k\right]$, where $\underline{x}_j$, $\overline{x}_j$, $\underline{y}_k$, $\overline{y}_k$ are given real numbers.

Until now we have assumed that traveling times a_{ij}, b_{ik} are given positive numbers. In real situations, however, the traveling times cannot usually be prescribed exactly. We shall assume that the traveling times may vary in intervals, i.e. that $a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$ and $b_{ik} \in \left[\underline{b}_{ik}, \overline{b}_{ik}\right]$, where $\underline{a}_{ij}, \overline{a}_{ij}, \underline{b}_{ik}, \overline{b}_{ik}$ are given real numbers, i.e. a_{ij}, b_{ik} are given as interval input parameters in the sense of interval mathematics. In the further part of this contribution, we shall propose how to proceed in this situation, i.e. how to define an appropriate concept of synchronization and how to find "synchronized" departure times x_j, y_k.

2. Notations and Problem Formulation

Similarly to Cechlárová [2] we define matrix intervals $\mathbf{A}$, $\mathbf{B}$ as follows:

$$\mathbf{A} \equiv \left[\underline{A}, \overline{A}\right] \equiv \left\{ A;\, \underline{A} \leq A \leq \overline{A} \right\}\,, \tag{1}$$

where $\underline{A}$, $\overline{A}$ are given (m, n)-matrices with real elements $\underline{a}_{ij}$, $\overline{a}_{ij}$, and

$$\mathbf{B} \equiv \left[\underline{B}, \overline{B}\right] \equiv \left\{ B;\, \underline{B} \leq B \leq \overline{B} \right\}\,, \tag{2}$$

where $\underline{B}$, $\overline{B}$ are given (m,k)-matrices with real elements $\underline{b}_{ip}$, $\overline{b}_{ip}$ for $i = 1,\dots,m$, $p = 1,\dots,k$.

Let $X \subseteq R^n$, $Y \subseteq R^k$ be given sets and let vector functions $f : \mathbf{A}\times X \to R^m$, $g : \mathbf{B} \times Y \to R^m$ be defined for $A \in \mathbf{A}$, $x \in X$, $B \in \mathbf{B}$, $y \in Y$ 'by rows' as follows

$$f(A,x) = \big(f_1(a_1,x),\dots,f_m(a_m,x)\big)^T,$$
$$g(B,y) = \big(g_1(b_1,y),\dots,g_m(b_m,y)\big)^T,$$

where a_i, b_i for $i = 1,\dots,m$ denote the rows of matrices A, B, respectively.

We will assume that functions $f(A,x)$, $g(B,y)$ are non-decreasing in A, B, respectively. Namely, if $A^{(1)}, A^{(2)} \in \mathbf{A}$, and $A^{(1)} \leq A^{(2)}$, then $f(A^{(1)},x) \leq f(A^{(2)},x)$, and similarly, if $B^{(1)}, B^{(2)} \in \mathbf{B}$, and $B^{(1)} \leq B^{(2)}$, then $g(B^{(1)},y) \leq g(B^{(2)},y)$ for any fixed $x \in X, y \in Y$. Further, we shall assume that functions $f_i(a_i,x), g_i(b_i,y), i = 1,\dots,m$ are continuous in a_i, b_i for any fixed $x \in X$, $y \in Y$.

Definition 2.1. Interval system of equations

$$f(\mathbf{A},x) = g(\mathbf{B},y), \tag{3}$$

is defined as the set of systems

$$f(A,x) = g(B,y), \tag{4}$$

with $A \in \mathbf{A}$, $B \in \mathbf{B}$. A question arises how a solution (x,y) for interval systems (3) can be defined. In the next section several concepts of solutions for the interval systems will be proposed.

3. Solution Concepts for Interval Systems of Equations

Definition 3.1. (Compare Cechlárová, [2]) A pair $(x,y) \in X \times Y$ is called *weak solution* of (3) if there exist $A \in \mathbf{A}$, $B \in \mathbf{B}$ such that equation (4) holds.

Definition 3.2. (Compare Cechlárová, [2]) A pair $(x,y) \in X \times Y$ is called:

1. *left tolerance solution* of (3) if for any $A \in \mathbf{A}$ there is $B \in \mathbf{B}$ such that equation (4) holds,

2. *right tolerance solution* of (3) if for any $B \in \mathbf{B}$ there is $A \in \mathbf{A}$ such that equation (4) holds,

3. *tolerance solution* of (3) if it is both left and right tolerance solution.

Definition 3.3. A pair $(x,y) \in X \times Y$ is called:

1. *left strong solution* of (3) there is $B \in \mathbf{B}$ such that for any $A \in \mathbf{A}$

equation (4) holds,

2. *right strong solution* of (3) if there is $A \in \mathbf{A}$ such that for any $B \in \mathbf{B}$ equation (4) holds,

3. *strong solution* of (3) if for each $A \in \mathbf{A}$, $B \in \mathbf{B}$ equation (4) holds.

4. Properties of the Solution Concepts

Theorem 4.1. *A pair $(x, y) \in X \times Y$ is a weak solution of (3) if and only if relations*

$$f(\underline{A}, x) \leq g(\overline{B}, y), \quad f(\overline{A}, x) \geq g(\underline{B}, y) \tag{5}$$

hold true.

Proof. Let (x, y) be a weak solution of (3), i.e. $f(A, x) = g(B, y)$ for some $A \in \mathbf{A}$, $B \in \mathbf{B}$. We have

$$(\underline{A}, x) \leq f(A, x) = g(B, y) \leq g(\overline{B}, y)$$

and

$$g(\underline{B}, y) \leq g(B, y) = f(A, x) \leq f(\overline{A}, x)$$

so that (5) holds.

Let us assume now that inequalities (4) hold for some fixed (x, y). We have to prove that (x, y) is a weak solution of (3). We will introduce for $i = 1, \ldots, n$ the following notations to simplify the explanations

$$\underline{f}_i(x) \equiv f(\underline{a}_i, x), \overline{f}_i(x) \equiv f(\overline{a}_i, x), \quad \underline{g}_i(y) \equiv g(\underline{b}_i, y), \overline{g}_i(y) \equiv g(\overline{a}_i, y)\,,$$

$$J_i \equiv \left[\underline{f}_i(x), \overline{f}_i(x)\right] \cap \left[\underline{f}_i(x), \overline{g}_i(y)\right] \cap \left[\underline{g}_i(y), \overline{g}_i(y)\right] \cap \left[\underline{g}_i(y), \overline{f}_i(x)\right]\,.$$

It follows from (5), and from $\underline{A} \leq \overline{A}$, $\underline{B} \leq \overline{B}$, that the interval J_i is non-empty. Let us recall that we have assumed that $f_i(a_i, x)$, $g_i(b_i, y)$ are continuous functions of a_i, b_i respectively. Interval J_i is a common part of their image sets, therefore if c_i is an arbitrary element of J_i, then there exist $a_i^{(c_i)}$ and $b_i^{(c_i)}$ such that

$$f_i(\underline{a}_i, x) \leq f_i(a_i^{(c_i)}, x) \leq f_i(\overline{a}_i, x), \quad g_i(\underline{b}_i, y) \leq g_i(b_i^{(c_i)}, y) \leq g_i(\overline{b}_i, y)\,,$$

$$f_i(a_i^{(c_i)}, x) = c_i, \quad g_i(b_i^{(c_i)}, y) = c_i$$

hold true. Hence, $f_i(a_i^{(c_i)}, x) = c_i = g_i(b_i^{(c_i)}, y)$ holds for every $i \in \{1, \ldots, m\}$. In the other words, if $c = (c_1, \ldots, c_m)$, $c_i \in J_i$ for all $i = 1, \ldots, m$, and if $A^{(c)}$ is the matrix, the i-th row of which is equal to $a_i^{(c_i)}$ and similarly $B^{(c)}$ is the matrix, the i-th row of which is equal to $b_i^{(c_i)}$, then $A^{(c)} \in \mathbf{A}$, $B^{(c)} \in \mathbf{B}$, and $f(A^{(c)}, x) = g(B^{(c)}, y)$ holds, and therefore (x, y) is a weak solution of (3), which completes the proof. $\square$

Theorem 4.2. *A pair $(x, y) \in X \times Y$ is:*

1. *a left tolerance solution of (3) if and only if*

$$f(\underline{A}, x) \geq g(\underline{B}, y), \quad f(\overline{A}, x) \leq g(\overline{B}, y); \tag{6}$$

2. *a right tolerance solution of (3) if and only if*

$$f(\underline{A}, x) \leq g(\underline{B}, y), \quad f(\overline{A}, x) \geq g(\overline{B}, y); \tag{7}$$

3. *a tolerance solution of (3) if and only if*

$$f(\underline{A}, x) = g(\underline{B}, y), \quad f(\overline{A}, x) = g(\overline{B}, y). \tag{8}$$

Proof. We will prove only part (i) of the theorem, since part (ii) can be proved by analogy and part (iii) is an evident consequence of parts (i), (ii) and of Definition 3.3(iii).

If (x, y) is a left tolerance solution, then for any $A \in \mathbf{A}$ there exists a matrix $B(A) \in \mathbf{B}$ such that $f(A, x) = g(B(A), y)$. Therefore also for matrix $\underline{A}$ there exists a matrix $B(\underline{A}) \in \mathbf{B}$ such that $f(\underline{A}, x) = g(B(\underline{A}), y)$ and we have

$$f(\underline{A}, x) = g(B(\underline{A}), y) \geq g(\underline{B}, y)\,.$$

Similarly for matrix $\overline{A}$ there exists a matrix $B(\overline{A}) \in \mathbf{B}$ such that

$$f(\overline{A}, x) = g(B(\overline{A}), y) \leq g(\overline{B}, y)\ .$$

Therefore inequalities (6) are fulfilled.

To prove the opposite assertion let us assume that inequalities (6) are fulfilled. We have to prove that (x, y) is a left tolerance solution of (3). Let $A \in \mathbf{A}$ be arbitrarily chosen. We obtain according to (6) that

$$f(A, x) \geq f(\underline{A}, x) \geq g(\underline{B}, y)$$

and further

$$f(A, x) \leq f(\overline{A}, x) \leq g(\overline{B}, y)\,,$$

and thus

$$g(\underline{B}, y) \leq f(A, x) \leq g(\overline{B}, y)\,.$$

Let us set for any i, $1 \leq i \leq m$, $f_i(a_i, x) = c_i(A)$. We have then

$$g_i(\underline{b_i}, y) \leq c_i(A) \leq g_i(\overline{b_i}, y).$$

Since $g_i(b_i, x)$ is a continuous function of b_i, there exists $b_i^{(c_i(A))}$ such that

$$g_i(b_i^{(c_i(A))}, y) = c_i(A) = f_i(a_i, x)\,.$$

Let $c(A) = (c_1(A), \ldots, c_m(A))$ and let $B^{(c(A))}$ denote the matrix, the i-th row of which is equal to $b_i^{(c_i(A))}$. We obtain that for arbitrary matrix $A \in \mathbf{A}$ there exists a matrix $B(A) \equiv B^{(c(A))} \in \mathbf{B}$ such that

$$f(A, x) = g(B(A), y)\ .$$

This proves that (x, y) is a left tolerance solution of (3). □

Theorem 4.3. *A pair $(x, y) \in X \times Y$ is:*

1. *a left strong solution of (3) if and only if*

$$g(\underline{B}, y) \le f(\underline{A}, x) = f(\overline{A}, x) \le g(\overline{B}, y); \tag{9}$$

2. *a right strong solution of (3) if and only if*

$$f(\underline{A}, x) \le g(\underline{B}, y) = g(\overline{B}, y) \le f(\overline{A}, x); \tag{10}$$

3. *a strong solution of (3) if and only if*

$$f(\overline{A}, x) = g(\underline{B}, y), \quad f(\underline{A}, x) = g(\overline{B}, y). \tag{11}$$

Proof. We shall first prove part (i) of the theorem, the part (ii) can be proved by analogy.

If (x, y) is a left strong solution, then there exists a matrix $B \in \mathbf{B}$ such that for any $A \in \mathbf{A}$, the equality $f(A, x) = g(B, y)$ holds. Therefore, for matrices $\underline{A}$, $\overline{A}$, we have $f(\underline{A}, x) = g(B, y) = f(\overline{A}, x)$. Hence

$$g(\underline{B}, y) \le g(B, y) = f(\underline{A}, x) = f(\overline{A}, x) = g(B, y) \le g(\overline{B}, y)\,.$$

Thus, inequalities (9) are fulfilled.

To prove the opposite assertion, let us assume that inequalities (9) are fulfilled. We have to show that (x, y) is a left strong solution of (3). By assumption, for any i, $1 \le i \le m$, the inequalities $g_i(\underline{b}_i, y) \le f_i(\underline{a}_i, x) = f_i(\overline{a}_i, x) \le g_i(\overline{b}_i, y)$ hold. Since $g_i(b_i, x)$ is a continuous function of b_i, there exists $b_i^\star$, $\underline{b}_i \le b_i^\star \le \overline{b}_i$, such that

$$g_i(b_i^\star, y) = f_i(\underline{a}_i, x) = f_i(\overline{a}_i, x)\,.$$

Let $A \in \mathbf{A}$ be arbitrarily chosen. Then we have

$$g_i(b_i^\star, y) = f_i(\underline{a}_i, x) \le f_i(a_i, x) \le f_i(\overline{a}_i, x) = g_i(b_i^\star, y)\,.$$

If we denote by $B^\star$ the matrix, the i th row of which is $b_i^\star$, for every i, $1 \le i \le m$, then we have shown that there exists a matrix $B^\star \in \mathbf{B}$ such that for arbitrary matrix $A \in \mathbf{A}$

$$f(A, x) = g(B^\star, y)\,.$$

This proves that (x, y) is a left strong solution of (3).

For the proof of part (iii), let us assume that (x, y) is a strong solution of (3). Then equalities (11) are fulfilled by the definition of the strong solution. To prove the opposite direction, let us assume that equalities (11) are fulfilled. We have to prove that (x, y) is a strong solution of (3). Let $A \in \mathbf{A}, B \in \mathbf{B}$ be arbitrarily chosen. Using equalities (11) and the monotonicity of f, g, we obtain:

$$g(\underline{B}, y) \le g(B, y) \le g(\overline{B}, y) = f(\underline{A}, x) \le f(A, x) \le f(\overline{A}, x) = g(\underline{B}, y)\,,$$

so that $f(A, x) = g(B, y)$. Since A, B were arbitrarily chosen, vector (x, y) is a strong solution of (3), and the proof is completed. □

Theorem 4.4. *Following statements hold true for any pair $(x, y) \in X \times Y$.*

1. (x, y) is a strong solution of (3) if and only if (x, y) is a left strong solution and a right strong solution of (3).

2. (x, y) is a tolerance solution of (3), if and only if (x, y) is a left tolerance solution and a right tolerance solution of (3).

3. if (x, y) is a strong solution (left strong solution, right strong solution) of (3), then (x, y) is a tolerance solution (left tolerance solution, right tolerance solution) of (3).

4. if (x, y) is a left tolerance solution, or a right tolerance solution of (3), then (x, y) is a weak solution of (3).

Proof. We shall only prove the 'if' implication in part (i). The converse implication in (i) and parts (ii)–(iv) of the theorem follow easily from definitions, under assumption $\underline{A} \le \overline{A},\ \underline{B} \le \overline{B}$.

Let us assume that (x, y) is a left strong solution, and also a right strong solution. We shall show that then (x, y) is a strong solution. By Theorem 4.3 we have $g(\underline{B}, y) \le f(\underline{A}, x) = f(\overline{A}, x) \le g(\overline{B}, y)$ and $f(\underline{A}, x) \le g(\underline{B}, y) = g(\overline{B}, y) \le f(\overline{A}, x)$. Thus, we get

$$f(\overline{A}, x) = f(\underline{A}, x) \le g(\underline{B}, y) = g(\overline{B}, y) \le f(\overline{A}, x)\,.$$

This implies

$$f(\underline{A}, x) = f(\overline{A}, x) = g(\underline{B}, y) = g(\overline{B}, y)\,.$$

Hence, (x, y) is a strong solution, in view of Theorem 4.3. □

The relations between solution concepts described in Theorem 4.4 are shown graphically in Figure 1.

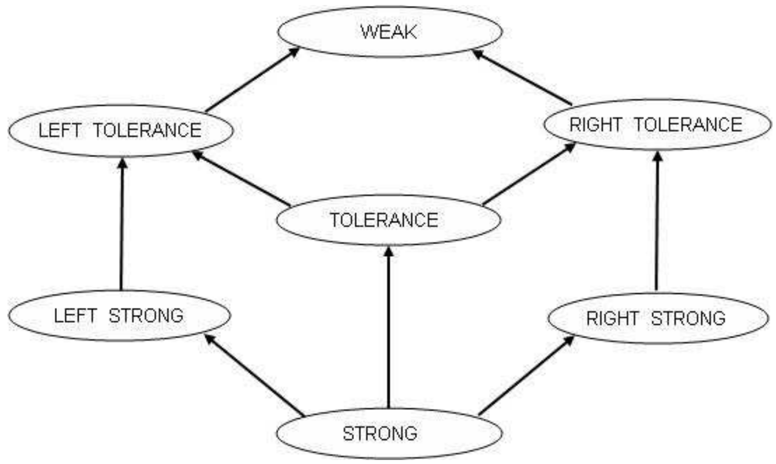

Figure 1: Relations between solution concepts

In the sequel, we will bring some examples to the theoretical results proved in this section.

Remark 4.1. (i) Similar results as in Theorems 4.1–4.4 can be obtained for interval inequalities by introducing slack variables in an appropriate way.

(ii) Necessary and sufficient conditions for other solution concepts considered in Myšková [6] could be derived in a similar way.

Example 4.1. In the motivating example in the Introduction we have for $i = 1, \dots, m$

$$f_i(a_i, x) = \max_{j \in J}(a_{ij} + x_j), \quad g_i(b_i, y) = \max_{k \in K}(b_{ik} + y_k)\,,$$

$$X = \Big\{x \in R^n;\, \underline{x} \le x \le \overline{x}\Big\}, \quad Y = \Big\{y \in R^k;\, \underline{y} \le y \le \overline{y}\Big\}\,.$$

It can be easily seen that every function $f_i(a_i, x)$ $\big(g_i(b_i, y)\big)$ is nondecreasing and continuous in a_i (b_i). Therefore, all the assumptions that are necessary for proving the above theorems are fulfilled. Let us note that a weak, strong or tolerance solution of the corresponding interval system of equations can be found by solving the systems of inequalities in the corresponding theorems from the preceding section by the algorithm proposed in Butkovič et al [1].

Example 4.2. Let us set for $i = 1, \dots, m$

$$f_i(a_i, x) = \max_{j \in J}\big(\min(a_{ij}, x_j)\big), \quad g_i(b_i, y) = \max_{k \in K}\big(\min(b_{ik}, y_k)\big)\,,$$

$$X = \Big\{x \in R^n;\, \underline{x} \le x \le \overline{x}\Big\}, \quad Y = \Big\{y \in R^k;\, \underline{y} \le y \le \overline{y}\Big\}\,,$$

where sets J, K are given finite index sets. Since functions $f_i(a_i, x)$ $\big(g_i(b_i, y)\big)$ are nondecreasing and continuous in a_i (b_i) for any fixed $x \in X, y \in Y$, all necessary assumptions for the validity of Theorems 4.1–4.4 are fulfilled again. Therefore the theorems remain valid for $(\max, \min)$-linear systems in the so-called fuzzy algebra considered e.g. in Myšková [6].

We shall bring two further examples, in which all assumptions made above are satisfied.

Example 4.3. We set for $i = 1, \dots, m$

$$f_i(a_i, x) = \max_{j \in J}(a_{ij} \cdot x_j), \quad g_i(b_i, y) = \max_{k \in K}(b_{ik} \cdot y_k)\,,$$

$$X = \Big\{x \in R^n_+;\, \underline{x} \le x \le \overline{x}\Big\}, \quad Y = \Big\{y \in R^k_+;\, \underline{y} \le y \le \overline{y}\Big\}\,,$$

where R^n_+, R^k_+ are non-negative orthants. In this example, the continuity of $f_i(a_i, x)$ $\big(g_i(b_i, y)\big)$ with respect to a_i (b_i) for any fixed x, y is evident. The monotonicity of $f_i(a_i, x)$ $\big(g_i(b_i, y)\big)$ in a_i (b_i) follows from the appropriate choice of X and Y, which are subsets of non-negative orthants R^n_+, R^k_+.

Example 4.4. Let us set for $i = 1, \dots, m$

$$f_i(a_i, x) = \sum_{j \in J}(a_{ij} \cdot x_j), \quad g_i(b_i, y) = \sum_{k \in K}(b_{ik} \cdot y_k)\,,$$

$$X = \Big\{x \in R^n_+;\, \underline{x} \le x \le \overline{x}\Big\}, \quad Y = \Big\{y \in R^k_+;\, \underline{y} \le y \le \overline{y}\Big\}\,.$$

Here the continuity and monotonicity of $f_i(a_i, x)$ $\big(g_i(b_i, y)\big)$ in a_i (b_i) are again ensured. The system can be characterized as a 'mixed' system with usual linear functions on the left hand sides and $(\max, +)$-linear functions on the right hand sides.

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